CHAPTER 3. PROPERTIES OF CHAOS

This chapter will introduce several ways to test and predict for chaos. Since the approaches have been well established in many chaos and time series textbooks (for instance, Brock, et al. 1991; Hilborn, 1994; Kantz and Schreiber, 1997; Alligood, et al. 2000; Sprott, 2003), we only briefly outline them in the following parts. The chapter is organized as follows: In section 3.1, defines the chaos. In section 3.2, presents some properties of chaos. In section 3.3, introduces the promising indexes, geometric plots and statistical tests, to distinguish the chaos from other dynamical systems. Prediction of chaotic time series based on Takens's embedding theorem is introduces in section 3.4.

3.1 Definition of Chaos

Chaos is one subject area in the field of nonlinear dynamics, which is part of the broader field of dynamical systems. A dynamical system, one that evolves in time, can be stochastic or deterministic (Sprott, 2003). A stochastic system will change with time according to some random[1] processes, including uncorrelated (white) and correlated (colored) noises. A deterministic[2] system, by contrast, will evolve under some deterministic governing rules (or mathematical equations) in such a way that the present state is uniquely determined by the past states. Such deterministic chaos can only occur when the governing rules or equations are nonlinear. There are several definitions of chaos in use. A definition similar to the following is commonly found in the literature (for instance, Adrangi, *et al*. 2001; Barnett, *et al*. 1995; Hilborn, 1994; Kantz and Schreiber, 1997).

> *"The series a_t has a chaotic explanation if there exists a system (h, F, x_0) where $a_t = h(x_t)$, $x_{t+1} = F(x_t)$, x_0 is the initial condition at $t = 0$, and where h maps the n-dimensional phase space, R^n to R^1 and F maps R^n to R^n. It is also required that all trajectories x_t lie on an attractor A*

[1] It means breakdown of cause and effect, i.e. given exact knowledge of the state of a random system at one time, it is impossible to predict which set of alternatives will occur as the state of the system at the next instant.

[2] A system is deterministic if precise knowledge of the time evolution equations and the initial conditions completely determine the future behavior of the system.

and nearby trajectories diverge so that the system never reaches equilibrium (i.e., not eventually locating at fixed points) nor exactly repeats its path (i.e., it is aperiodic). For the chaotic time series, if one knows (h, F) *and could measure* x_t *without error, one could forecast* x_{t+i} *and thus* a_{t+i} *perfectly. With the divergence property and attractor A, in order that F generates stochastic-looking behavior, nearby trajectories must diverge (repel) exponentially. Moreover, in order that F generates deterministic behavior, locally diverging trajectories must eventually fold back (attract) on themselves. The attractors may be thought of as a subset of the phase space towards which sufficiently close trajectories are asymptotically attracted."*

3.2 Some Properties of Chaos

According to Sprott (2003), chaotic systems have several important features: (1) they are aperiodic, namely trajectories or orbits never repeat (Strange attractor); (2) they exhibit sensitive dependence on initial conditions (SDIC) and hence they are unpredictable[3] in the long run; (3) they are governed by one or more control parameters, a small change in which can cause the chaos to appear or disappear; (4) their governing equations are nonlinear; (5) they exhibit an apparent randomness; (6) they exist order within disorder; (7) In addition, the geometry with non-integer dimensionalities plays an essential role in the chaotic systems. Such geometries have been named "fractals" because of the non-integer dimensionalities (Mandelbrot, 2000). The fractals have the property of "self-similarity," which characterizes that a small section of an object or time series, suitably magnified, is resemble to the original one.

The above properties of chaos are probably better appreciated in the framework of a chaotic function. Here we briefly illustrate some of these properties in the framework of the Logistic function, a function commonly employed to demonstrate the chaos phenomenon (Baumol and benhabib, 1989; Hsieh, 1991; Adrangi, 2001). Consider the nonlinear equation (Logistic function) with a single parameter, *w*

$$x_{t+1} = F(x_t) = wx_t(1 - x_t)$$

Figure 3-1 graphs the relationship (x_{t+1}, x_t) for *w*=3.75, x_0=0.10. It should be

3 Predictability means if given an initial condition to within a small uncertainty range, we know the subsequent evolution of the system to within, more or less, the same order range of uncertainty.

apparent that (x_{t+1}, x_t) oscillations from a distinctive phase diagram (the bounding parabolic curve). As the oscillations expand, they encounter and "bounce off" the phase curve, moving closer to an apparent equilibrium on the negative slope of the phase curve. However, the convergence towards any equilibrium in that vicinity can only be temporary, since the slope of the phase curve ($\partial x_{t+1} / \partial x_t = w(1-2x_t)$) is less than -1. Figure 3-1 also illustrates the property of period folding of trajectories in chaotic systems, and demonstrates the concept of low dimension: the chaotic map of x_{t+1} against x_t give us a series of points in the phase curve. Even in the limit, these points would only from a one dimension set –a curve. On the other hand, had the x_{t+1} and x_t relationship been random, the points would have been scattered about the two-dimensional phase space. Figure 3-2 demonstrates the Lorenz attractor from the X-Z plane, which is also a good example of strange attractor.

