Chapter 3

DESCRIPTION OF CONTINUUM TRAFFIC FLOW MODELS AND RIEMANN PROBLEMS

In this chapter, the conservation equation of traffic flow was derived, and then this study implemented the formulation of the simple and high order continuum models for numerical discretization. Finally, Riemann problems in continuum traffic flow models were described briefly.

3.1 Derivation of Conservation Equation of Traffic Flow

Consider some segment of roadway without entrances or exits, the number of cars N is the integral of the traffic density $k(x,t)$

$$N = \int_{x_0}^{x_0+\Delta x} k(x,t)dx \,. \tag{3.1}$$

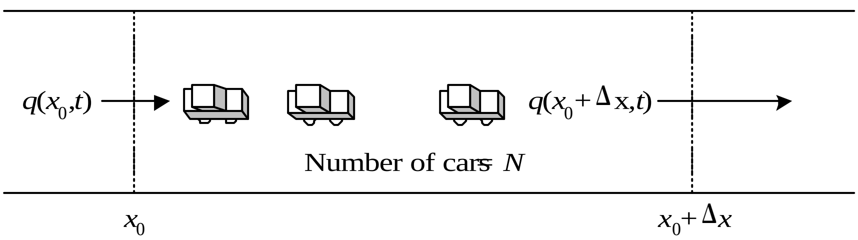

Figure 3.1. An interval of roadway without entrances or exits.

The difference in number of cars between time t and t+Ät equals the number crossing at x_0 between t and t+Ät minus the number crossing at x_0+Äx between t and t+Ät

$$N(t+\Delta t) - N(t) \approx \Delta t\big(q(x_0,t) - q(x_0+\Delta x,t)\big). \tag{3.2}$$

where $q(x_0,t)$ denotes traffic flow rate at x_0. Dividing Eqn. (3.2) by Δt and taking the limit as $\Delta t \to 0$ produces

$$\frac{dN}{dt} = q(x_0,t) - q(x_0 + \Delta x, t) . \tag{3.3}$$

Combine Eqns. (3.2) and (3.3) to obtain

$$\frac{d}{dt}\int_{x_0}^{x_0+\Delta x} k(x,t)dx = q(x_0,t) - q(x_0 + \Delta x, t) . \tag{3.4}$$

Eqn. (3.4) indicates that changes in the number of cars are only induced by the flow across the boundary. Assume that no cars are created or destroyed, and thus the number of cars is conserved. Eqn. (3.4) is called the integral conservation law. Provided that the endpoints of the roadway section, x_0 and $x_0+\Delta x$ are considered as additional independent variables, a partial derivative with respect to time must be employed instead of the full derivative in Eqn. (3.4),

$$\frac{\partial}{\partial t}\int_{x_0}^{x_0+\Delta x} k(x,t)dx = q(x_0,t) - q(x_0 + \Delta x, t) . \tag{3.5}$$

Then dividing Eqn. (3.5) by $-\Delta x$ and taking the limit as $\Delta x \to 0$ yields

$$\lim_{\Delta x \to 0} \frac{\partial}{\partial t} \frac{1}{-\Delta x}\int_{x_0}^{x_0+\Delta x} k(x,t)dx = \lim_{\Delta x \to 0} \frac{q(x_0,t) - q(x_0 + \Delta x, t)}{-\Delta x} . \tag{3.6}$$

The right-hand side of Eqn. (3.6) signifies the partial derivative of $q(x_0,t)$ with respect to x_0. Since Δx is small, the number of cars between x_0 and $x_0+\Delta x$ could be approximated by the traffic density at x_0, $k(x_0,t)$ times the distance of the segment Δx.

$$\lim_{\Delta x \to 0} \frac{1}{-\Delta x}\int_{x_0}^{x_0+\Delta x} k(x,t)dx \approx -k(x_0,t) \tag{3.7}$$

Hence, coupling Eqn. (3.6) with Eqn. (3.7) yields

$$\frac{\partial}{\partial t} k(x_0,t) + \frac{\partial}{\partial x_0} q(x_0,t) = 0 . \tag{3.8}$$

Finally, replacing x_0 by x provides the conservation equation of traffic flow

$$\frac{\partial}{\partial t}k(x,t)+\frac{\partial}{\partial x}q(x,t)=0\,. \tag{3.9}$$

3.2 Formulation of Continuum Traffic Flow Models for Numerical Discretization

In this section, continuum traffic flow models including the simple and high order continuum models were formulated for the numerical discretization.

3.2.1 LWR Model

LWR model contains following three equations:

$$\frac{\partial k}{\partial t}+\frac{\partial q}{\partial x}=g(x,t)\,, \tag{3.10}$$

$$q=ku\,, \tag{3.11}$$

$$u=u_e(k)\,. \tag{3.12}$$

Eqn. (3.10) can be derived form Section 3.1 by introducing the generation rate of flow $g(x,t)$, which is equal to zero in roadway segments without entrances or exits. Eqn. (3.11) implies that the flow rate of the traffic stream equals the traffic density times the velocity. Eqn. (3.12), which is obtained either from empirical observations or from theoretical models, represents the equilibrium relationship between the velocity and the traffic density.

