Chapter 2

LITERATURE REVIEW

This chapter reviewed existing macroscopic continuum traffic flow models and the simulation methods applied to solve them. A few recently developed numerical methods for hyperbolic PDEs were also surveyed in this study.

2.1 Review of Macroscopic Continuum Traffic Flow Models

2.1.1 The Simple Continuum Model

The landmark papers of Lighthill and Whitham (1955), as well as Richards (1956) mark the birth of dynamic macroscopic modeling of traffic flow. In this macroscopic model, three aggregate variables, density, flow rate, and space-mean speed, are used. This Lighthill-Whitham-Richards (LWR) model consists essentially of the conservation equation

$$\frac{\partial k}{\partial t} + \frac{\partial q}{\partial x} = g(x,t) \tag{2.1}$$

supplemented by the fundamental equation of traffic flow

$$q = ku \tag{2.2}$$

and a speed-density (u-k) relationship

$$u = u_e(k), \tag{2.3}$$

where k represents the traffic density; q the flow rate of the traffic stream; u the space-mean speed. t and x is time and space, respectively. The equilibrium relationship between the speed and the traffic density is denoted as $u_e(k)$, and $g(x,t)$ represents the generation of flow.

Since there is only one partial differential equation, the LWR model henceforth is called "the simple continuum model." The most significant result of the simple continuum model is manifestation of shock waves in traffic flow. However, since the PDE is nonlinear and dominated by the convective term, the simple continuum model easily produces discontinuous solutions even when the initial condition is arbitrary smooth.

In addition, since speed in the simple continuum model is totally determined by the statistical equilibrium speed-density relationship (2.3), no fluctuations of speed around the equilibrium values are allowed. Moreover, the simple continuum model does not have the ability to explain the amplification of small disturbances in heavy traffic because no stable condition can be derived from the model. Therefore, from theoretical point of view, the simple continuum model does not adequately describe traffic flow dynamics, especially at transient flow conditions.

Although LWR mode is not perfect, it is still the best available description of kinematical waves in traffic, particularly if the intensity of perturbations is not too large. It was used by Lighthill and Whitham to provide a fair description of the behavior of traffic in front of bottlenecks and the periodic disturbances caused by traffic light.

Newell (1993) evaluated flows or densities by cumulative flow $A(x,t)$ past any point x by time t. It is shown how a formal solution for $A(x,t)$ can be evaluated directly from boundary or initial conditions. Daganzo (1994) presented a model on a highway with a single entrance and exit. The model proposed can be used to predict traffic's evolution over time and space, including transient phenomena such as the building, propagation, and dissipation of queue. The well-solved difference equations were derived from a special case of the hydrodynamic model of traffic flow. Daganzo (1995a) extended the difference equations to network traffic.

2.1.2 The High Order Continuum Models

In order to overcome the defects in the simple continuum model, Payne (1971) and Whitham (1974) developed the first high-order dynamic and macroscopic traffic flow model based on a nonlinear car-following model by means of Taylor's series expansions. Payne-Whitham (PW) model is commonly called "the original high-order model". The state equations of the original high-order model are

$$\frac{\partial k}{\partial t}+\frac{\partial q}{\partial x}=g(x,t)\,, \tag{2.4}$$

$$\frac{\partial u}{\partial t}+u\left(\frac{\partial u}{\partial x}\right)=-\frac{1}{k}\frac{\partial}{\partial x}\left(P_e(k)\right)+\frac{1}{\tau}\left(u_e(k)-u\right), \tag{2.5}$$

$$q=ku\,. \tag{2.6}$$

The term

$$-\frac{1}{k}\frac{\partial}{\partial x}\left(P_e(k)\right)=-\frac{P_e'(k)}{k}\left(\frac{\partial k}{\partial x}\right) \tag{2.5}$$