To illustrate the concept of SDIC, we graph in Figure 3-3 (a) and (b) the time paths (x_t, t=1, 2,.., 60) for the Logistic equation with *w*=3.750, x_0 =0.10, and *w*=3.753, x_0 =0.10, respectively. It is immediately apparent that the Logistic equation has produced fairly complex time paths. Note that the same change (an "error") of only 0.003 introduced in *w* has caused the time path to be vastly different after only a few time periods. For instance, for the first nine periods, the time path in Figure 3-3 (a) "looks" almost identical to that in Figure 3-3 (b). However, the paths after t=10 diverge substantially. While we employ the Logistic equation to demonstrate SDIC here, the sort of behavior holds for a very wide set of chaotic relations.

The above illustration suggests that the presence of chaos will hamper the success of technical analysis and long-range forecasting models. For instance, it is hard to imagine how to imagine how any forecasting technique that relies on extrapolation could have correctly predicted the relative calm between points A and B in Figure 3-3 (b). Of course, one could forecast x_t perfectly if one could measure *w* and x_0 with infinite accuracy. Given that such measurement is not practical, both basic forecasting devices - extrapolation and estimation of structural forecasting models - become highly questionable in chaotic systems (also see Baumol and benhabib, 1989; Hsieh, 1991; Adrangi, 2001).

It should be noted, however, that chaotic systems may provide some advange for forecasting/technique analysis in the very-short run (say a few days when dealing with chaotic daily data).

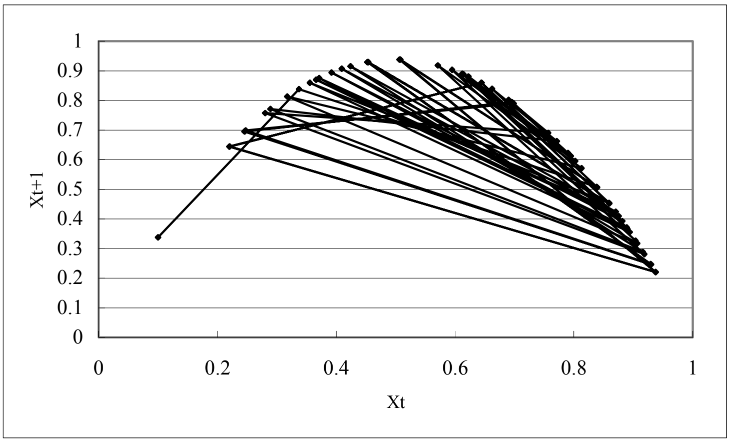


Figure 3-1 Logistic function $x_{t+1}=3.750x_t(1- x_t)$, $x_0=0.10$ (60 iterations)

