

# Chapter 1

## Introduction

### 1.1 Motivation and Objective

Planning, designing, and operating transportation systems require fundamental traffic flow characteristics and associated analytical techniques. Traffic flow characteristics include time headway, flow, time-space restriction, speed, distance headway, and density. This analytical process can vary from a simple equation to a complex simulation model. Usually, we can develop traffic analytical techniques using two different points of view, microscopic and macroscopic analysis. Microscopic analysis may be selected for moderate-sized systems where the number of transport units passing through the system is relatively small and there is a need to study the behavior of individual units in the system. A macroscopic system may be selected for higher-density, large-scale systems in which a study of the behavior of groups of units is sufficient. Traffic flow model can be dependent on time periods or not. A static model is independent of the time, and is used to plan, and design highway systems. A dynamic model depends on

the time, and is used to operate highway systems such as ramp metering and signal control.

A new traffic flow model based on spatial and capacity is proposed. Using field theory, the flow is depicted a conservative vector field. The existence and uniqueness of the solution for this model is established using Green's identity. A general analytical solution formula is discussed for various boundary conditions.

An objective of traffic and transportation engineering is to control the traffic streams on a set of roads (a network) so as to reduce delay or improve flow without inducing undesirable side effects to society [[31]]-[[33]]. This is attempted for example when engineers change the signal-timing plan on a network in order to reduce both congestion and vehicular emissions. At other times the objective is to achieve a societal goal, such as preventing traffic from flowing through neighborhoods, while inconveniencing those who must travel as little as possible. In these two cases, and others as well, transportation engineering objectives are usually pursued by means of system control and redesign.

Clearly, in order to be capable of developing effective system design and control strategies, engineers must thoroughly understand how the system in question might respond to possible engineering changes. In particular, they should be able to predict (e.g., by means of mathematical models) any figures of merit that are relevant to the affected public, and should also have an intuitive 'feel' for the likely response of the system to a control or redesign. The latter skill is important when the time comes to develop a short list of potential improvements for further evaluation.

The macroscopic continuous transportation model within a scalar velocity field was developed and applied to the study of transportation capacity allocation in congested areas[[17]][[26]]. Congestion is defined to be the breakdown of fluent traffic flow due to the inefficiency of the transport link to handle the traffic volume. This is precipitated by a multitude of disruptive phenomena within the dynamics of traffic flow. Congestion and traffic flow will be dealt with within a framework of three dimensions, i.e., geographic space  $(x; y)$ , carrying capacity  $(c)$  under steady state assumptions. Capacity is measured as a third dimension orthogonal to the plane and is defined as the ability to transfer traffic in space. Traffic flow increases or decreases dependent on the net rate of outflow per unit of traffic volume at a point[[4]].

## 1.2 Methodology and Approach

In this study, a transonic dispersion relation model is considered using the continuity equation using mass conservation in transonic flow system.

In order to study the transonic interaction behaviors for two motion objectives, the gravity function model is also discussed in this thesis. Furthermore, the main governing equation, the so called Poisson Equation is established using the potential theory proposed by Sheppard.

We derive a general solution formula for the proposed equation for the applications in transonic flow engineering. Numerical solution using the finite difference method is also presented for a more complete model study.

The transonic flow dispersion relation for various empirical transonic assumptions is discussed and compared for the proposed model.

### 1.3 Study Flowchart

As shown in the flowchart, Figure 1.1, first of all the study issue for this thesis is confirmed. The second frame reviews some transonic flow properties and transonic flow models. The third frame shows our main mathematical definitions that include, the conservative vector field theory and the analytical solution technique. In the fourth step, we establish our proposed model, the transonic dispersion mode. We proposed a transonic dispersion model that describes the relationship between transonic flow, density, spatial variation as well as transonic capacity. Furthermore, analytical and numerical solution techniques are presented in this study. Some comparisons and remarks are also given at the conclusion of this study.

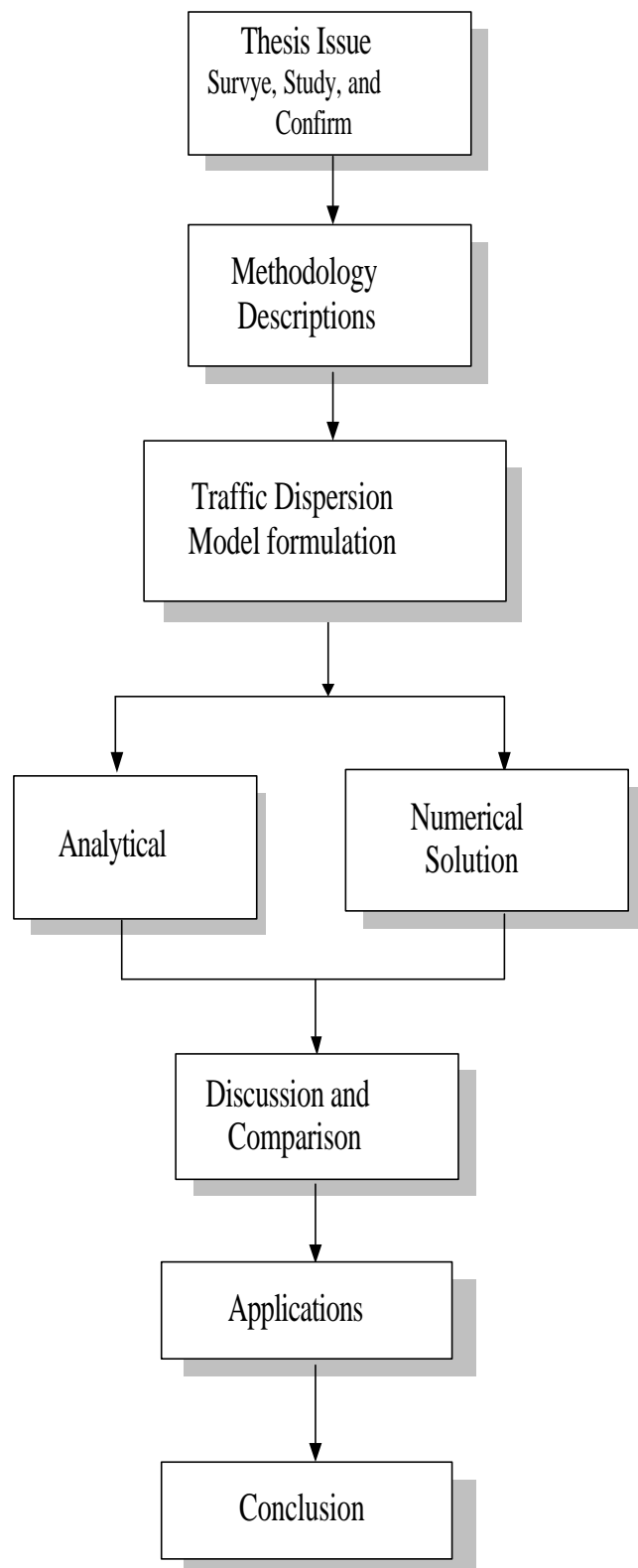


Figure 1.1: study °owchart

## 1.4 Outline

This thesis is organized as follows: in chapter 2, some relative literature is reviewed and our main methodology is described. In chapter 3, some mathematical methodologies are described. We formulate the proposed model using the conservative vector field theory in chapter 4. A well-posed analysis of the propose model is given in chapter 5. A general solution technique is presented in chapter 6 for various model problems. Some numerical results are given in chapter 7. The conclusion and remarks are provided in the last chapter.





# Chapter 2

## Historical Developments

In this chapter, we briefly review the traffic flow basic definition, relation and characterizations. General traffic flow models are then described, including microscopic traffic flow models and macroscopic traffic flow models. Section 2.1 reviews traffic flow fundamentals, microscopic traffic flow models are stated in section 2.2. In section 2.3, the macroscopic traffic flow model is also given.

### 2.1 Traffic Flow Fundamentals

According to the Highway Capacity Manual [[23]], the operational state of any given traffic stream is defined by three primary measures:

1. Flow ( $q$ )

Flow is defined as the number of vehicles (N) passing a specific point or short section in a given period of time (T) in a single lane. Flow is computed as

$$q = \frac{N}{T} \quad (2.1)$$

This is usual traffic engineering practice. It is customary to start the timing of counting intervals at random with respect to traffic. Flow is expressed as an hourly rate on a per lane basis (veh/hr/lane).

## 2. Speed (u)

Speed is defined as the average rate of motion and is expressed in miles per hour (mile/hr). The average speed is an important measure of the traffic performance at a particular point or along a particular route. There are two principal average speeds, the time mean speed and the space mean speed

(a) Time mean speed (spot speed) : It has been common practice among traffic engineers to report the "spot speed" for a given location. This is computed as the arithmetic mean of the observed speeds

$$u_t = u_{\text{spot}} = \frac{1}{N} \sum_{i=1}^N u_i$$

In theoretical discussions on traffic flow, this value is referred to as the "time mean speed".

(b) Space mean speed (harmonic mean speed) : Cars are traversing a length D and average travel time would be  $t = \frac{1}{N} \sum_{i=1}^N \frac{D}{u_i}$ . The average speed representing this travel time would be

$$u = \frac{D}{t}$$

that is "space mean speed".

It is noted that the space mean speed is lower than the time mean speed.

### 3. Density (k)

Density is defined as the number of vehicles occupying a section of roadway in a single lane. Density is expressed on a per mile and a per lane basis (veh/mile/lane).