is an anticipation term taking into account awareness of the drivers for the traffic condition ahead, where $P_e(k)$ is an equilibrium traffic pressure. The anticipation term used by Payne is determined by $P_e'(k)=(1/2\tau)\left|u_e'(k)\right|$. In general, $P_e(k)=k\theta_e(k)$, where $\theta_e(k)$ denotes speed variance. Analogous to the situation for $u_e(k)$, there have been different suggestions for the function $\theta_e(k)$. For example, Kühne (1991), and Kerner and Kohnhäuser (1993, 1994) suggested a constant value $\theta_e(k)=c_0^2$, whereas Phillips (1977) proposed a linear relation $\theta_e(k)=\theta_m(1-k/k_m)$. The dependence of speed $u_e(k)$ and variance $\theta_e(k)$ on the density in the equilibrium case can be derived by means of kinetic theory from the equilibrium distribution function.

Furthermore, due to the original high order model having two different characteristic speeds $\lambda_1=u+\sqrt{\frac{\nu}{T}}$ and $\lambda_2=u-\sqrt{\frac{\nu}{T}}$, the original high order model allows negative speeds at the tail of congested regions which violates an essential characteristic of traffic flow (Daganzo, 1995b), i.e. traffic flow cannot move at a negative speed.

Since 1971, several high order models have been developed by some researchers in traffic flow theory. All those high order models have made some changes to the original high order model. For example, Papageorgiou (1983) made some improvements to the Euler discretized form of the PW model based on the computational experiments. The equations of Papageorgiou' s improved high order model are

$$\frac{\partial k}{\partial t}+\frac{\partial (ku)}{\partial x}=g(x,t)\ , \tag{2.6}$$

$$\frac{\partial u}{\partial t}+u\boldsymbol{z}\frac{\partial u}{\partial x}=\frac{1}{T}[u_e(k)-u]-\frac{\boldsymbol{n}}{T(k+\boldsymbol{k})}\frac{\partial k}{\partial x}-\boldsymbol{h}|g|\frac{u}{k}\ , \tag{2.7}$$

where, in addition to the previous notation, æ and ê are the two new parameters which were introduced to improve the computational results of the PW model. The third term, $\boldsymbol{h}|g|\frac{u}{k}$, was introduced to handle the traffic friction between the main flow and the merging(or diverging) flow. However, the third term did not correctly describe the traffic friction because it turns to the maximum value when the density on the freeway is close to zero, which is not realistic.

Phillips (1979) derived a high order continuum traffic flow model form a kinematic model by taking moments of the associated Boltzmann equation. The equations for Phillips' high order model are

$$\frac{\partial k}{\partial t}+\frac{\partial (ku)}{\partial x}=0\ , \tag{2.8}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\frac{1}{T(k)}[u_e(k)-u]-\frac{1}{k}\frac{dP}{dk}\frac{\partial k}{\partial x}\ , \tag{2.9}$$

where, in addition to the previous notation. $T(k)$ is the relaxation time with the form:

$$T(k)=\frac{2bk}{3k_j u_l(k_j-k)}$$

where b is the number of lanes, k_j is the jam density and u_l is an experimentally determined proportionality constant. $P(k)$ is a traffic pressure function with the form:

$P(k) = Ak(k_j - k)$,

where A is a constant. It seems that Phillips' model accounts for passing and lane changing on multilane highways due to his model involving vehicular interaction. Comparing with the original high order model, this model also has a smooth solution under certain conditions and is able to explain the amplification of small disturbances on heavy traffic. Moreover, this model is consistent with the simple continuum model in the equilibrium limit.

Kühne (1984) developed a high order model based on the Navier-Stokes equations for a viscosity term including in the momentum equation. The equations for Kühne's model are

$$\frac{\partial k}{\partial t} + \frac{\partial (ku)}{\partial x} = 0 , \tag{2.10}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{1}{T}[u_e(k) - u] - \frac{\boldsymbol{n}}{k}\frac{\partial k}{\partial x} + \boldsymbol{u}\frac{\partial^2 u}{\partial x^2} , \tag{2.11}$$

where the third term $\boldsymbol{u}\frac{\partial^2 u}{\partial x^2}$ is the viscosity term. Thus, this model seems to be more realistic because all the properties of a continuous flow behavior can be described by the Navier-Stokes equations. Due to viscosity effects, discontinuous solutions that exist in other high order models do not exist in Kühne's model. The model was used to capture stop-start waves.