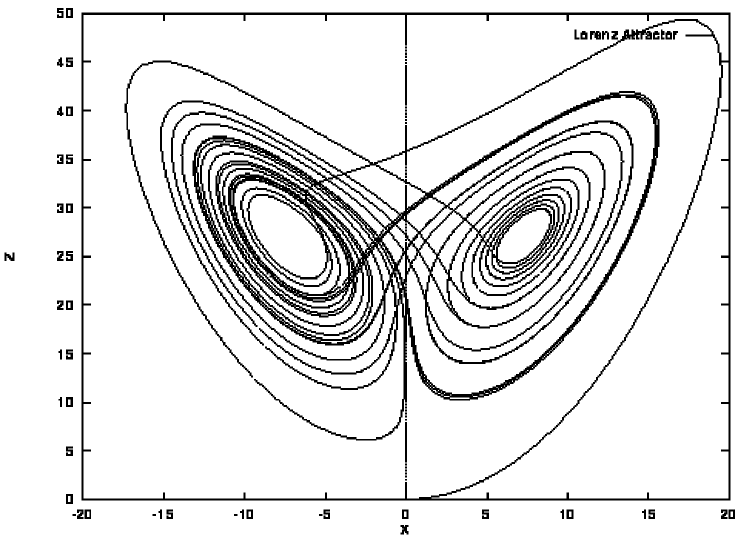
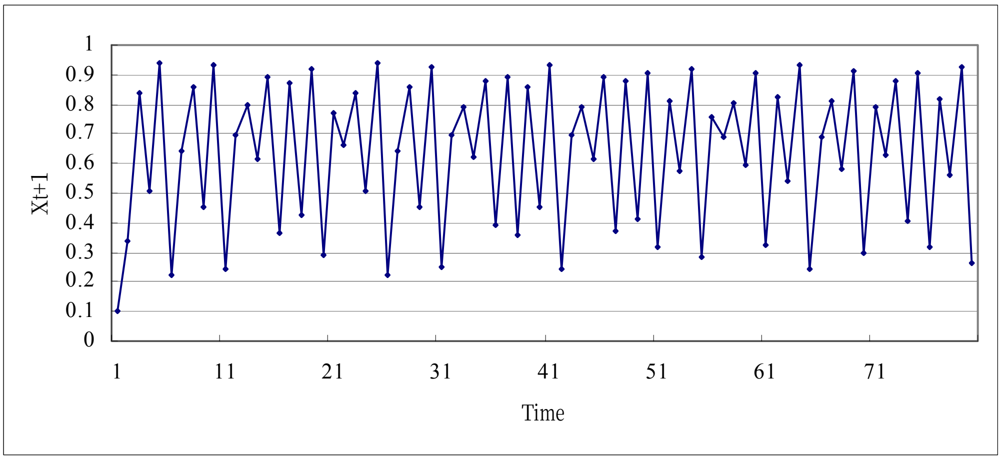
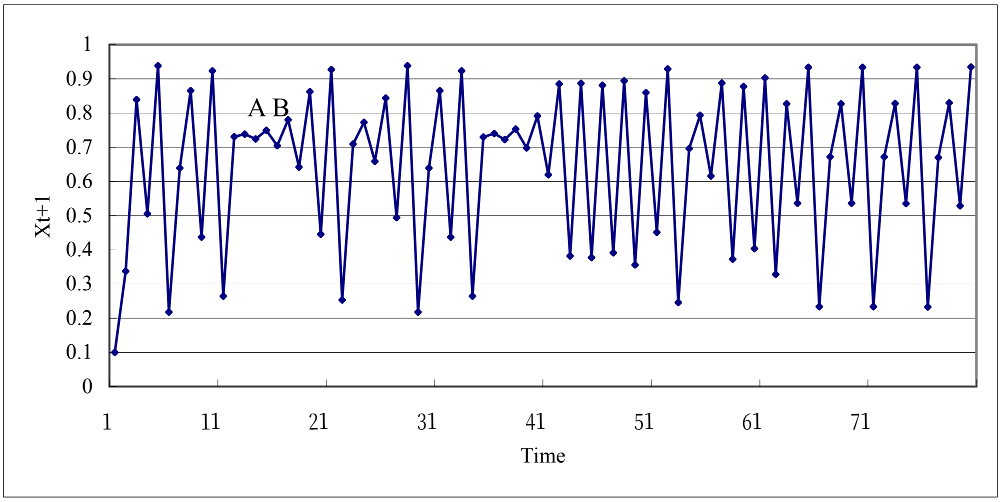


Figure 3-2 Lorenz attractor from the X-Z plane



(a) $x_{t+1}=3.750x_t(1- x_t)$, $x_0=0.10$



(b) $x_{t+1}=3.753x_t(1- x_t)$, $x_0=0.10$

Figure 3-3 Logistic function x_{t+1} vs. time

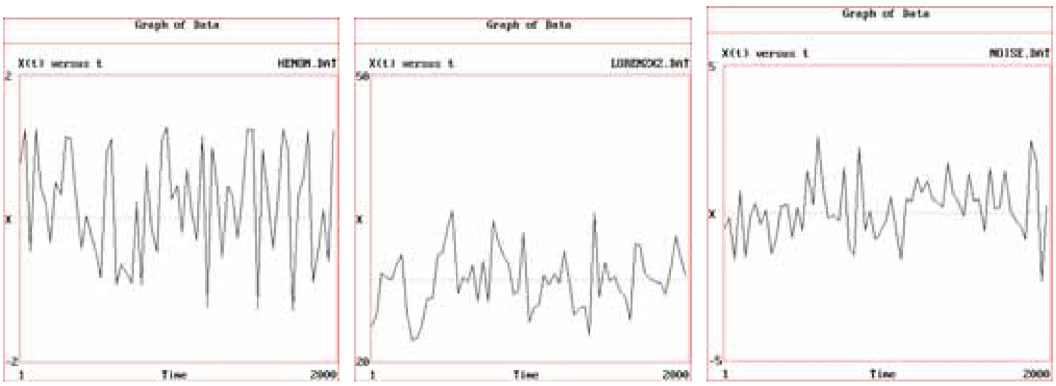
A chaotic time series appears stochastic (feature 5) but it is actually generated by a deterministic system. However, it is difficult to make distinction between stochastic data and deterministic chaos because both have very similar irregularity (feature 1). To elucidate this feature, we deliberately generate three well-known time series data: the Henon-type time series (2000 iterates) generated by eq. (3-1), the Lorenz-type time series (2000 data points at intervals of $\triangle t=0.1$) generated by eq. (3-2), and the Gaussian white noise data generated by eq. (3-3).