### 4. Relationships among speed, density, and flow

The relationship among the three variables  $u$ ,  $k$ , and  $q$  is called a traffic flow model. Let the three flow characteristics be functions which depend on distance( $x$ ) and time( $t$ ). That are  $q(x; t)$ ,  $u(x; t)$ , and  $k(x; t)$ . We consider the number of vehicles ( $N$ ) passing an observer during the small interval ( $\Delta t$ ). Since the vehicles don't move too far afield, we assume the speed  $u(x; t)$  to agree with density  $k(x; t)$ . The number of vehicles that are in area of  $u(x; t)\Delta t$  pass an observed point during  $\Delta t$ , then the number of vehicles that pass the observed point are  $u(x; t)\Delta t k(x; t)$ : By the definition of flow,

$$q(x; t) = \frac{u(x; t)\Delta t k(x; t)}{\Delta t} = u(x; t) k(x; t): \quad (2.2)$$

The flow (2.2) is shown to be equivalent to the product of speed and density. (2.2) must be on a three-dimensional surface. It is usually more convenient to show the model as one or more of the three separate relationships in two dimensions as shown in Figure 2.1. Figure 2.1 was based on the assumption of a linear speed-density relationship. Speed has two unique speed parameters,

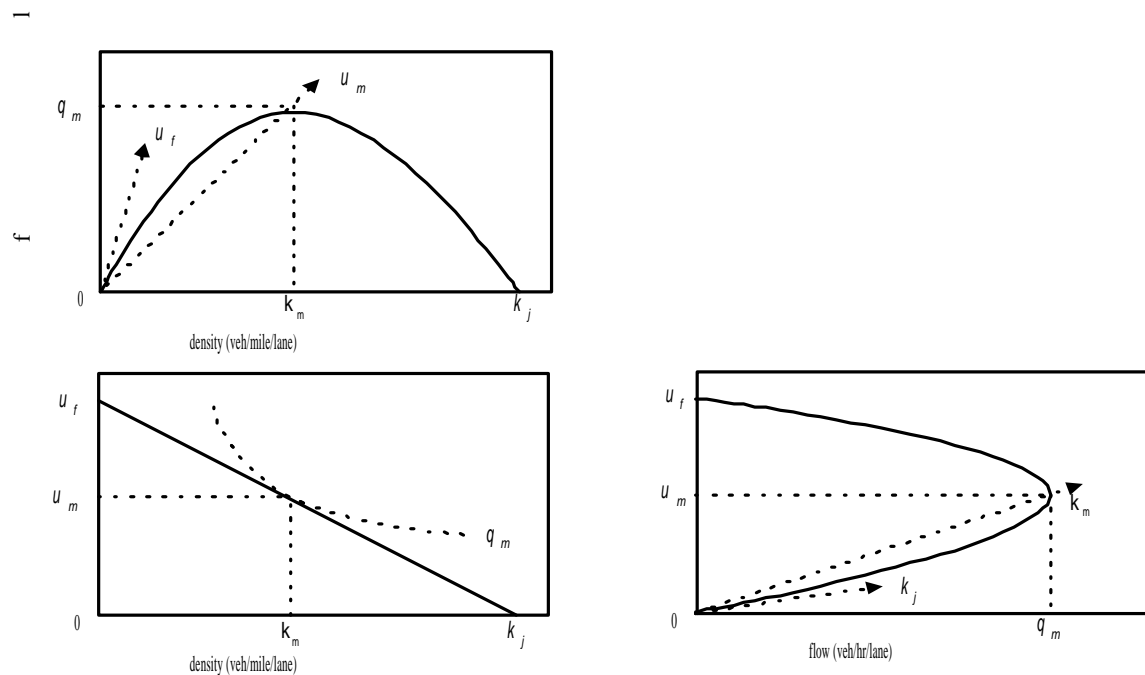


Figure 2.1: Basic stream flow diagrams

free-flow speed ( $u_f$ ) and optimum speed ( $u_m$ ). Free-flow speed is that speed which exists when the flows approach zero under free-flow conditions while the optimum speed is that speed which exists under maximum flow conditions. Density has two unique density parameters, jam density ( $k_j$ ) and optimum density ( $k_m$ ): Jam density is that density that occurs when both flow and speed approach zero while optimum density occurs under maximum flow conditions.

5. Design capacity ( $c$ ): A single capacity value representing the maximum traffic volume that may pass a cross section of a road with a certain probability under predefined road and weather conditions. This value is used for planning and designing roads and carriage ways, called "design capacity"[[23]].

Traditionally, traffic stream models primarily discuss uninterrupted and macroscopic traffic stream characteristics. This shows the relationships between flow, speed, and density under uninterrupted traffic flow. The earlier models assumed a single regime phenomenon over the complete range of flow conditions including free-flow and congested flow situations. Later models attempted to improve on the earlier models by considering two separate regimes (free-flow regime and congested-flow regime) and attempted to generalize by introducing additional parameters that could be used to distinguish between roadway environments. We arrange the first single-regime model in Table 2-1[[31]].

Table 2-1. Equation of single-regime model

Single-Regime Models	Equation	Scope and characteristics
Greenshields model(1934)	$u = u_f \left(1 - \frac{k}{k_j}\right)$	- linear speed-density relationship
Greenberg model(1959)	$u = u_0 \frac{k_j}{k}$	- logarithm traffic stream model relationship -unsuitable to low density
Underwood model(1961)	$u = u_f \exp\left(-\frac{k}{k_0}\right)$	- exponential traffic stream models ; - It has better suitable distribution in low density; -unsuitable in high density
Northwestern's model(1967)	$u = u_f \exp\left(-\frac{1}{2} \frac{k}{k_0}\right)$	- unsuitable to describe jam flow
Drew model(1968)	$u = u_f \left(1 - \frac{k}{k_j}\right)^{\frac{(n+1)}{2}}$	- Conducting common equation of single-regime model based on Greenshield Model.
Pipes-Munjal model(1967)	$u = u_f \left(1 - \frac{k}{k_j}\right)^n$	-Conducting more common equation of single-regime model.
parameter	$u$ : speed	$k$ : density
	$u_f$ : free-flow speed	$k_j$ : jam density
	$u_0$ : optimum speed	$k_0$ : optimum density

In Table 2-1, the Greenberg and Underwood models have the lowest maximum flow predictions. In regard to free-flow speeds, the Greenberg and Underwood models predict much higher values than would be expected from the data set. The Greenberg model appears to underestimate optimum speed. The Greenshields model significantly underestimates jam density, while the Underwood and Northwestern models predict jam densities to infinity. The Greenberg model slightly overestimates optimum density.

In summary, single-regime models have been described and then applied to a freeway data set. Each model has deficiencies over some portion of the density range. The most disconcerting feature of these models is their inability to track faithfully the measured field data under near capacity conditions. One can observe a discontinuity in the flow-speed-density relationships as depicted by measured field data in the vicinity of capacity conditions. This has led to several researchers proposing two-regime models with separate formulations for the free-flow and congested-flow regimes.

Supporting the idea of the multiregime, a Northwestern University research team produced three additional model formulations[[12]]. The first difficulty in multiregime models is determining the breakpoint between regimes. The Northwestern researchers applied the work of Quandt [[33]]-[[34]] on likelihood functions to identify the breakpoints between regimes for all multiregime models. Then using regression analysis, the best model was selected for each regime. The resulting equation and breakpoints for the multiregime models are presented in Table 2-2[[12]].

Table 2-2. Equations and breakpoints for multiregime models

Multiregime Model	Free-Flow Regime	Transitional- Flow Regime	Congested- Flow Regime
Eddie model	$u = 54.9 \exp\left(\frac{-1}{163.9}k\right)$ ( $k < 50$ )	-	$u = 26.8 \ln \frac{162.5}{k}$ ( $k > 50$ )
Two-regime linear model	$u = 60.9 - 0.515k$ ( $k < 65$ )	-	$u = 40 - 0.265k$ ( $k > 65$ )
Modified Greenberg model	$u = 48$ ( $k < 35$ )	-	$u = 32 \ln \frac{145.5}{k}$ ( $k > 35$ )
Three-regime linear model	$u = 50 - 0.098k$ ( $k < 40$ )	$u = 81.4 - 0.913k$ ( $40 < k < 65$ )	$u = 40 - 0.265k$ ( $k > 65$ )

The above discussed traffic flow models were macroscopic traffic flow. The macroscopic model conducts groups of vehicles' flow, speed, and density. The other researchers described traffic flow model using individual vehicles, time headways, and other characteristics.. This is called microscopic traffic flow (ex. car-following theory). This study involves traffic dispersion characteristics from the traffic dispersion equation, which is an analysis from the macroscopic traffic flow. It is not necessary to discuss microscopic traffic flow.



## 2.2 Reviews on Microscopic Traffic Flow Models

Microscopic models focus on individual behavior, the main characteristics they describe and predict are time headway, individual speeds, and distance headway. Static microscopic models discuss the distribution of time headway, individual speeds, and distance headway by investigation. Dynamic models can be derived from the interaction between vehicles on the road. The basic point in contention is discussed well within the context of the development of Gazis, Herman, and Potts[[16]] as it relates to car-following models which describe the behavior of each vehicle modeled in relation to the vehicle ahead. As this definition indicates, this theory concentrates on single lane situations where a driver reacts to the movements of the vehicle ahead of him. The equations of motion are

$$M\ddot{x}_{n+1}(t) = -s(x_n - x_{n+1})_{t-t_0} \quad (2.3)$$

where  $x_n(t)$  is the position of the  $n$ th car in the line,  $M$  is the mass of each car,  $s$  is the sensitivity, and  $t_0$  is the time-lag of driver-car system. The general form can be noted as

$$\ddot{x}_{n+1}(t) = -\frac{(x_{n+1}(t))^m}{(x_n - x_{n+1})_{t-t_0}^l} (x_n - x_{n+1})_{t-t_0} \quad (2.4)$$

where  $l, m$  are sensitivity parameters. Mathematically, parts of this theory are very similar to the treatment of atomic movements in crystals and give result about the stability of chains of cars (platoons) in follow-the-leader situations. One of the achievements of traffic theory is that flow density relations are derived from the between car-following models.

## 2.3 Reviews on Macroscopic Traffic Flow Models

Macroscopic traffic flow models describe higher-density, larger-scale systems. The basic variables in macroscopic traffic flow are  $u$  (speed),  $k$  (density), and  $q$  (flow) and the relationship between them is  $q = k \cdot u$ .

Macroscopic models historically search for time-independent relations between  $q$ ;  $k$ ;  $u$ . The form for such a relation is, though, still debated in the traffic flow literature. The problem arises mainly from the fact that reality measurements are done in nonstationary conditions. Therefore, only short time averages make sense and they usually show large fluctuations. This type of model always uses long time averages and assumes that the traffic is in stationary condition.

The most famous dynamic traffic flow model is the traffic stream model or so called kinematic wave theory. Some researches investigated speeds, flows and densities from low quality time lapse film and acquired information on vehicle tracking, which proved that vehicular platoons can be treated as a stream of fluid. Gerlough and Huber[[17]], Haberman[[21]] described "the relationship among the three variable" using the term "traffic stream model". The basic equation is

$$\frac{\partial k(x; t)}{\partial t} + \frac{\partial q(x; t)}{\partial x} = 0 \text{ or } \frac{\partial k(x; t)}{\partial t} + \frac{\partial}{\partial x}(ku) = 0 \quad (2.5)$$

Lighthill and Whitham[[30]] and Richards introduced this to traffic flow using a kinematic wave. This model is also called LWR model. They combined the law of vehicle concentration and traffic stream models to provide the ability to predict the temporal and spatial variation in traffic flow. Payne[[33]] seems to be the first

publication following up on that suggestion, although a very similar theory appears in the book by Whitham[[44]].

The kinematic wave theory offers an alternative approach that provides a ready answer to what should be used as the underlying statistical distribution, and that provides the development of a traffic stream model that does not suggest that the validity of the model is restricted to static conditions. There is another family of models which are called the "Boltzmann-like model" developed by Prigogine and Herman (1971). The basic form for this model is

$$\frac{df}{dt} = \frac{df}{dt} + v \frac{df}{dx} = 0 \quad (2.6)$$

where  $f(x;v;t)$  is a velocity distribution function. They derived LWR model as a limiting case of the kinetic theory, and they extended it to many of the phenomena presented in later works.



## Chapter 3

# Mathematical Preliminaries

This chapter states some important mathematical preliminaries that we will use in the following studies. The definitions of gradient, divergence, and curl are given first. Their mathematical and physical meanings for transport engineering are stated respectively. In section 3.4, we point out some basic facts about partial differential equations for our next applications. The boundary conditions and potential theory for transport application are examined in section 3.5 and 3.6 respectively.