Rewriting LWR model by combining Eqns. (3.10), (3.11), and (3.12), the following equation is acquired:

$$\frac{\partial k}{\partial t}+\frac{\partial k\cdot u_e(k)}{\partial x}=g\,. \tag{3.13}$$

Let U denote the state variable, $F(U)$ the flux function, and $S(U)$ the source flux. The formulation of continuum models for numerical discretization could be expressed as follows:

$$\frac{\partial U}{\partial t}+\frac{\partial F(U)}{\partial x}=S(U)\,. \tag{3.14}$$

Consequently, the state variable, flux function, and source flux for the LWR model are

$U=k\,,\ \ F(U)=k\cdot u_e(k)=f_e(k)\,,\ \ S(U)=g\,.$

3.2.2 High Order Continuum Models

During the past several decades, various high order models have been presented to conquer the deficiencies in LWR model. Two of those high order models, PW model and Jiang's improved model, were chosen to implement numerical simulation by high resolution schemes. PW model was selected due to being the original high order model. By reason of novelty and the capability of overcoming the backward travel problem, this study preferred Jiang's improved model. Furthermore, shocks, rarefaction waves, stop-and-go waves, and local cluster effects, which are consistent with the diverse nonlinear dynamical phenomena observed in the freeway traffic, could be obtained from Jiang's improved model (Jiang et al., 2002).

3.2.2.1 PW Model

By coupling the conservation equation and a dynamic speed equation (often named "momentum equation"), PW model can be represented as follows:

$$\frac{\partial k}{\partial t}+\frac{\partial ku}{\partial x}=g\,, \tag{3.15}$$

$$\frac{\partial u}{\partial t}+u\left(\frac{\partial u}{\partial x}\right)=-\frac{c_0^2}{k}\frac{\partial k}{\partial x}+\frac{1}{\tau}\left(u_e(k)-u\right). \tag{3.16}$$

Rewriting Eqns. (3.15) and (3.16) as a system, the formulation of PW model for numerical discretization is obtained:

$$\frac{\partial \boldsymbol{U}}{\partial t}+\frac{\partial \boldsymbol{F}(\boldsymbol{U})}{\partial x}=\boldsymbol{S}(\boldsymbol{U})\,, \tag{3.17}$$

where the state variable $\boldsymbol{U}$, flux function $\boldsymbol{F}(\boldsymbol{U})$, and source flux $\boldsymbol{S}(\boldsymbol{U})$ are vectors:

$$\boldsymbol{U}=\begin{bmatrix} k \\ u \end{bmatrix},\quad \boldsymbol{F}(\boldsymbol{U})=\begin{bmatrix} ku \\ \frac{u^2}{2}+c_0^2\ln k \end{bmatrix},\quad \boldsymbol{S}(\boldsymbol{U})=\begin{bmatrix} g \\ (u_e-u)/\boldsymbol{t} \end{bmatrix}.$$

By transferring Eqn. (3.17) to the following formulation:

$$\frac{\partial \boldsymbol{U}}{\partial t}+[\boldsymbol{J}]\frac{\partial \boldsymbol{U}}{\partial x}=\boldsymbol{S}(\boldsymbol{U}), \tag{3.18}$$

the Jacobian matrix of the system $[\boldsymbol{J}]$ could be procured as follows:

$$[\boldsymbol{J}]=\begin{bmatrix} u & k \\ \frac{c_0^2}{k} & u \end{bmatrix}. \tag{3.19}$$

3.2.2.2 Jiang's Improved Model

Based on improved car-following model, Jiang's improved model replaces the density gradient with speed gradient in the momentum equation to avoid wrong-way problem proposed by Daganzo (1995b).

Jiang's improved model, which is consistent with other high order models, consists of two PDEs as follows:

$$\frac{\partial k}{\partial t}+\frac{\partial ku}{\partial x}=g, \tag{3.20}$$

$$\frac{\partial u}{\partial t}+u\left(\frac{\partial u}{\partial x}\right)=\frac{1}{\boldsymbol{t}}\left(u_e(k)-u\right)+c\frac{\partial u}{\partial x}. \tag{3.21}$$

With the procedure of formulating PW model in Section 3.2.2.1, the formulation of Jiang's improved model for numerical discretization can be also obtained.

The state variable $\boldsymbol{U}$, flux function $\boldsymbol{F}(\boldsymbol{U})$, source flux $\boldsymbol{S}(\boldsymbol{U})$, and Jacobian matrix $[\boldsymbol{J}]$ of Jiang's improved model are as follows:

$$\boldsymbol{U}=\begin{bmatrix} k \\ u \end{bmatrix},\quad \boldsymbol{F}(\boldsymbol{U})=\begin{bmatrix} ku \\ \frac{u^2}{2}-cu \end{bmatrix},\quad \boldsymbol{S}(\boldsymbol{U})=\begin{bmatrix} g \\ (u_e-u)/\boldsymbol{t} \end{bmatrix},\quad [\boldsymbol{J}]=\begin{bmatrix} u & k \\ 0 & u-c \end{bmatrix}.$$

3.3 Riemann Problems in Continuum Traffic Flow Models

It turns out that continuum traffic flow models comprise hyperbolic PDEs, with LWR model a single PDE and high order models systems of PDEs. As such, discontinuous solutions, called shocks, arise from even smooth initial data in these models. A particular problem, called a Riemann problem, is often used to characterize the solutions of hyperbolic PDEs. A Riemann problem is nothing more than hyperbolic PDEs with a special set of initial data, called Riemann data, in which a single jump discontinuity separates two infinitely long regions of constant states (e.g. traffic density, velocity, and flow rate). Riemann data for LWR model, for example, are shown as follows:

$$k(x,0) = \begin{cases} k_l, & x < 0 \\ k_r, & x \geq 0 \end{cases},$$

where the upstream density k_l and the downstream density k_r are constants.

The solutions to hyperbolic PDEs with Riemann initial data, called Riemann solutions, are usually of three basic types: shocks, rarefaction waves, and contacts. Not every one of these waves is present in continuum traffic flow models. Contacts, for example, do not exist in continuum models. Because of the existence of shocks, hyperbolic PDEs such as those of LWR and high order models are notoriously difficult to solve numerically. Special care must be taken when these equations are converted into formulation of numerical discretization. Consequently, Riemann problems would be the main issue to solve in this study.