$$X_{n+1}=1-1.4X_n^2+Y_n;\ Y_n=0.3X_n \tag{3-1}$$

$$dX/dt=10(Y-X);\ dY/dt=28X-Y-XZ;\ dZ/dt=XY-8Z/3 \tag{3-2}$$

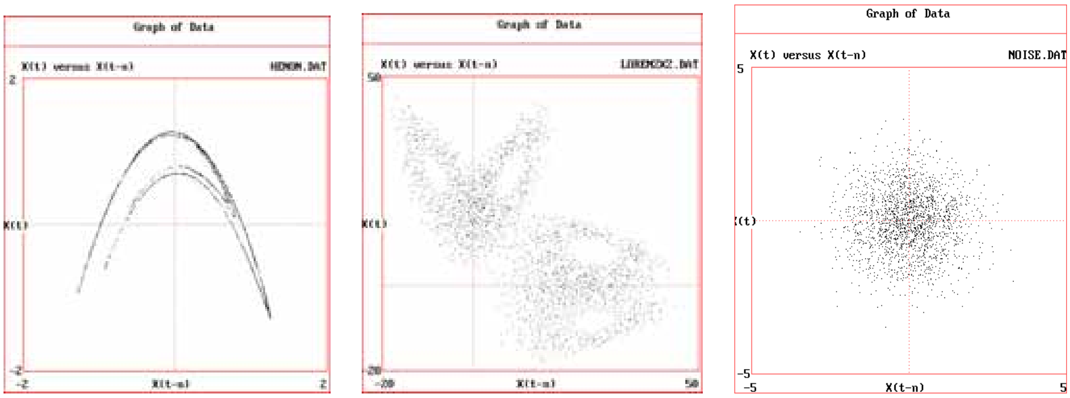
$$P(X)\frac{1}{\sqrt{2\pi}}e^{-X^2/2} \tag{3-3}$$

The Henon-type and Lorenz-type time series data with parameters given in eqs. (3-1) and (3-2) have been proven as deterministic chaotic systems (Henon, 1976; Tucker, 1999); while the Gaussian white noise given in eq. (3-3) is known as a stochastic random system (Press, *et al*. 1992). For the one-dimensional plots (trace), X(t) versus t, we notice that Henon-type and Lorenz-type chaotic time series (Figures 3-4(a) and 3-4 (b)) are almost indistinguishable from a Gauassian white noise (Figure 3-4 (c)). Such one-dimensional plots conclude that it is almost impossible to distinguish, by visualization method, between a stochastic system and a deterministic chaos because both have very similar irregularity. However, if we reconstruct these time series in higher dimensional state space, we would see the difference. For instance, Figure 3-5 presents their two-dimensional plots, X(t) versus X(t-n), where n is the delay time; and Figure 3-6 shows the three-dimensional plots, X (t) versus X(t-n) versus X(t-2n). Notice that both chaotic systems have shown discernible structures (Figures 3-5 (a) and 3-5 (b); Figures 3-6(a) and 3-6(b)), which are intrinsically governed by different deterministic rules. In contrast, the random system does not reveal any structure at all, which plots just fill up the entire plane as shown in Figure 3-5 (c) and look like a “fuzzy ball” as shown in Figure 3-6(c). These three examples show that a very simple deterministic equation of trajectory motions or time series data, which is essentially a chaotic system, can reveal very irregular trace similar to a stochastic system, and it is also a good example to demonstrate how to find order within disorder. Figure 3-7 demonstrates the fractal umbrella trees, which is a good example of geometric self-similarity.