### 3.1 Gradient

The gradient expression corresponds to the derivative itself and describes the variations of a scalar field at a certain point  $(x; y; z)$  in space. We assume the concepts of partial and total derivatives to be known. Let us start from the point  $(x; y; z)$  and move along the vector  $d\mathbf{r}$ , which may have the length  $dr$  and the components  $dx; dy; dz$  shown as figure 3.1.

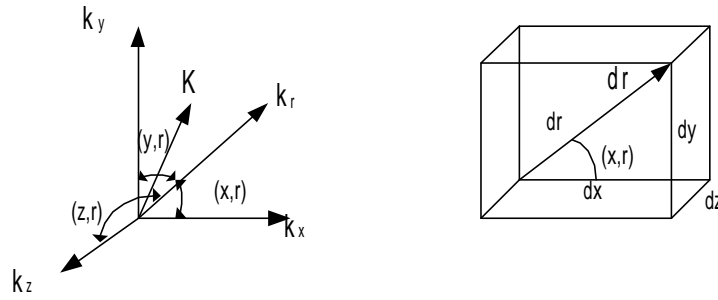


Figure 3.1: Vector components and element of path and components

Then if  $\phi$  is any scalar function, we have

$$\frac{\partial \phi}{\partial r} dr = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz \quad (3.1)$$

or

$$\frac{\partial \phi}{\partial r} = \frac{\partial \phi}{\partial x} \cos(x; r) + \frac{\partial \phi}{\partial y} \cos(y; r) + \frac{\partial \phi}{\partial z} \cos(z; r)$$

Thus the derivative in any direction can be expressed by the three quantities  $\frac{\partial \phi}{\partial x}$ ;  $\frac{\partial \phi}{\partial y}$ ;  $\frac{\partial \phi}{\partial z}$ ; and this expression has the same form as the expression of a component  $K_r$  of a vector  $K$  by means of three perpendicular components  $K_x$ ;  $K_y$ ;  $K_z$ . The direction of the vector  $K$  is the direction for which the component in this direction has a maximum absolute attainable value. This length is equal to the Pythagoric sum of the three components. In the same way we can easily calculate that in a certain direction  $r_0$ ,  $\frac{\partial \phi}{\partial r}$  has a maximum value and is equal to zero in all directions perpendicular to this. In the direction opposite to  $r_0$ , we have a minimum of the same absolute value as the maximum. Thus a vector in the three-dimensional space corresponds to the derivative in the case of one variable. This vector corresponds in length and in direction to the maximum rate of increase of the scalar  $\phi$  from the point under consideration. Its

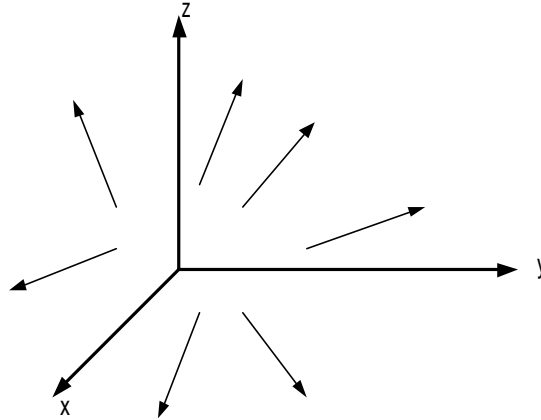


Figure 3.2: Velocity vector field in  $\mathbb{R}^3$

component in any direction is equal to the partial derivative in this direction. This vector is called the "gradient" (grad).

**Definition 1** Let  $\phi$  be a function of three variables such that  $\phi_x$  and  $\phi_y$  and  $\phi_z$  exist at a point  $x=(x; y; z)$ . Then the gradient of  $\phi$  at  $x$ , denoted by  $\nabla \phi(x)$ , is given by

$$\begin{aligned} \nabla \phi(x) &= \left( \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right) \\ &= \phi_x(x; y; z)\mathbf{i} + \phi_y(x; y; z)\mathbf{j} + \phi_z(x; y; z)\mathbf{k} \end{aligned} \quad (3.2)$$

Note that the gradient of  $\phi$  is a vector function. That is, for every point  $x$  in  $\mathbb{R}^3$  for which  $\nabla \phi(x)$  is defined.

**Remark 1** The typical physical meaning for the gradient is a derivative taken in the direction of maximum rate of change, as in figure 3.2.

## 3.2 Divergence

Assume that a vector field in which there is a closed surface dividing the space into two parts {the interior and the exterior of the surface. For each element of this surface, the product "element of surface  $df$  times the normal component of the field vector (exterior normal)" has a value which is independent of the coordinate system. This can be represented by the scalar product of the field vector with another vector whose length is equal to the area of the element and the direction of which is that of the direction of the exterior normal to the element. The integral of this product over the whole closed surface has a physical significance. If the field vector is the velocity of a fluid flow, then the product is the volume of liquid which flows through the element from the interior to the exterior in a unit of time. Thus the integral represents the volume of fluid leaving the interior. At points on the surface where  $v$  is directed into the interior, the integrand becomes negative, of course. The integral thus represents the total strength of all sources in the interior. If the latter is not equal to zero, this can be caused either by streamlines originating in the interior or by streamlines along which the flow rate increases or decreases. In the hydrodynamic example, the first case corresponds to a flow with a source in the interior, as shown in Figure 3.3, while the second case corresponds to the stationary flow of traffic in which the density decreases in the direction of the flow, shown as Figure 3.4. In the latter case, the same mass of traffic enters the region bounded by the two boundary,  $\partial V_1$  and  $\partial V_2$ , and two cross sections, as shown in Figure 3.4, in unit time as it leaves. Thus the product "density times velocity times area of entrance or exit" must have the same value for



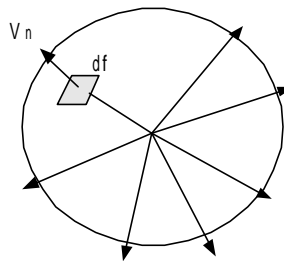


Figure 3.3: Element of an arbitrary surface

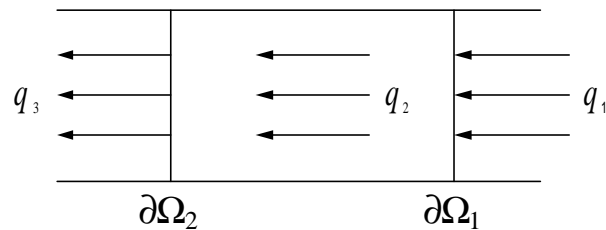


Figure 3.4: Flow with variable density

both cross sections. Since the density is different on the two sections, the products "velocity times area of cross section" are different, so that the integral, which in this case is equal to the difference of the two products, is different from zero.

In order to find the mathematical expression for the "magnitude of a source" at a point in space, we have to take a closed surface which contains an infinitely small neighborhood of this point. In this way the integral becomes an infinitesimal of third order (the surface itself is of second order, but the difference between the field vector at the point of entrance and point of exit, also becomes infinitesimal). Thus if we divide by the volume  $V$  of the enclosed region, we shall have a finite limit. We call

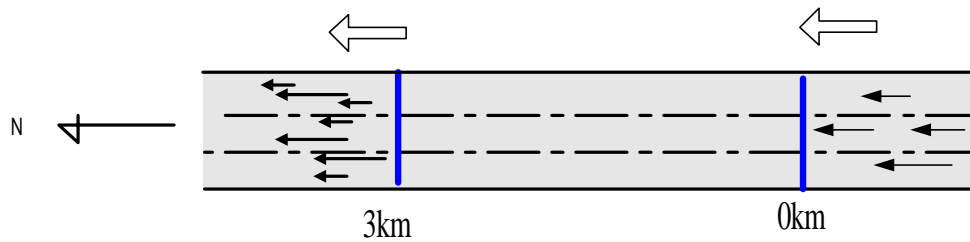


Figure 3.5: The divergence is greater than zero.

this limit the "divergence",  $\text{div}$ , of the field,

$$\text{div } \mathbf{A} = \lim_{V \rightarrow 0} \frac{1}{V} \oint \mathbf{A} \cdot d\mathbf{f} \quad (3.3)$$

The divergence is a scalar.

**Definition 2** The calculation of the divergence in various coordinate systems is now simple. In rectangular systems we use an infinitesimal cube in the field  $\mathbf{A}$ , the edges of which are a length parallel to the axes of the system. In this case the normal components of the vector on the surface elements of the cube are either  $A_x$  or  $A_y$  or  $A_z$ .

$$\text{div } \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}; \quad (3.4)$$

this is the Mathematical expression of the divergence.

**Remark 2** The traffic physical meaning for the divergence of a vector field is equivalent to "source strength". If source is equal to zero, then divergence is also equal zero. If exit source is greater than enter source, then divergence is greater than zero, as figure 3.5.

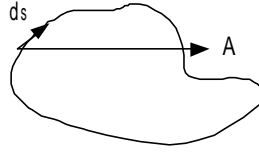


Figure 3.6: Line element and vector

### 3.3 Curl

Let  $C$  be a closed plane curve in the vector field. Along this curve we shall form the integral of the scalar product of the field vector and the element  $ds$  of the curve. Naturally we have to determine the direction in which we shall integrate along the curve. This integral is independent of the coordinate system. If  $A_s$  is the vector component in the direction of the tangent to the curve and  $ds$  the length  $ds$ , of an element of the curve, the value of the integral is  $\oint A_s ds$ . If the curve is now subjected to a limiting process where its length tends to 0, the value of the integral becomes small of the second order. For on one hand the length of the curve becomes small, and on the other hand the differences of the field vector on different parts of the curve are small. Dividing the integral by the area  $f$  of the plane surface enclosed by the curve, we shall therefore obtain a finite limit. This limit is called the "curl" of the vector  $A$ :

$$\text{curl } A = \lim_{f \rightarrow 0} \frac{1}{f} \oint A_s ds \quad (3.5)$$

**Definition 3** Curl in  $R^3$ . Let the vector field  $A$  is given by

$$A(x; y; z) = A_x(x; y; z)i + A_y(x; y; z)j + A_z(x; y; z)k,$$

where  $A_x, A_y, A_z$  are differentiable. Then the curl of  $A$  (  $\text{curl } A$  ) is given by

$$\begin{aligned}
 \text{curl } A &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \\
 &= \frac{\partial A_z}{\partial y} \mathbf{i} - \frac{\partial A_y}{\partial z} \mathbf{i} + \frac{\partial A_x}{\partial z} \mathbf{j} - \frac{\partial A_z}{\partial x} \mathbf{j} + \frac{\partial A_y}{\partial x} \mathbf{k} - \frac{\partial A_x}{\partial y} \mathbf{k} \\
 &= \text{curl } A.
 \end{aligned} \tag{3.6}$$

Note :  $\text{curl } A$  is a vector field.

**Remark 3** The true physical meaning of the curl is an expression of the way the vector changes with distance in terms of the rates at which its components change in directions perpendicular to themselves.