Michalopoulos et al. (1991b, 1993a, 1993b) developed two high order continuum models. One is the semi-viscous high order model, and the other is the viscous high order model. The semi-viscous model are shown as follows

$$\frac{\partial k}{\partial t} + \frac{\partial (ku)}{\partial x} = g(x,t) , \tag{2.12}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{1}{T(k)}[u_f(x) - u] - \frac{\boldsymbol{n}}{k}\frac{\partial k}{\partial x} - \boldsymbol{m}k^a g , \tag{2.13}$$

where ì is a geometry parameter depending on the type of road geometry and á is a dimensionless constant. $T(k)$ is the relaxation time with the form

$$T(k) = T_0(1 + \frac{gk}{k_{jam} - gk})$$

where T_0>0 and 0<ã<1 are constants. The third term, $mk^a g$, was introduced to handle the traffic friction. Since the traffic friction turns to maximum value when the density on the freeway approaches jam density, the third term did not correctly describe the traffic friction. While the density on the freeway approaches jam density, the flow speed is already low. Thus traffic friction as defined earlier is likely negligible. In addition, since there is no absolute value of g in the third term, the speed of the main flow will be increased in diverging situations, which is also unrealistic.

The viscous model was formed as follows

$$\frac{\partial k}{\partial t} + \frac{\partial (ku)}{\partial x} = g(x,t)\ , \tag{2.14}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{\boldsymbol{n}}{k}\frac{\partial k}{\partial x} + \boldsymbol{u}k^b \frac{\partial^2 k}{\partial x^2}\ , \tag{2.15}$$

where $\hat{a}$ is a dimensionless constant. Because of no relaxation time, this model is no longer consistent with the simple continuum model.

The main feature of the semi-viscous and viscous models is that the equilibrium speed-density relationship is never required. However, since the equilibrium speed is replaced by free flow speed, incorrect trends for speed might be produced by the semi-viscous model under the varying distribution of free flow speed on a highway. In addition, the semi-viscous model is not consistent with the simple continuum model in the equilibrium limit. In the viscous model, since the relaxation process is totally replaced by viscosity, the accuracy of the viscous model describing traffic dynamics could be a problem in congested situations(Liu et al., 1997).

It is clear to see that these high order models mentioned above are still based on the fundamental structure of the original high order model. Thus, they have two different characteristic speeds: one is always positive and greater than the flow velocity; another is negative under congestion conditions. Therefore, these high order models suffer from the same serious defects as the PW model.

Helbing (1996b) extended the PW model by introducing an additional PDE for the velocity variance q. His macroscopic model is derived from gas-kinetic equations and consists of the conservation of vehicles equations and the velocity dynamics. Helbing's model

$$\frac{\partial q}{\partial t}+u(\nabla \cdot q)=-2q(\nabla \cdot u)+2\frac{m}{k}(\nabla \cdot u)^2+\frac{2}{t}(q_e(k)-q)+\frac{m}{k}\nabla \cdot (\nabla u)+\frac{k}{k}\nabla \cdot (\nabla q) \qquad (2.15)$$

describing the dynamics of the variance q with q_e depending on k is the same way as u_e in theose PW-like models.

Zhang (1998, 1999, 2000, 2001) proposed a new non-equilibrium traffic flow model that is shown to be devoid of "wrong way" travel. Upon further examination, Zhang found that PW-like models always behave isotropically because the material derivative of travel speed depends on density gradient.