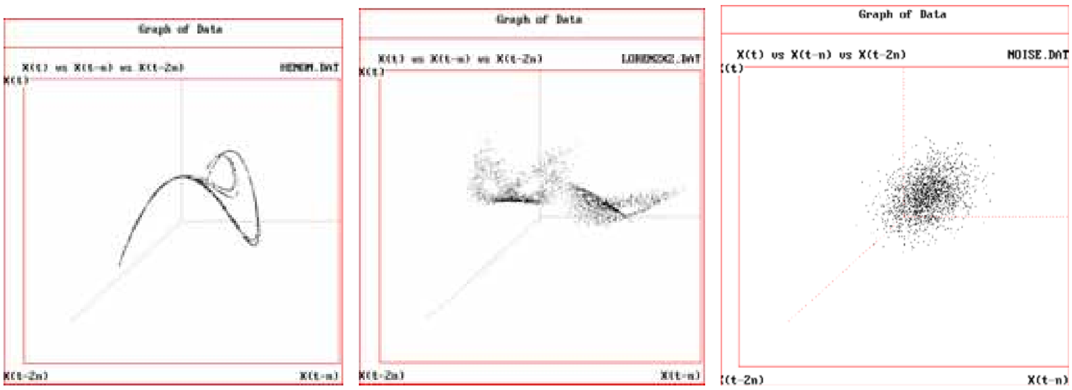
(a) Henon (b) Lorenz (c) White noise

Figure 3-4 One-dimensional state-space plots for the examples



(a) Henon (b) Lorenz (c) White noise

Figure 3-5 Two-dimensional state-space plots for the examples



(a) Henon (b) Lorenz (c) White noise

Figure 3-6 Three-dimensional state-space plots for the examples

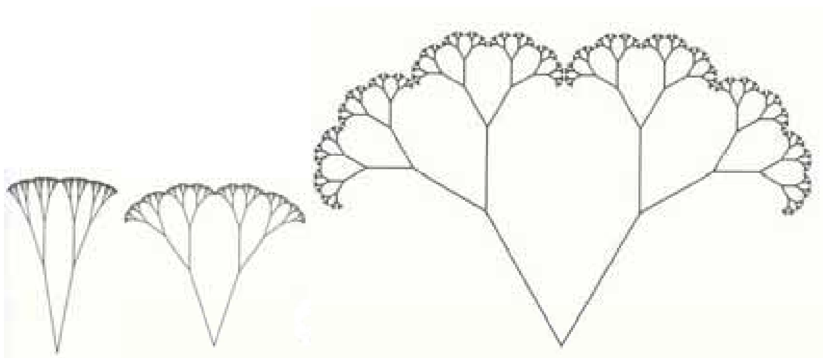


Figure 3-7 Fractal umbrella trees

3.3 Geometric Plots and Statistics

The above illustrations from eqs (3-1) through (3-3) suggest that it is very easy to incorrectly think a random system as chaos or a chaotic system as random by only visualizing their time series dynamics in the one-dimensional state space (or trace) because they are very much alike. Therefore, we must make use of other effective indexes that could noticeably distinguish them. We know that the simplest determinism of chaotic time series has each value dependent solely on its immediate predecessor; hence, through the reconstruction of the state space, some of its spatial plots would reveal very unique patterns, which can be served for distinction purposes. The two- or three-dimensional state space plots in Figures 3-4 through 3-6 are good examples of such "promising" plots.

This research would attempt batteries of promising indexes, including geometric plots and statistics, and choose the most crucial ones to develop a parsimony procedure to test for chaos. Other known geometric plots in chaos and time series literatures include return maps (plots of each local maximum versus the previous maximum), phase-space plots (slopes of the trajectories), Poincare maps (or Poincare movies), iterated function systems (IFS) clumpiness maps, autocorrelation function plots, probability distributions, and power spectra. The well-known statistics include the largest Lyapunov exponent, Kolmogorov entropy, Hurst exponent, relative complexity, capacity dimension, embedding dimension, correlation dimension, and delay time. To facilitate the comparison, we summarize the main properties of these indexes in Table 3-1.

Table 3-1 Summary of geometric plots and statistics for time series data

Index	Periodic and quasi-periodic data	Stochastic data (white or colored noises)	Chaotic data
State-space plots	a closed loop for periodic; a fuzzy loop for quasi-periodic	no apparent structure and the plots fill up the entire plane or space for white noise; may exhibit a structure for colored noise	A simple chaotic system can produce a plot with discernible structure; however, more complicated cases will fill two- and three- dimensional regions, respectively, with no discernible structure.
Phase-space plots	reveal a closed curve	no apparent structure and the plots fill up the entire plane or space for white noise; may exhibit a structure for colored	may exhibit a structure

		noise	
Return maps	may exhibit a pattern	fill the two-dimensional plane for white noise; may exhibit a pattern for colored noise	may yield discernible patterns
Poincare movies	nearby points move together	no discernible patterns would emerges	repeated stretching and folding, causing the nearby points to separate
IFS clumpiness map	with some localized clumps	White $1/f^0$ noise is a space-filled uncorrelated process that uniformly fills its space of representation. At the other extreme, Brown $1/f^2$ noise accumulates over the diagonals and some of the sides of the square leaving most of the representation space empty. Pink $1/f^1$ noise produces self-similar repeating triangular structure of different sizes and accumulates, albeit in a dispersed way, near the diagonals.	with some localized clumps
Correlation function plots	varied with delay time (τ) with amplitude slowly decreasing	dropped abruptly to zero for white noise; varied withτ with amplitude slowly decreasing for colored noise	tend to have little correlation; however, chaotic data from differential equations may be highly correlated if the sample time is small, since adjacent data points have similar values.
Probability distribution	a simple histogram with sharp edges	a Maxwellian distribution	likely to be a fractal
Power spectrum	with a few dominant peaks	broadband spectrum(on a linear scale); the steepness of the slope (on a log-log scale): Brown $1/f^2$ noise has a steep slope; Pink $1/f^1$ noise has a shallow slope; white $1/f^0$ noise with a flat spectrum.	broadband spectrum (on a linear scale); power spectra that are straight lines on a log-linear scale are thought as good candidates for chaos