### 3.4 Partial Differential Equation (PDE)

Hopf[[28]] describes the difference between ordinary and partial differential equations. A differential equation is called ordinary if there is only one independent variable, and partial if there are several independent variables. In an ordinary differential equation only ordinary derivatives appear, while a partial differential equation contains partial derivatives. The difference between these two types of differential equations appears more significant from the standpoint of physics, and is related to the deepest physical problems.

The key defining property of a partial differential equation (PDE) is that there is more than one independent variable  $x; y; \dots$ . There is a dependent variable that is an unknown function of these variables  $u(x; y; \dots)$ [[41]]. Its derivatives by subscripts; thus  $\frac{\partial u}{\partial x} = u_x$ , and so on. A PDE is an identity that relates the independent variables, the dependent variable  $u$ , and the partial derivatives of  $u$ . It can be written as

$$F(x; y; u(x; y); u_x(x; y); u_y(x; y)) = F(x; y; u; u_x; u_y) = 0: \quad (3.7)$$

This is the most general PDE in two independent variables of the first order. The order of an equation is the highest derivative that appears. The most general second-order PDE in two independent variables is

$$F(x; y; u; u_x; u_y; u_{xx}; u_{yy}) = 0: \quad (3.8)$$

A solution of a PDE is a function  $u(x; y; \dots)$  that satisfies the equation identically, at least in some region of the  $x; y; \dots$  variables.

## 3.5 Boundary Conditions

Let  $\Omega$  denote a bounded region in the plane with "smooth" boundary  $\partial\Omega$ . The bounded  $\Omega$  is a region in  $\mathbb{R}^3$  or even  $\mathbb{R}^N$ . However, our model has three independent variables. We are not specific about what is meant by a smooth boundary for a region  $\Omega$  other than to say that included in this class are "reasonable" regions such as disks or sectors of disks, rectangles or unions of finitely many rectangles, and other nonpathological sets.

Following problem that we discuss our model on boundary  $\partial\Omega$ .

### 1. Dirichlet Problem

The Dirichlet Problem, also called the first boundary value problem, asks for a solution  $u$  in  $\Omega$  which takes on prescribed values

$$u = f$$

at the boundary  $\partial\Omega$  of that region.

### 2. Neumann Problem

The Second boundary value problem, more often referred to as the Neumann problem, consisting of the determination in some region  $\Omega$  of a solution  $u$  that possesses the prescribed normal derivatives

$$\frac{\partial u}{\partial n} = f$$

on the surface  $\partial\Omega$  bounding  $\Omega$ . To be specific, we take  $n$  in this context to stand for the unit normally directed into the interior of  $\Omega$ , and we assume, of course, that  $\partial\Omega$  is smooth enough to have a meaningful normal.

### 3. Robin Problem

Finally, Robin problem that imposes given values

$$\frac{\partial A}{\partial n} + \alpha A = f$$

on a linear combination of  $A$  and  $\frac{\partial A}{\partial n}$  along  $\partial\Omega$ , rather than on either of them separately, defines what is known as the third boundary value problem. We shall see that the latter question is well posed only for suitably restricted choices of the coefficient  $\alpha$ , for example, for  $\alpha \geq 0$ .

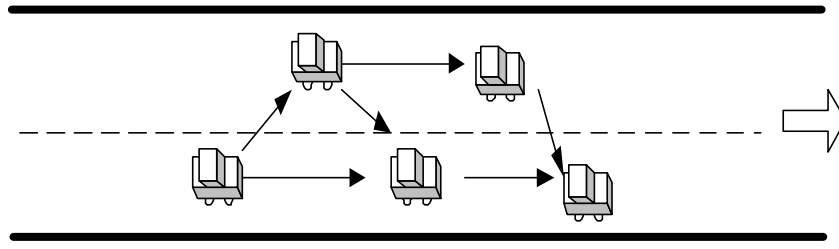


Figure 3.7: Interaction generated between places by the activity.

## 3.6 Potential Theory

Sheppard[[40]] introduces potential theory to describe the influence between a set of points in infinite space, given some models of interaction between them, as shown in Figure 3.7. Influence is defined as the potential exerted by one place on another. Potential on occasion has also been described as a measure of influence. Geographers also originally drew from the mathematics of potential theory.

Assume that activities take place at a series of points in an unbounded space. Let the interaction generated between places by this activity be defined as  $I_{ij}$ . Then a vector  $I_{ij}$ , may be constructed representing the size and orientation of interaction between any two places in this space. Define :

$$I_{ij} = \frac{r_{ij}}{d_{ij}} \quad (3.9)$$

where  $r_{ij}$  is the vector of direction from  $i$  to  $j$ ; and  $r_{ij}$  has a magnitude of  $d_{ij}$ .

The mathematics of potential theory tells us that the spatial gradient of the potential or influence function between two points is given by the interaction function.



Defining  $u_{ij}$  as the potential function between  $i$  and  $j$ , then:

$$5u_{ij} = \frac{\partial}{\partial x} u_{ij} p + \frac{\partial}{\partial y} u_{ij} q = \oint I_{ij} \quad (3.10)$$

Therefore:

$$u_{ij} = \oint^Z I_{ij} dr \quad (3.11)$$

where  $u_{ij}$  is a scalar and  $p; q$  are unit vectors.

Equation (3.11) represents the general derivation for potential, and demonstrates that in order to calculate the potential, a theory of spatial interaction is necessary to specify  $I_{ij}$ ;  $u_{ij}$  may be interpreted as the influence of  $i$  on  $j$  generated by spatial interaction.

In general, Potential also was used in electric field[[22]]. The electric field around a charged rod can be described not only by a (vector) electric field strength  $E$  but also by a scalar quantity, the electric potential  $V$ . These quantities are intimately related, and often which is used is only a matter of convenience in a given problem.

$$E = - \frac{dV}{dl} \quad (3.12)$$

This equation says: If we travel through an electric field along a straight line and measure  $V$  as we go, the rate of change in  $V$  with distance that we observe, when changed in sign, is the component of  $E$  in that direction.

Potential was also be defined in mathematics[[14]]. If  $A$  and  $B$  are two points in an open region  $\mathcal{R}$  in space, the work  $\int_A^B \mathbf{F} \cdot d\mathbf{r}$  done in moving a particle from  $A$  to  $B$  by a field  $\mathbf{F}$  defined on  $\mathcal{R}$  usually depends on the path taken. Let  $\mathbf{F}$  be a field defined on an open region  $\mathcal{R}$  in space, and suppose that for any two points  $A$  and  $B$  in  $\mathcal{R}$  the

work  $\int_A^B \mathbf{F} \cdot d\mathbf{r}$  done in moving from A to B is the same over all paths from A to B. Then the integral  $\int \mathbf{F} \cdot d\mathbf{r}$  is path independent in  $\mathcal{R}$  and the field  $\mathbf{F}$  is conservative on  $\mathcal{R}$ . So it can be defined as follow.

**Definition 4** If  $\mathbf{F}$  is a field defined on  $\mathcal{R}$  and  $\mathbf{F} = \nabla f$  for some scalar function  $f$  on  $\mathcal{R}$ , then  $f$  is called a potential function for  $\mathbf{F}$ .

Given the vector-valued function  $\mathbf{F}$ , the task of finding a function  $f$  for which  $\mathbf{F}(x; y) = \nabla f(x; y)$  is therefore referred to as finding a potential for  $\mathbf{F}$ : This is also what we mean by "reconstructing the function  $f$  from its gradient".

## Chapter 4

# Tra $\pm$ c Dispersion Model

## Formulation

In this study, we adapted vector field methodology to analyze the tra $\pm$  c flow property, and establish a model for tra $\pm$  c flow motion behavior using divergence theorem. According to the vehicle number conservation law, we derive the flow continuity equation and the gravity function is applied to study the interaction between individual and group motions. The proposed model is also suggested as a way to master the local interaction relationships. Furthermore, using the potential theory studied by Sheppard[[40]], an novel tra $\pm$  c dispersion model is presented.

### 4.1 Fundamental Assumptions

First of all, the divergence for tra $\pm$  c flow is defined. Suppose  $q(x; y; c)$  is a vector function of tra $\pm$  c flow that has three components  $q_x; q_y; q_c$  of  $q$  which are functions

of a coordinate system  $x; y; c$ , namely:

$$\mathbf{q}(x; y; c) = i q_x(x; y; c) + j q_y(x; y; c) + k q_c(x; y; c) \quad (4.1)$$

where  $x; y$  are spatial dependent variables,  $c$  is a capacity function. In general, the capacity is a maximum flow rate, we note that the function  $c$  is not only a design capacity, but also it is an operational capacity in the model. Where  $i, j$ , and  $k$  are unit vectors.

According to chapter 3, If a vector operator  $\nabla$  is defined as

$$\nabla = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial c} \quad (4.2)$$

We can form two useful combinations of  $\nabla$  and  $\mathbf{q}$ . We define the divergence of  $\mathbf{q}$ , abbreviated  $\text{div } \mathbf{q}$  or  $\nabla \cdot \mathbf{q}$ , by

$$\nabla \cdot \mathbf{q} = \text{div } \mathbf{q} = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_c}{\partial c} \quad (4.3)$$

The term  $\nabla \cdot \mathbf{q}$  is a scalar and can be used if the result of the vector operator on the trans flow has just magnitude.

We define the curl of  $\mathbf{q}$ , written  $\nabla \times \mathbf{q}$ , by

$$\begin{aligned} \nabla \times \mathbf{q} &= \text{curl } \mathbf{q} \\ &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial c} \\ q_x & q_y & q_c \end{vmatrix} \\ &= \frac{\partial q_c}{\partial y} i - \frac{\partial q_y}{\partial c} i + \frac{\partial q_x}{\partial c} j - \frac{\partial q_c}{\partial x} j + \frac{\partial q_y}{\partial x} k - \frac{\partial q_x}{\partial y} k \end{aligned} \quad (4.4)$$

If  $\mathbf{q}$  is a trans flow then  $\nabla \times \mathbf{q}$  is a trans flow vector, where the magnitude is  $|\nabla \times \mathbf{q}|$  and direction is given by  $\nabla \times \mathbf{q}$ .

The divergence of  $q$  is related to the variation in outward traffic flow. Hence, for a congestion zone the traffic flow variation is indeed a divergence problem.

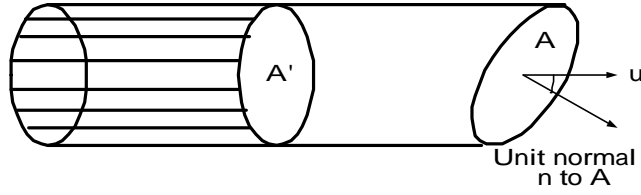


Figure 4.1:

## 4.2 Tra± c Continuity Equation in 3D Space

In this section, the tra± c continuity and its congestion will be discussed. When a moving tra± c °ow region is considered, for each point, a vector  $u$  could be drawn equal to the velocity of the tra± c °ow at that point. The vector function  $u$  describes a vector field. The divergence of  $u$  is related to the variation in tra± c which °ows out of a given capacity volume. This could be different from zero because a change in density or there is a source or sink in tra± c volume.