Based on improved car-following model, Jiang (2002) replaces the density gradient with speed gradient in the momentum equation to avoid wrong-way problem mentioned above. Jiang's model, which is consistent with other high order models, consists of two PDEs as follows:

$$\frac{\partial k}{\partial t}+\frac{\partial ku}{\partial x}=g \,, \qquad (2.16)$$

$$\frac{\partial u}{\partial t}+u\left(\frac{\partial u}{\partial x}\right)=\frac{1}{t}(u_e(k)-u)+c\frac{\partial u}{\partial x} \,. \qquad (2.17)$$

2.2 Review of Numerical Simulation of Continuum Traffic Flow Models

While numerical methods are widely studied and implemented in the field of computational fluid dynamics, they are less well comprehended in traffic flow. This has stimulated studies of reliable numerical methods for the solution of continuum traffic flow models based on PDEs. So far, there are four types of numerical methods used in the continuum traffic flow models. They are the Lax method (Michalopoulos et al., 1985), the explicit Euler method (Payne, 1971, 1979; Papageorgiou, 1983; Papageorgiou et al., 1989) and the upwind scheme with flux vector splitting (Michalopoulos te al., 1991b, 1993a, 1993b; Lytinzis et al., 1994a, 1994b) as well as Roe's flux difference splitting method (Leo et al., 1992). Among these four numerical methods, the explicit Euler method is an unstable numerical method. The other three methods are stable under the Courant-Friedriches-Lewy (CFL) condition. These methods are briefly reviewed in this section.

Michalopoulos et al. (1985) have used the Lax method (Lax, 1954) to the simple continuum model because the Lax method is simple and easily implemented. However, the Lax method introduces a strong numerical dissipation effect to the simple continuum model. Thus, the Lax method cannot capture the correct shock intensities.

Payne (1979) first used the explicit Euler method to the original high-order model due to its simplicity. However, this numerical method cannot make the original high order model work at the smaller values of the density because there is a term including the reciprocal of the density. The model cannot produce the correct shock intensities because the model was discretized based on the non-conservation form (Hirsch, 1990). Moreover, this method is unstable from the computational point of view because this method yields a negative diffusion coefficient in the truncation error terms (Lyrintzis, 1994b).

Michalopoulos et al. (1991b, 1993a, 1993b) first applied the upwind scheme with flux vector splitting to the semi-viscous model. Later, Lyrintzis et al. (1994b) also used the upwind scheme with flux vector splitting to their proposed high order model. Since the upwind scheme with flux vector splitting is a stable numerical method under the CFL condition and introduces wave propagation properties (i.e., the sign of the eigenvalues) in the discretization process, i.e. a forward difference is used for an upstream moving wave and a backward difference for a downstream moving wave, the models with this upwind method capture correct shock waves approximately and produce more accurate results than the models with the Euler method (Lyrintzis et al., 1994b). However, this upwind method is totally dependent on the fact that fluxes are homogeneous of degree one in the conservative variables. One cannot directly apply this scheme to a general flux function. Moreover, since this upwind method introduces only information on the sign of the eigenvalues, one cannot introduce other physical properties such as shock and rarefaction waves into the discretization process through this method.

Leo and Pretty (1992) derived Roe's flux difference splitting method (Roe, 1981) for PW system of equations and applied the algorithm to some numerical examples. They also tested out the LWR model with the Murman scheme that is an equivalent Roe's algorithm for a single equation.

Furthermore, Cho and Lin (2000) used and compared finite difference methods and finite element methods to solve the LWR model. They summarized that the finite element methods are more accurate than finite difference approximations.

2.3 Brief Review of Numerical Methods for Hyperbolic PDEs

Hyperbolic PDEs are time-dependent systems of PDEs with a particularly simple structure. In one space dimension, hyperbolic PDE can be shown as the form

$$\frac{\partial k(x,t)}{\partial t} + \frac{\partial f(k(x,t))}{\partial x} = 0 . \tag{2.18}$$