Largest Lyapunov exponent (LE)	LE<0 fixed point; LE=0 periodic; (quasi-periodic, with LE>0	LE→∞	LE>0
Kolmogorov entropy (KE)	periodic KE=0; quasi-periodic KE>0	KE→∞	∞ >KE>0
Hurst exponent (HE)	HE >0	HE=-0.5 white noise; HE＞0.5 black noise; HE = 0.5 Brown noise (random walk); HE=0 pink noise; HE＜-0.5 blue noise	differ from 0 and 0.5
Relative complexity (LZC)	LZC→0 (perfect predictability has a value of 0)	LZC=1 white noise	0<LZC<1
Capacity dimension (CAD)	1	N/A	not integer
Embedding dimension (ED)	N/A	ED> 5	ED≤5
Correlation dimension (COD)	N/A	COD> 5	COD≤5
Delay time (DT)	DT>0	DT→0 white noise; colored noise DT>0	DT>0

N/A: not available in the textbooks such as Brock, *et al*. 1991; Hilborn, 1994; Kantz and Schreiber, 1997; Alligood, *et al*. 2000; Sprott, 2003.

3.3.1 Geometric Plots

1. State-space plots

These plots illustrate how a multidimensional space can be constructed from a time series without the necessity of taking derivatives of the data. The simplest chaotic determinism would have each value dependent only on its immediate predecessor. For the one-dimensional plot, X (t) versus t, very often looks like random. For the two-dimensional delay-time plot, X (t) versus X (t+τ), a chaotic sequence might show remarkable structure. A simple chaotic system can produce a plot with discernible structure; however, more complicated cases will fill two- and three-dimensional regions respectively with no discernible structure. A white noise sequence should fill up the entire plane with no apparent structure; Colored noise may exhibit a structure. A periodic system will exhibit a closed loop. A fuzzy loop means the system is qusi-periodic on a long time scale.

2. Phase-space plots

A two-dimensional phase-space plot is the time derivative x'(t) plotted with respect to x(t) at each data point. The first derivative is taken by half of the two data points adjacent to each point. A three-dimensional phase-space plot is the second derivative x''(t) plotted along with x'(t) and x(t) on the three axes. The second derivative is taken as the difference between the slopes of the lines connecting each data point with its two nearest neighbors. Some cases that are not obviously periodic in two dimensions may reveal their periodicity in three-dimension.

Periodic data should appear as a closed curve on such plot. White noise should appear no apparent structure and the plots fill up the entire plane or space; Colored noise may exhibit a structure. Chaotic may exhibit a structure.

3. Return maps

A two-dimensional phase-space plot generally will not distinguish between random and chaotic data. For this purpose, it is useful to take some sort of cross section of the phase plane in order to reduce its dimension by one. After such an operation, chaotic data will often appear in the form of a strange attractor having a fractal structure with fractional dimension. Periodic and quasi-periodic data may exhibit a pattern. White noise should fill the two-dimensional plane; Colored noise may exhibit a pattern.

4. Poincare movies

The structure of a time series trajectories can often be revealed in a Poincare section (also called a surface of section). It reduces the dimension of the attractor by one. The dimension of the Poincare section is usually independent of the section taken as long as it includes the attractor. There are infinitely many such sections, and some are more revealing than others. The map produced in this way (called a Poincare map) has a subset of the same dynamics as the corresponding trajectories, including Lyapunov exponents and bifurcation behavior, except that it is missing the zero Lyapunov exponents corresponding to the direction of the trajectories. The Poincare movies show an animated display of every n-th data point versus its m-th predecessor as if viewed under a strobe light that flashes every n-th time step (in 2-D or 3-D).

For the random data dominated by noise, no discernible pattern would emerge. Periodic data nearby points move together. Chaotic data one can observe repeated stretching and folding of the trajectories, causing nearby points to separate.

5. IFS clumpiness maps

Iterated function systems (IFS) suggest a data-analysis method (Peak and Frame, 1994). The IFS clumpiness is highly sensitive to determinism in the data; however, it does not very well distinguish chaos from colored (correlated) noise. We can shuffle the data points (randomizing their order but preserving their distribution) and look for the difference of the IFS clumpiness in the original and surrogate plots.

Periodic data may with some localized clumps. White $1/f^0$ noise is a space-filled uncorrelated process that uniformly fills its space of representation. At the other extreme, Brown $1/f^2$ noise accumulates over the diagonals and some of the sides of the square leaving most of the representation space empty. Pink $1/f^1$ noise produces self-similar repeating triangular structure of different sizes and accumulates, albeit in a dispersed way, near the diagonals. Chaotic data should with some localized clumps.

6. Correlation function plots

The Fourier transform of the power spectrum in the time domain, according to the Wiener-Khinchin theorem, is given by the serial correlation function (or called autocorrelation function):

$$G(k) \quad \frac{\sum_{n=1}^{N-k}(X_n - \langle X \rangle)(X_{n+k} - \langle X \rangle)}{\sum_{n=1}^{N-k}(X_n - \langle X \rangle)^2}$$

The correlation function measures how strongly on average each data point is always correlated with one k time steps. It is the ratio of the auto covariance to the variance of the data, and G(k) is normalized so that $G(0)=1$. In general, the correlation function falls from a value of 1 at k=0 to zero at large k. The value of k at which it falls to 1/e 37% is called the correlation time τ.