If relation  $q=ku$  holds where  $k$  is the tra± c density. During time  $t$  the amount of tra± c crossing an area  $A^0$  which is perpendicular to the direction of °ow, as shown in Figure 4.1, is the amount of tra± c in a cylinder of cross section  $A^0$  and length  $ut$ . This amount of tra± c is

$$(ut) A^0 (k) \quad (4.5)$$

Further along the tra± c °ow at a cross area  $A$  as shown in Figure 4.1 whose normal is inclined at angle  $\mu$  to  $u$ . since  $A^0 = A \cos \mu$ ,

$$utA^0 k = utkA \cos \mu \quad (4.6)$$

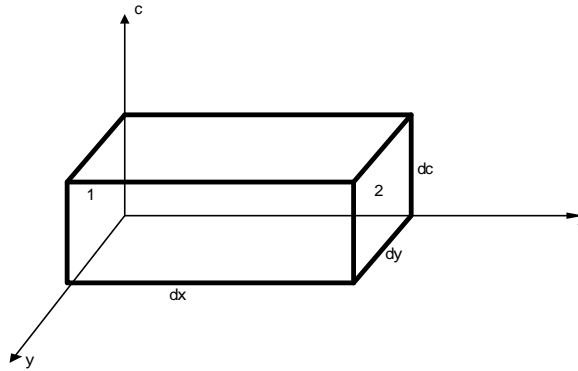


Figure 4.2:

When  $\text{tra}_{\pm} c$  is moving in the direction  $u$  making an angle  $\mu$  with the normal  $n$  to the surface, then the carrying capacity or the amount of  $\text{tra}_{\pm} c$  crossing per unit area of the transport surface in unit time is

$$u \cos \mu = q \cos \mu = q \cdot n \quad (4.7)$$

where  $n$  is a unit vector.

Now consider an element of volume  $dx dy dz$  in the region through which  $\text{tra}_{\pm} c$  is moving as shown in Figure 4.2.  $\text{Tra}_{\pm} c$  motion is either in or out of the volume  $dx dy dz$  through each of the six surfaces of the volume element. We shall calculate the net outward flow. In Figure 4.2, the rate at which  $\text{tra}_{\pm} c$  flows into  $dx dy dz$  through surface 1 is, by (4.7),  $q \cdot n$  per unit area, or  $(q \cdot n) dy dz$  through the area  $dy dz$  of surface 1. Since  $q \cdot n = q_x$ , we find that the rate at which  $\text{tra}_{\pm} c$  flows across surface 1 is  $q_x dy dz$ . A similar expression gives the rate at which  $\text{tra}_{\pm} c$  flows out through surface 2, except that  $q_x$  must be the  $x$  component of  $q$  at surface 2 instead of at surface 1. We find the difference in the two  $q_x$  values at two points, one on surface 1 and one

on surface 2, directly opposite each other, that is, for the same  $y$  and  $c$ . These two values of  $q_x$  differ by  $\delta q_x$  which can be approximated by  $dq_x$ . For constant  $y$  and  $c$ ,  $dq_x = \left(\frac{\partial q_x}{\partial x}\right)dx$ . Then the net outflow through these two surface is the outflow through surface 2 minus the inflow through surface 1 can be written

$$[(q_x)_2 - (q_x)_1] dydc = \frac{\partial q_x}{\partial x} dx dydc; \quad (4.8)$$

and for the other spatial surface

$$[(q_y)_2 - (q_y)_1] dxdc = \frac{\partial q_y}{\partial y} dy dxdc; \quad (4.9)$$

Likewise, the net change in carrying capacity can be described as

$$[(q_c)_2 - (q_c)_1] dxdy = \frac{\partial q_c}{\partial c} dc dxdy; \quad (4.10)$$

Therefore, the total net rate of loss of traffic and capacity from  $dxdydc$  is

$$\begin{aligned} \left[ \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_c}{\partial c} \right] dxdydc &= \text{div } q \, dxdydc \\ &= -\delta q \, dxdydc \end{aligned} \quad (4.11)$$

which reduces to (4.3).

The divergence of  $q$  may be different from zero because of variable density or because of sources and sinks in traffic flow. If we divide (4.11) by  $dxdydc$ , we have the rate of loss of traffic per unit volume. Denote following notations,  $\tilde{A}$  is source density minus sink density, (i.e. net mass of flow being created per unit time per unit volume);  $k$  is density of traffic (i.e. mass per unit volume);  $\frac{\partial k}{\partial t}$  is time rate of increase of traffic mass per unit volume.



Then the rate of increase of traffic mass in  $dx dy dz$  equals rate of creation minus the rate of outward flow. This relationship can be expressed as

$$\begin{aligned}\frac{\partial k}{\partial t} dx dy dz &= \tilde{A} dx dy dz + (\sum \phi_q dx dy dz) \\ &= \tilde{A} dx dy dz - \sum \phi_q dx dy dz\end{aligned}\quad (4.12)$$

Integrate (4.12) with respect to  $dx dy dz$ , we obtain

$$\sum \phi_q = \tilde{A} - \frac{\partial k}{\partial t} \quad (4.13)$$

If there are no sources or sinks then

$$\frac{\partial k}{\partial t} + \sum \phi_q = 0 \quad (4.14)$$

The equation (4.14) is traffic continuity equation and it defines traffic flow equilibrium.

If the carrying capacity is reduced to a point where it cannot carry any further traffic, then it can be considered incompressible and so the traffic density  $k$  will be constant.

In this case  $\frac{\partial k}{\partial t} = 0$  and equation (4.13) reduces to

$$\sum \phi_q = \tilde{A} \quad (4.15)$$

Equation (4.15) describes maximum traffic congestion. This will occur when the net mass of traffic added to the system is equal to the rate of outflow per unit volume at a point.

### 4.3 Main Theorem for Tra± c Dispersion Model

According to section 3.2, Sheppard[[40]] discussed Potential Theory that describes the notion of influence between a set of points given a model of interaction between motorists. In this problem when the interaction between two points in tra± c space produces some degree of restriction of movement then this function is termed velocity potential. It describes the inability of the motorist to move along a transport path.

If a tra± c vector  $q = q \cdot r$  then

$$\begin{aligned} q &= q \cdot (x \hat{x} + y \hat{y} + c \hat{c}) \\ &= q(x \hat{x} + y \hat{y} + c \hat{c}) \end{aligned}$$

or

$$q = q^x \hat{x} + q^y \hat{y} + q^c \hat{c} : \quad (4.16)$$

Equation (4.16) can be expressed as a gravity function

$$q = q \cdot \frac{r}{d} : \quad (4.17)$$

Then

$$q = q \cdot \left( \frac{x}{x^2 + y^2 + c^2} \hat{x} + \frac{y}{x^2 + y^2 + c^2} \hat{y} + \frac{c}{x^2 + y^2 + c^2} \hat{c} \right), \quad (4.18)$$

where  $r = x \hat{x} + y \hat{y} + c \hat{c}$ ,  $d = \sqrt{x^2 + y^2 + c^2}$ , and  $\hat{x}, \hat{y}, \hat{c}$  are orthogonal components of this vector.

**Theorem 5** The differentiable vector field  $q$  is the gradient of a function  $\phi$  if and only if  $\text{curl } q = 0$ . Then  $q$  is conservative vector field and exists  $\phi$  which is potential function such that  $q = -\nabla \phi$  [[20]].

Theorem 6 curl  $q$  equals zero, that is  $q$  in equation (4.18) is conservative vector field.

Proof.  $\text{curl } q = 0$  :

$$\begin{aligned}
 \text{curl } q &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial c} \\ q_x & q_y & q_c \end{vmatrix} \\
 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial c} \\ -\frac{x}{x^2+y^2+c^2} & -\frac{y}{x^2+y^2+c^2} & -\frac{c}{x^2+y^2+c^2} \end{vmatrix} \\
 &= 0:
 \end{aligned} \tag{4.19}$$

Curl  $q = 0$  by theorem 3, then we have checked the  $q$  is conservative vector field. The theorem 2 tells us that a potential function  $\hat{A}$  exists. According to the mathematics of potential theory indicating the spatial gradient of the potential or the influence function between two points is given by the interaction function. Defining  $\hat{A}$  as the potential function in tra±c vector field, and is described as:

$$\nabla \hat{A} = \frac{\partial \hat{A}}{\partial x} \hat{x} + \frac{\partial \hat{A}}{\partial y} \hat{y} + \frac{\partial \hat{A}}{\partial c} \hat{c}; \tag{4.20}$$

then

$$\hat{A} = \int^z q \, dr$$

or

$$\hat{A} = \frac{q}{d} \int^z r \, dr; \tag{4.21}$$

We form the integral of the equation (4.21), then obtained

$$q = \nabla \hat{A}; \tag{4.22}$$

## 4.4 Relations and Physical Meaning for New Model and Tra $\pm$ c Flow

If  $\hat{A}$  is a velocity potential function and we know equation (4.15) describing maximum tra $\pm$  c congestion. So the equation of maximum tra $\pm$  c congestion can be written as

$$\nabla^2 \hat{A} = \tilde{A} \quad (4.23)$$

We write  $\tilde{A}$  as density of tra $\pm$  c k, then we can obtain

$$\frac{\partial^2 \hat{A}}{\partial x^2} + \frac{\partial^2 \hat{A}}{\partial y^2} + \frac{\partial^2 \hat{A}}{\partial c^2} = k$$

or

$$\nabla^2 \hat{A} = \phi \hat{A} = k \quad (4.24)$$

In this paper, the tra $\pm$  c dispersion equation, (4.24), is so called a poisson equation.

## Chapter 5

# A Well-Posed Analysis of the Proposed Model

The proposed model is an elliptic partial differential equation. Before solving the model, we must prove the model to have a solution and this solution is unique. In this chapter we introduce some theory to show exist and uniqueness for transverse dispersion equation(4.24).

As we discussed in the above section, the basic mathematical problem is to solve Poisson's equation in a given domain  $\Omega$  with a condition on boundary  $\partial\Omega$  - [[41]]:

$$\begin{aligned} \Delta u &= k \quad \text{in } \Omega \\ u &= f \quad \text{or} \quad \frac{\partial u}{\partial n} = f \quad \text{or} \quad \frac{\partial u}{\partial n} + \alpha u = f \quad \text{on } \partial\Omega \end{aligned} \quad (5.1)$$

According to, we know that a partial differential equation must be a well-posed problem if it has a solution. A well-posed problem is one for which a solution not only exists, but in addition, that solution is unique and depends continuously on the data.

In this section we prove for some of these problems that the solution is also unique and depends continuously on the data. This will prove that these problems are well-posed problems. The proofs of uniqueness and continuous dependence will be based on one or another technique that is discussed in the appendix.