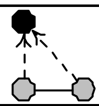
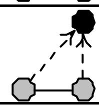
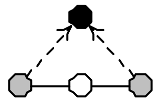
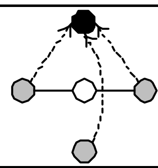
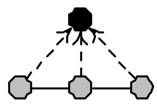
Here $k : \Re \times \Re \rightarrow \Re^m$ is an m-dimensional vector of conserved quantities, or state variable, and $f : \Re^m \rightarrow \Re^m$ is called flux function. The system (2.18) is 'hyperbolic' provided that the eigenvalues of $f'(k)$, the $m\times m$ Jacobian matrix of the flux function, are real, and the matrix is diagonalizable for each value of k. By the chain rule, (2.18) can be written in the quasilinear form

$$\frac{\partial k(x,t)}{\partial t}+a(k)\frac{\partial k(x,t)}{\partial x}=0, \tag{2.19}$$

where $a=f'$.

There have been many finite difference applied to solve hyperbolic conservation laws. Such as the upwind, Godunov's, Hyman, Lax-Wendroff, MacCormack's, Rusanov, and Glimm's methods, a random choice method, were surveyed by Sod (1978). He also discussed the hybrid scheme of Harten and Zwas (1972), the antidiffusion method of Boris and Book (1973), and the artificial compression method of Harten. The antidiffusion method of Boris and Book, can be viewed as a flux-limiter method, is called the flux-corrected transport (FCT) method. The numerical results were compared and demonstrated that Glimm's method has several advantages. Without the use of corrective procedures, Gonudov's and Hyman's methods produce the best results of all the schemes tested. Glimm' scheme gives the best resolution of the shocks and contact discontinuities. The hybrid methods of Harten and Zwas combines the first and high order schemes in such a way as to extract the best features of both. Furthermore, high order accurate methods, such as artificial viscosity, slop-limiter, and ENO schemes were developed recently. ENO schemes proposed by Harten, Osher, Engquist, and Chakravarthy (1987) was the first successful attempt to obtain self similar (i.e. no mesh size dependent parameters), uniformly high order accurate, yet essentially non-oscillatory interpolation (i.e. the magnitude of the oscillations decays as $O(\Delta x^k)$ where k is the order of accuracy) for piecewise smooth functions. The vital idea of ENO schemes is to use the "smoothest" stencil among several candidates to approximate the fluxes at cell boundaries to a high order accuracy and at the same time to avoid spurious oscillations near discontinuities.

Table 2.1. Finite Difference Methods for hyperbolic conservation laws $k_t + f(k)_x = 0$.

Scheme	Order	Difference Equations	Stencil
Left One-sided	1	$k_j^{n+1} = k_j^n - \frac{\Delta t}{\Delta x}(f_{j+1}^n - f_j^n)$	
Right One-sided	1	$k_j^{n+1} = k_j^n - \frac{\Delta t}{\Delta x}(f_j^n - f_{j-1}^n)$	
Lax-Friedrichs	1	$k_j^{n+1} = \frac{1}{2}(k_{j-1}^n + k_{j+1}^n) - \frac{\Delta t}{2\Delta x}(f_{j+1}^n - f_{j-1}^n)$	
Leapfrog	1	$k_j^{n+1} = k_j^{n-1} - \frac{\Delta t}{2\Delta x}(f_{j+1}^n - f_{j-1}^n)$	
Lax-Wendroff	2	$k_{j+0.5}^{n+0.5} = \frac{1}{2}(k_j^n + k_{j+1}^n) - \frac{\Delta t}{2\Delta x}(f_{j+1}^n - f_j^n)$ $k_j^{n+1} = k_j^n - \frac{\Delta t}{\Delta x}(f_{j+0.5}^{n+0.5} - f_{j-0.5}^{n+0.5})$	
MacCormack	2	$\bar{k}_j^{n+1} = k_j^n - \frac{\Delta t}{\Delta x}(f_{j+1}^n - f_j^n)$ $k_j^{n+1} = \frac{1}{2}(k_j^n + \bar{k}_j^{n+1}) - \frac{\Delta t}{2\Delta x}(\bar{f}_j^{n+1} - \bar{f}_{j-1}^{n+1})$