Random data will have no correlation and its autocorrelation function will drop abruptly to 0, implying small correlation time. Highly correlated data will have a correlation function that varies with τ but whose amplitude only slowly decreases. Chaotic data tend to show little correlation; However, chaotic data from differential equations may be highly correlated if the sample time is small, since adjacent data points have similar values.

7. Probability distributions

The probability distribution, P(X), or probability density distribution, is a plot of the probability that X is within some "bin" ΔX of X. By iterating many times from an arbitrary initial condition, one can plot a histogram of the resulting X values. Normalizing the histogram such that the area under the curve is 1 leads to an approximate probability distribution P(X) that a point X is within the bin of X. The largest peak (mode) represents a value would spend most of its time near its extrema. A highly non-uniform probability is a result of the non-uniform stretching of the trajectories with enough iteration.

8. Power spectra

For stationary or distrended data with inherent periodicities, Fourier analysis (also called spectral analysis, frequency analysis, or harmonic analysis) is useful (Newland; 1993). A Fourier transform on the time series data displays the power (mean square amplitude) as a function of frequency. Random and chaotic data give rise to broad spectra. Periodic and quasi-periodic data will produce a few dominant peaks in the spectrum. Power spectra that are straight lines on a log-linear scale are thought to be good candidates for chaos since noise tends to have a power-law spectrum. The cumulative period gram is the integral of the power spectrum over frequency. It should follow the 45-degree line if the power spectrum is flat indicating white noise.

The steepness of the slope (on a log-log scale): Brown $1/f^2$ noise has a steep slope; Pink $1/f^1$ noise has a shallow slope; white $1/f^0$ noise with a flat spectrum.

Note that noise with a power spectrum that varies with frequency as $1/f^{\alpha}$ is called correlated (or colored). White noise has α=0 in which power spectrum is independent of frequency. White noise is uncorrelated since the correlation function is zero for all nonzero time lags. The case of α=1 is called pink (or flicker) noise. The case with α=2 is Brownian motion (or called brown noise, named after Robert Brown but has nothing to do with color). Cases with α>2 are sometimes called black noise. The case with α<0 is called blue noise.

3.3.2 Statistics

1. Lyapunov exponent

The Lyapunov exponents for a dynamical system are measures of the average rate of divergence or convergence of typical trajectories in the phase space. A positive Lyapunov exponent is a measure of the average exponential divergence of two nearby trajectories. If a discrete nonlinear system is dissipative, a positive Lyapunov exponent is an indication that the system is chaotic (Gencay, 1996). Chaotic trajectories should have at least one positive Lyapunov exponent. For fixed data, all Lyapunov exponents are negative. For periodic trajectories, the largest Lyapunov exponent is equal to zero. White noise the largest Lyapunov exponent is equal to infinite.

The definition of Lyapunov exponent is expressed as follows:

$$\lambda = \frac{1}{n}\ln\frac{d_n}{d_0}$$

where

$x_i = j^{th}$ value from the time series data

$x_j = i^{th}$ value that is close to x_i

$d_0 = \left| x_j - x_i \right|$

$d_n = \left| x_{j+n} - x_{i+n} \right|$

2. Kolmogorov entropy

Similar to Lyapunov exponent, Kolmogorov-Sinai invariant (or Kolmogorov entropy) also focuses on the concept of SDIC for chaos (Hilborn, 1994). Consider two trajectories representing time paths that are extremely close so as to be indistinguishable. However, these trajectories diverge so that they become distinguishable as time passes. The Kolmogorov entropy (*K*) measures the speed with which this takes place and is given by

$$K = \lim_{\varepsilon \to \infty} \lim_{m \to \infty} \lim_{N \to \infty} \ln\left(\frac{C^M(\varepsilon)}{C^{M+1}(\varepsilon)}\right)$$

If a time series is non-complex and completely predictable, *K* will approach to zero. If a time series is completely random, the value will tend to be very large. That is, the lower the value of *K*, the more predictable the system is. For chaotic systems, one would expect small *K* values.

3. Hurst exponent

Hurst exponent is a measure of the extent to which the data can be represented by a random walk, or Brownian motion. In such a case that a time series trajectory x_t, on average, moves away from its initial position by an amount proportional to the square root of time, we say that its Hurst exponent is 0.5 (Kantz and Schreiber, 1997). Thus, we can judge whether the time series is random or not by this test. It is determined from the square root relation between increments and time intervals as follows:

$$\left(\Delta x^2\right) \propto \Delta t^{2HE}$$

where *HE*= Hurst exponent. For a time series data, *HE* greater than 0.5 indicates the time series data is positively correlated (or persistence). *HE* less than 0.5 indicates the time series data is negatively-correlated (or anti-persistence). Note that Hurst exponent can be estimated by $HE=(\alpha -1)/2$, where α is the exponent of the noise with a power spectrum that varies with frequency as $1/f^{\alpha}$.