## 5.1 Existence and Uniqueness for the Tra± c Dispersion Model with Dirichlet Boundary-Value Problem

We begin with a study of one of the simplest boundary value problems for second-order partial differential equations. On a bounded, three-dimensional domain  $\Omega$  with boundary  $\partial\Omega$ , we pose the problem of determining a function  $\hat{A}(x; y; c)$  which is twice differentiable in  $\Omega$ , is continuous on  $\bar{\Omega} = \Omega \cup \partial\Omega$ , and satisfies the equation

$$\nabla^2 \hat{A} = \frac{\partial^2 \hat{A}}{\partial x^2} + \frac{\partial^2 \hat{A}}{\partial y^2} + \frac{\partial^2 \hat{A}}{\partial c^2} = k(x; y; c) \quad \text{in } \Omega \quad (5.2)$$

and the boundary condition

$$\hat{A} = f(s) \quad \text{on } \partial\Omega : \quad (5.3)$$

Equation(5.2) is known as the tra± c dispersion model also called a Poisson Equation. The function  $k$  is prescribed throughout  $\Omega$ , and the function  $f$ , given in terms of the arc length  $s$ , is prescribed along  $\partial\Omega$ .

The problem of determining such a solution  $\hat{A}$  is known as the Dirichlet Problem. We will not investigate the conditions on  $\Omega$  and those on  $k$  and  $f$  which are

sufficient to guarantee the existence of a solution for the Dirichlet problem.. However by means of the maximum principle[[35]] alone, it is possible to show that if a solution for the first boundary value does exist, then it must be unique. That is, we prove that there can be at most one solution for the problem.

To establish this result, we suppose that  $\hat{A}_1$  and  $\hat{A}_2$  are two functions which satisfy (5.2) and (5.3) with the same  $k$  and  $f$ . Following

$$\begin{aligned} \nabla \hat{A}_1 &= k & \text{in } \Omega & \quad \nabla \hat{A}_2 = k & \text{in } \Omega \\ \hat{A}_1 &= f & \text{on } \partial\Omega & \quad \hat{A}_2 = f & \text{on } \partial\Omega \end{aligned} \quad (5.4)$$

and Defining

$$u = \hat{A}_1 - \hat{A}_2$$

we see that  $u$  satisfies

$$\begin{aligned} \nabla u &= 0 & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega \end{aligned}$$

According to the maximum principle(Appendix A.1),  $u$  cannot have a maximum in the interior of  $\Omega$ . However, the maximum of a continuous function on a closed bounded set must be attained. Since  $u$  is continuous on  $\bar{\Omega} = \Omega \cup \partial\Omega$  and since  $u = 0$  on  $\partial\Omega$ , we conclude that  $u \leq 0$  in  $\Omega$ . Applying the same reasoning to  $-u$ , we obtain  $u \geq 0$  in  $\Omega$ . Hence

$$u = \hat{A}_1 - \hat{A}_2 \leq 0 \quad \text{in } \Omega;$$

And by the maximum principle[[35]],

$$0 = u(x_m) \cdot u(x) \cdot u(x_M) = 0 \quad \text{for all } x \in \Omega :$$

Therefore, both the maximum and minimum of  $u(x)$  are zero. This also means  $u \geq 0$  and  $\hat{A}_1 = \hat{A}_2$ :

In other words, by the maximum principle, the Dirichlet Problem is one for which a solution not only exists, but in addition, the solution is unique and depends continuously on the data. We can obtain the following theorem:

#### Theorem 7 Dirichlet Problem

$$\Delta \hat{A}(x; y; c) = k(x; y; c) \quad \text{in } \Omega \quad (5.5)$$

$$\hat{A} = f \quad \text{on } \partial \Omega$$

This problem has one and only one solution and that solution depends continuously on the data  $k$  and  $f$ .



## 5.2 Existence and Uniqueness for the Transc Dispersion Model with Neumann Boundary-Value Problem

The other boundary value problems for second-order partial differential equations will now be proven. On a bounded, three-dimensional domain  $\Omega$  with boundary  $\partial\Omega$ , we pose the problem of determining a function  $\hat{A}(x; y; c)$  which is twice differentiable in  $\Omega$ , is continuous on  $\bar{\Omega} = \Omega \cup \partial\Omega$ , and satisfies the equation

$$\nabla^2 \hat{A} = \frac{\partial^2 \hat{A}}{\partial x^2} + \frac{\partial^2 \hat{A}}{\partial y^2} + \frac{\partial^2 \hat{A}}{\partial c^2} = k(x; y; c) \quad \text{in } \Omega \quad (5.6)$$

and the boundary condition

$$\frac{\partial \hat{A}}{\partial n} = f(s) \quad \text{on } \partial\Omega : \quad (5.7)$$

Equation(5.2) is known as the transc dispersion model that is also called a Poisson Equation. The function  $k$  is prescribed throughout  $\Omega$ , and the function  $f$ , given in terms of the arc length  $s$ , is prescribed along  $\partial\Omega$ . The problem of determining such a solution  $\hat{A}$  is known as the Neumann Problem.

This problem has no solution unless the data functions  $k$  and  $f$  satisfy the compatibility condition[[13]],

$$\int_{\Omega} k d\Omega = \int_{\partial\Omega} f dS: \quad (5.8)$$

If we apply the divergence theorem (see appendix A.2) in the special case that

$$f = \hat{A} \text{grad } v$$

for smooth, scalar-valued functions  $u$  and  $v$ , then

$$\operatorname{div}[\hat{A} \operatorname{grad} v] = \hat{A} \Delta v + \operatorname{grad} \hat{A} \cdot \operatorname{grad} v \quad (5.9)$$

This identity (5.9) follows from many applications of the usual product rule (5.10) for derivatives.

$$(\hat{A} v_x)_x = \hat{A}_x v_x + \hat{A} v_{xx} \quad (5.10)$$

Then (A.3) becomes

$$\iint_{\partial \Omega} u \frac{\partial v}{\partial n} dS = \iint_{\Omega} u \Delta v + \iint_{\Omega} \operatorname{grad} \hat{A} \cdot \operatorname{grad} v d\Omega \quad (5.11)$$

where  $\frac{\partial v}{\partial n} = \nabla v \cdot \mathbf{n}$  = outward normal derivative

= directional derivative of  $v$  in direction of  $\mathbf{n}$

The identity (5.11) is known as Green's first identity[[13]].

We apply Green's first identity with  $\hat{A} = 1$  and  $v = \hat{A}$ . Then  $\operatorname{grad} \hat{A} = \nabla \hat{A}$  is zero, and

$$\iint_{\Omega} \Delta \hat{A} d\Omega = \iint_{\partial \Omega} \frac{\partial \hat{A}}{\partial n} dS \quad (5.12)$$

According to the result from (5.12), if  $u$  satisfies (5.6), (5.7), and (5.8), the solution exists and is also unique.

Note that when the compatibility condition is satisfied, the solution exists but is unique only up to an additive constant.

#### Theorem 8 Neumann Problem

$$\begin{aligned} \Delta \hat{A}(x; y; c) &= k(x; y; c) \quad \text{in } \Omega \\ \frac{\partial \hat{A}}{\partial n} &= f \quad \text{on } \partial \Omega \end{aligned} \quad (5.13)$$

Here we use the notation  $\frac{\partial A}{\partial n} = 5A \phi n$ , where  $n$  denotes the unit outward normal to  $\partial\Omega$ . This problem has a solution. This solution is unique up to an additive constant and depends continuously on the data  $k$  and  $f$ .

### 5.3 Existence and Uniqueness for the Tra± c Dispersion Model with Robin Boundary-Value Problem

Finally, we prove the third boundary value problems for second-order partial differential equations. On a bounded, three-dimensional domain  $\Omega$  with boundary  $\partial\Omega$ , we pose the problem of determining a function  $\hat{A}(x; y; c)$  which is twice differentiable in  $\Omega$ , is continuous on  $\bar{\Omega} = \Omega \cup \partial\Omega$ , and satisfies the equation

$$\Delta \hat{A} = \frac{\partial^2 \hat{A}}{\partial x^2} + \frac{\partial^2 \hat{A}}{\partial y^2} + \frac{\partial^2 \hat{A}}{\partial c^2} = k(x; y; c) = P\hat{A} \quad \text{in } \Omega \quad (5.14)$$

and the boundary condition

$$\frac{\partial \hat{A}}{\partial n} + \alpha \hat{A} = f(s) \quad \text{on } \partial\Omega : \quad (5.15)$$

Equation (5.14) is known as the tra± c dispersion model that is also called a Poisson Equation. The function  $k = P\hat{A}$  is prescribed throughout  $\Omega$ , and the function  $f$ , given in terms of the arc length  $s$ , is prescribed along  $\partial\Omega$ . The problem of determining such a solution  $\hat{A}$  is known as the Robin Problem.

Let  $u, v$  be two solutions of (5.14) with Robin's condition (5.15). Let  $\hat{A} = u - v$ . Then  $\hat{A}$  satisfies

$$\begin{aligned} \Delta \hat{A}(x; y; c) &= P\hat{A} \quad \text{in } \Omega \\ \frac{\partial \hat{A}}{\partial n} + \alpha \hat{A} &= 0 \quad \text{on } \partial\Omega \end{aligned} \quad (5.16)$$

At appendix A.3, we discussed Green's identities[[13]]. In (A.11)

$$k\Delta k^2 + \int_{\partial^-} \Delta^2 dS = 0$$

So if  $\Delta > 0$ , then

$$k\Delta k^2 = 0 \text{ and } \Delta = 0 \text{ on } \partial^-$$

and we get

$$\Delta = 0 \text{ in } \Omega :$$

This proves the uniqueness  $u=v$  of the solution of the Robin Problem.

**Theorem 9 Robin Problem**

$$\Delta(x; y; c) = k(x; y; c) \text{ in } \Omega \tag{5.17}$$

$$\frac{\partial \Delta}{\partial n} + \Delta = f \text{ on } \partial^-$$

Here,  $\Delta(x; y; c)$  appearing in the boundary condition denote (continuous) functions defined on  $\partial^-$  and satisfying the condition  $\Delta > 0$  on  $\partial^-$ . Then this problem has a unique solution that depends continuously on the data.



## Chapter 6

# A derivation of General Solution Formula

This chapter is designed to derive some analytical solution formula for the proposed new model under some proper assumptions. The solution formula for Dirichlet Boundary-value problem is discussed in the first section. With the same approach on the solution formula for Neumann and Robin Boundary-value problem are also presented respectively.

$$\mathcal{L} u = P u \quad (6.1)$$

we define the Green's function  $G=G(x; y)$  of equation (6.1) with respect to the region  $\Omega$  to be a fundamental solution of (6.1) there which satisfies homogeneous boundary condition

$$G(x; y) = 0 \quad (6.2)$$

when the argument point  $x$  is located on  $\partial^-$ . Thus the Green's function has the form

$$G = S + u_S;$$

where  $u_S$  is a regular solution of 6.1 in  $\Omega$  that reduces to the particular fundamental solution  $S$  on the boundary  $\partial^-$ . The existence and uniqueness of such a term  $u_S$  are consequences of the solvability of Dirichlet's problem in  $\Omega$  for the linear elliptic equation 6.1, which is feasible when  $P \geq 0$ .