HE=-0.5 white noise; HE＞0.5 black noise; HE = 0.5 Brown noise (random walk); HE=0 pink noise; HE＜-0.5 blue noise

4. Relative Lempel-Ziv complexity

We can use symbolic dynamics to calculate the relative Lempel-Ziv complexity (LZC) relative to white noise. It is a measure of the algorithmic complexity of the time series. Using the algorithm of Kaspar and Schuster, each data point is converted to a single binary digit according to the fact that whether its value is greater or less than the median value. Maximal complexity (complete randomness) has a value of 1.0 and perfect predictability has a value of 0.

5. Capacity dimension

Self-similarity of sets is characterized by the Hausdorff dimension, although the box counting dimension is much more convenient to compute. It presents an upper bound on the Hausdorff dimension from which it is known to differ only for some constructed examples. Consider a point set located in R^m. If we cover it with a regular grid of boxes of length ε and call $M(\varepsilon)$ the number of boxes which contain at least one point, then for a self-similar set

$$M(\varepsilon) \propto \varepsilon^{-D_F}, \qquad \varepsilon \to 0.$$

D_F is then called the box counting or capacity dimension. A not integer dimension implies essentially chaotic data.

6. Embedding dimension

The false nearest-neighbors method is commonly used. Find the nearest x_l for each point x_n in a time-delay embedding m and call the separation between these points $R_n = \sqrt{(x_l - x_n)^2 + (x_{l-1} - x_{n-1})^2 + \ldots}$. Likewise, calculate the separation

$R_n(m+1)$ in a time-delay embedding *m+1*. If $R_n(m+1)$ significantly exceeds $R_n(m)$, then the neighbors are close only because of overlap; namely they are not truly close. The criterion for falseness is thus $\frac{|x_{l-m} - x_{n-m}|}{R_n(m)} \succ R_T$, where R_T is a threshold value. A correlation dimension greater than 5 implies essentially random data.

7. Correlation dimension

Correlation dimension, applied to characterize chaotic attractors, is widely used by physicists to test for chaos in time series data (Hilborn, 1994). Compared with the capacity dimension measure, it has a computational advantage because it uses the trajectory points directly and does not require a separate partitioning of the state space. Grassberger and Procaccia (1983) define the correlation dimension of a time series as

$$D^N = \lim_{\varepsilon \to 0} [\log C(\varepsilon) / \log \varepsilon]$$

where N is the embedding dimension and $C(\varepsilon)$ is the correlation integral. A correlation dimension greater than 5 implies essentially random data.

8. Delay time

Highly random data will have no correlation and its autocorrelation function will drop abruptly to 0, implying small correlation time, or called delay time τ. Highly correlated data like the output of a sine wave generator will have an autocorrelation function that varies with tau (τ) but whose amplitude only slowly decreases. Chaotic data from difference equations tend to show little correlation, but chaotic data from differential equations may be highly correlated if the sample time is small, since adjacent data points have similar.

3.4 Takens' Embedding Theorem

Building the chaotic prediction model, from a time series mainly involves two steps: (1) reconstruction of the phase space from data by time delay embedding; and (ii) development of a methodology for phase-space prediction. In Figure 1-2, we reconstruct the phase space with the Takens' embedding theorem by embedding the one-dimension time series data into the n-dimensional space. After plotting the time series data from the latest observations in the n-dimensional reconstructed phase space, we investigated the historical observations neighboring to the latest observation.

Generally, the order of a system is described by "N" state variables ("N" stands for the number of variables which affect a system) and can be represented as a trajectory in the N-dimensional state (phase) space. However, the observable time series data is only a part of "N" state variables usually. In this case, from the time series data of a single observed variable, its trajectory can be reconstructed in an N-dimensional space using delay time. From the observed time series data $\zeta(t)$, data vector $Z(t) = \{\zeta(t), \zeta(t-\tau), ..., \zeta(t-(N-1)\tau)\}$ is generated where N and τ represent embedding dimension and delay time. The vector indicates one point of an N-dimensional reconstructed state (phase) space. Therefore, a trajectory can be drawn in an N-dimensional reconstructed state (phase) space by changing t with τ fixed. When embedding dimension N is sufficiently large, we can say that the reconstructed trajectory is embedded in the reconstructed state (phase) space. To be concrete, it has been proven by Takens (1981) that retains the phase structure in the original O-dimensional state (phase) space, i. e.; the reconstructed trajectory to be embedded is as follows.

N≧2O+1

If an observed time series data is chaotic, then the trajectories of the time series will follow a certain deterministic regularity. Thus, if the deterministic regularity can be estimated, then the data in the near future (before the deterministic causality is lost) can be predicted. However, since chaos has a property of "sensitivity on the initial condition," we cannot make a long-term prediction for any time series data.