## 6.1 Dirichlet Boundary-Value Problem

The importance of the Green's function  $G$  with regard to Dirichlet's problem is that when we insert it into

$$u(x) = \int_{\partial^-} u(x) \frac{\partial S(x; y)}{\partial n} dy + \int_{\partial^-} S(x; y) \frac{\partial u(y)}{\partial n} dy \quad (6.3)$$

to play the role of  $S$ , the contribution from the normal derivative  $\frac{\partial u}{\partial n}$  disappears, by virtue of 6.2. Thus we obtain the specific representation

$$u(x) = \int_{\partial^-} u(y) \frac{\partial G(x; y)}{\partial n} dy \quad (6.4)$$

for the solution  $u$  of the Dirichlet problem as a definite integral of the prescribed boundary values, multiplied by the kernel  $\frac{\partial G}{\partial n}$ . Incidentally, since  $G \rightarrow +1$  as  $x \rightarrow \infty$ , and since the maximum principle shows that  $G$  cannot have a negative minimum in the interior of  $\Omega$ , its minimum must occur on the boundary  $\partial^-$ . Since 6.2 holds on  $\partial^-$ , we therefore have

$$G(x; y) > 0$$



inside  $\Omega$ . It follows that the kernel  $\frac{\partial G}{\partial n}$  featuring in the integral formula 6.4 is non-negative[[15]].

## 6.2 Neumann Boundary-Value Problem

For  $P > 0$  we can introduce the Neumann's function, or Green's function of the second kind, for equation 6.1 with respect to  $\Omega$ , which is a fundamental solution  $N = N(x; y)$  fulfilling the requirement

$$\frac{\partial N}{\partial n} = 0 \quad (6.5)$$

on  $\partial\Omega$ . It can be expressed as the difference

$$N = S - v_S$$

between an arbitrary fundamental solution  $S$  in  $\Omega$  and a regular function  $v_S$  which solves the Neumann problem for 6.1 defined by the boundary condition

$$\frac{\partial v_S}{\partial n} = \frac{\partial S}{\partial n}$$

Substitution of  $N$  for  $S$  in 6.3 eliminates the term involving boundary values of  $u$  and provides the integral representation

$$u(y) = \int_{\partial\Omega} N(x; y) \frac{\partial u(x)}{\partial n} d\Omega \quad (6.6)$$

for the solution of the general Neumann problem[[15]].

### 6.3 Robin Boundary-Value Problem

Introduce the Green's function of the third kind for equation 6.1 in any  $\Omega$  as a fundamental solution  $G_3 = G_3(x; y)$  there satisfying the mixed boundary condition

$$\frac{\partial G_3}{\partial n} + \alpha G_3 = 0$$

for  $x$  on  $\partial\Omega$ . It can be shown that a solution  $u$  for the Robin boundary value problem has the representation

$$u(y) = \int_{\partial\Omega} G_3(x; y) \frac{\partial u(x)}{\partial n} + \alpha u(x) \, d\Omega \quad (6.7)$$

in terms of this Green's function[[15]].



# Chapter 7

## Some Computed Model Problems and Related Comparisons

In this chapter, the analytical solution and finite difference numerical simulation approach are given, compared and discussed in detail. Section 7.1 demonstrates a traffic dispersion model without any source density in the lane. In this section we compute the problem using the analytical solution technique first and the results are compared with the numerically computed result. This shows a good consistence between these two approaches. In section 7.2, we extend the solution procedures to study a model problem includes source density in the lane. In section 7.3, a complete 3D problem are also discussed. Last section, we study a physical based model problem with Gaussian type source density to describe some traffic jam phenomena.

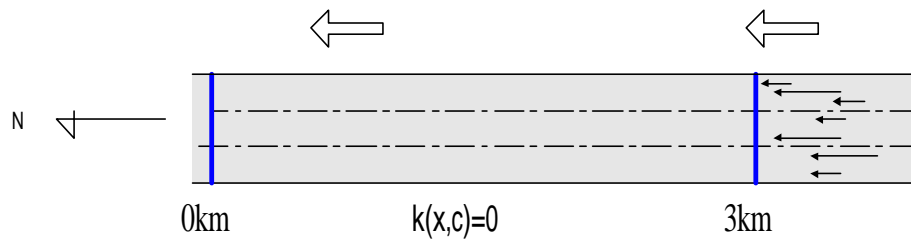


Figure 7.1: Traffic model without source density

## 7.1 A 2D-Traffic Model without Source Density

In this section we compute the problem using the analytical solution technique first and the results are compared with the numerically computed result. Consider a long section of roadway shown in Figure 7.1. We consider only flowing traffic on boundary, but with no source density in this region.

This problem can then be formulated as a traffic dispersion model. In order to find the analytical solution, we simplify this region into a unit square  $(0; 3) \times (0; 3)$  and consider the problem of finding an unknown function  $\hat{A} = \hat{A}(x; c)$  which satisfies the following model

$$\nabla \cdot \hat{A}(x; c) = 0 \quad \text{in } \Omega; \quad (7.1)$$

$$\hat{A}(0; c) = 0; \quad \hat{A}(3; c) = \sin \frac{1}{3}x \quad 0 \leq c \leq 3;$$

$$\hat{A}(x; 0) = 0; \quad \hat{A}(x; 3) = 0 \quad 0 \leq x \leq 3;$$

To find the unknown function  $\hat{A}(x; c)$ , we assume that

$$\hat{A}(x; c) = X(x)C(c); \quad (7.2)$$

From (7.1), we have the function  $C(c)$

$$\begin{aligned} C''(c) &= -C(c); \quad 0 < c < 1 \\ C(0) &= C(3) = 0; \end{aligned}$$

By fundamental methodology in differential equations [[20]], the eigenvalues and eigenfunctions of this problem are given by

$$\lambda_n = \frac{n^2\pi^2}{9}; \quad n = 1; 2; \dots$$

and

$$C_n(c) = \sin \frac{n\pi}{3}c; \quad n = 1; 2; \dots \quad (7.3)$$

The function  $X(x)$  has to satisfy

$$X''(x) + \lambda_n X(x) = 0 \quad 0 < x < 1;$$

Then

$$X(x) = A_n \cosh \frac{n\pi}{3}x + B_n \sinh \frac{n\pi}{3}x; \quad (7.4)$$

Then combining the  $X$  and  $C$  solutions,

$$\hat{A}_n(x; c) = \sum_{n=1}^{\infty} \left[ A_n \cosh \frac{n\pi}{3}x + B_n \sinh \frac{n\pi}{3}x \right] \sin \frac{n\pi}{3}c;$$

$\hat{A}_n(x; c)$  satisfies the equation (7.29) and all of the homogeneous boundary conditions of the problem we are trying to solve.

$$\hat{A}_n(0; c) = \sum_{n=1}^{\infty} [A_n + 0] \sin \frac{n\pi}{3}c = 0;$$

Then  $A_n = 0$ :

$$\begin{aligned}\hat{A}_n(3; c) &= \sum_{n=1}^{\infty} B_n \sinh \frac{n\sqrt{4}}{3} x \sin \frac{n\sqrt{4}}{3} c \\ &= \sin \frac{\sqrt{4}}{3} c\end{aligned}$$

$$\begin{aligned}B_n \sin n\sqrt{4} &= \frac{2}{3} \int_0^3 \sin \frac{n\sqrt{4}}{3} c \sin \frac{\sqrt{4}}{3} c \, dc \\ &= 0\end{aligned}$$

$$B_1 \sinh \sqrt{4} = \frac{2}{3} \int_0^3 \sin^2 \frac{\sqrt{4}}{3} c \, dc = 1$$

$$B_1 = (\sinh \sqrt{4})^{-1} :$$

The solution is  $\hat{A}$  given by

$$\hat{A}(x; c) = \frac{\sinh \frac{\sqrt{4}}{3} x \sin \frac{\sqrt{4}}{3} c}{\sinh \sqrt{4}} : \quad (7.5)$$

In this problem, the analytical solution is shown in Figure 7.2. We simulated the result using numerical method as shown in Figure 7.3, where the numerical solution procedure is described in Figure 7.4. Comparing Figure 7.2 with 7.3, we find that there exists almost no difference between these two solution methods. Thus we can use the numerical method to solve more complicated model problems.



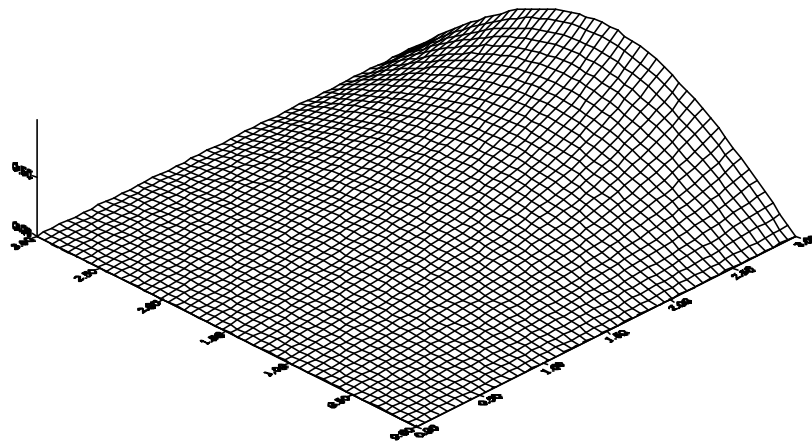


Figure 7.2: Analytic solution in 2D without source density

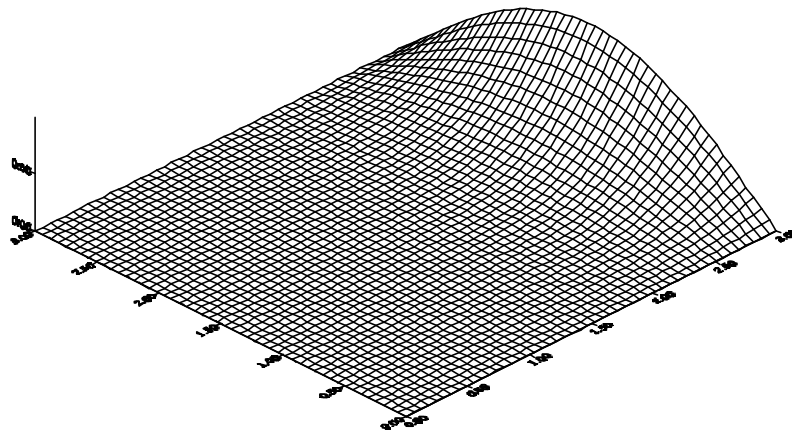


Figure 7.3: Numerical solution in 2D without source density

## 7.2 A 2D-Traffc Model with Source Density

In this section we compute the problem using the analytical solution technique first and the results are compared with the numerically computed result. Consider a long section of roadway, shown in figure 7.5. We considered only traffic flowing on the boundary, and with source density in this region.

This problem can be then transformed into another traffic dispersion model. In order to find the analytical solution, we simplify this region into a unit square  $(0; 1) \times (0; 1)$  and consider the problem of finding an unknown function  $\hat{A} = \hat{A}(x; c)$  which satisfies the following model

$$\nabla^2 \hat{A}(x; c) = k(x; c) \quad \text{in } \Omega; \quad (7.6)$$

$$\hat{A}(0; c) = f(c); \quad \hat{A}(1; c) = 0 \quad 0 \leq c \leq 1;$$

$$\hat{A}(x; 0) = \hat{A}(x; 1) = 0 \quad 0 \leq x \leq 1;$$

Here,  $k$ ,  $f$ , and  $g$  all denote given data functions that are at least square-integrable functions of their respective arguments. This is a "dirichlet problem" because specified condition data are imposed around the entire boundary of  $\Omega$  shown in figure 7.6:

As shown in figure 7.6, the potential on boundary  $\partial\Omega_1$  shows that the flow is given by  $5f$  and on boundary  $\partial\Omega_4$ ;  $\partial\Omega_2$ ; and  $\partial\Omega_3$  the flow variation is specified by zero respectively.

We shall split the original problem into two subproblems, one for each of the three inhomogeneous terms in the main problem. We proceed now to solve each of the two subproblems.

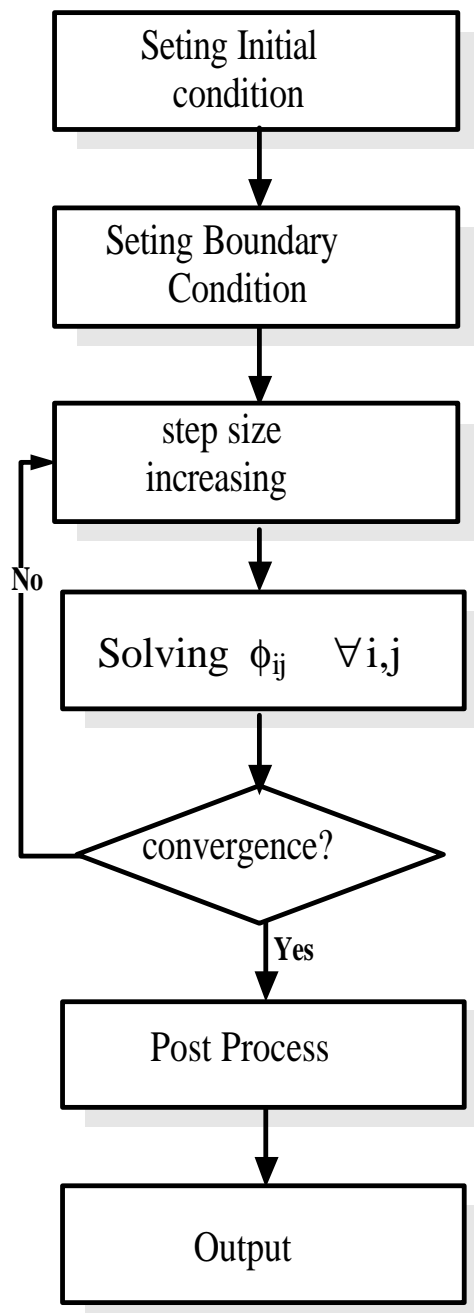


Figure 7.4: Finite Di®erence Method Process

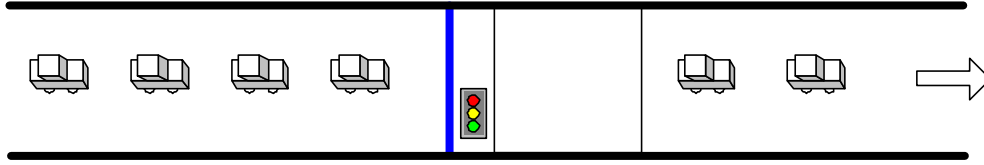


Figure 7.5: Tra $\pm$  c model with source density

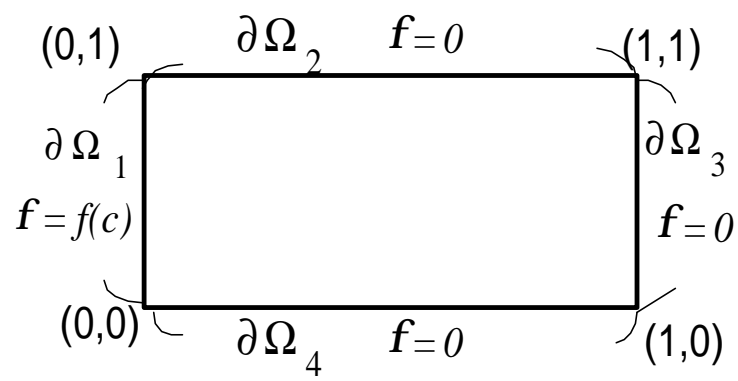


Figure 7.6: Dirichlet-type boundary conditions

## Subproblem A

$$\Delta \hat{A}^A(x; c) = 0 \quad \text{in } \Omega; \quad (7.7)$$

$$\hat{A}^A(0; c) = f(c); \quad \hat{A}^A(1; c) = 0 \quad 0 \leq c \leq 1;$$

$$\hat{A}^A(x; 0) = \hat{A}^A(x; 1) = 0 \quad 0 \leq x \leq 1;$$

To find the unknown function  $\hat{A}^A(x; c)$ , we assume that

$$\hat{A}^A(x; c) = X(x)C(c); \quad (7.8)$$

From (7.8), we have the function  $X(x)$

$$X''(x) = -\lambda_n X(x); \quad 0 < x < 1$$

$$X(1) = 0;$$

By fundamental methodology in differential equations[[39]], the eigenvalues and eigenfunctions of this problem are given by

$$\lambda_n = (n\pi)^2; \quad n = 1; 2; \dots$$

and

$$X_n(x) = B_n \sinh n\pi(1-x); \quad n = 1; 2; \dots \quad (7.9)$$

where  $B_n$  is arbitrary. For  $n = 0$  the  $X(x)$  problem has solution  $X_0(x) = B_0(1-x)$  for  $B_0$  arbitrary.

The function  $C(c)$  has to satisfy

$$C''(c) = -\lambda_n C(c) \quad 0 < c < 1;$$

$$C(0) = C(1) = 0;$$

The solution of C is expressed in the form

$$C_n(c) = \sin(n\pi c): \quad (7.10)$$

Then combining the X and C solutions,  $\hat{A}_n(x; c)$  satisfies the equation (7.29) and all of the homogeneous boundary conditions of the problem we are trying to solve.

Then by the principle of superposition, the solution is  $\hat{A}^A$  given by

$$\hat{A}^A(x; c) = B_0(1 - x) + \sum_{n=1}^{\infty} B_n \sinh n\pi(1 - x) \sin(n\pi c): \quad (7.11)$$

Assume that

$$f(c) = B_0 + \sum_{n=1}^{\infty} B_n \sinh n\pi \sin n\pi c; \quad 0 < c < 1; \quad (7.12)$$

Then we find

$$B_n = \frac{f_n}{\sinh(n\pi)}: \quad (7.13)$$

Using (7.13) into (7.11), we obtain the solution for problem (7.29).

subproblem C

$$\nabla^2 \hat{A}^C(x; c) = k(x; c) \quad \text{in } \Omega; \quad (7.14)$$

$$\hat{A}^C(0; c) = 0; \quad \hat{A}^C(1; c) = 0 \quad 0 < c < 1;$$

$$\hat{A}^C(x; 0) = \hat{A}^C(x; 1) = 0 \quad 0 < x < 1;$$

We can represent the solution in terms of the x eigenfunctions,

$$\hat{A}^C(x; c) = \sum_{n=1}^{\infty} V_n(c) \sin n\pi x:$$

We can also write the solution in terms of the c eigenfunctions,

$$\hat{A}^C(x; c) = \sum_{m=0}^{\infty} U_m(x) \sin m\pi c;$$

where the unknown functions  $V_n(c)$  and  $U_m(x)$  must be from the equation. we can choose to write the solution using both families of eigenfunctions,

$$\hat{A}^C(x; c) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \hat{A}_{mn} \sin n\frac{x}{4} \sin m\frac{c}{4}; \quad (7.15)$$

The function  $k(x; c)$  is represented as follows:

$$k(x; c) = \sum_{n=1}^{\infty} k_{mn} \sin n\frac{x}{4} \sin m\frac{c}{4}; \quad m = 0; 1; 2; \dots \quad n = 1; 2; \dots$$

We compute second partial differential to (7.15). Then substituting the expansions for  $\hat{A}^C(x; c)$  and  $k(x; c)$  into the subproblem C leads to

$$\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} k_{mn} + (m^2 + n^2)\frac{1}{4^2} \hat{A}_{mn} \sin n\frac{x}{4} \sin m\frac{c}{4} = 0;$$

where  $k_{mn} + (m^2 + n^2)\frac{1}{4^2} \hat{A}_{mn} = 0$ ; that is

$$\hat{A}_{mn} = \frac{i k_{mn}}{(m^2 + n^2)\frac{1}{4^2}};$$

Using this last result in (7.15), we have the so-called full eigenfunction expansion of the solution to subproblem C,

$$\hat{A}^C(x; c) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{i k_{mn}}{(m^2 + n^2)\frac{1}{4^2}} \sin n\frac{x}{4} \sin m\frac{c}{4}; \quad (7.16)$$

The sum of the solutions to the three subproblems is the solution to the original mixed-type model.

If the problem is

$$\begin{cases} \hat{A}(x; c) = 4xc & \text{in } -; \end{cases} \quad (7.17)$$

$$\hat{A}(0; c) = \sin \frac{c}{4}; \quad \hat{A}(1; c) = 0 \quad 0 \leq c \leq 1;$$

$$\hat{A}(x; 0) = \hat{A}(x; 1) = 0 \quad 0 \leq x \leq 1;$$

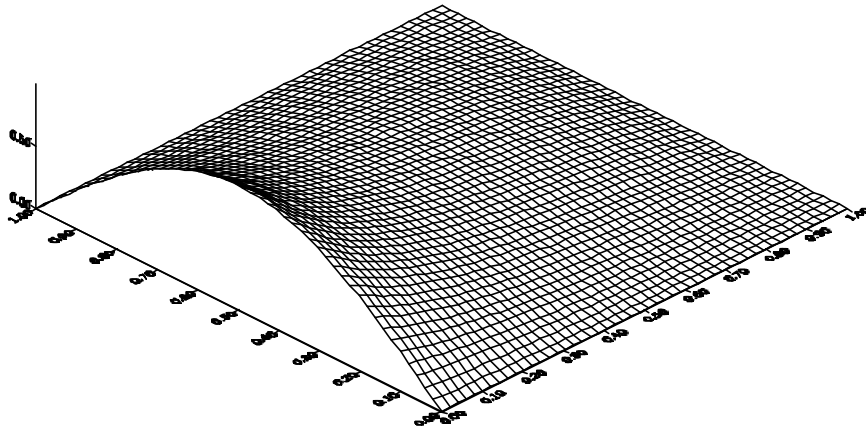


Figure 7.7: Analytic solution in 2D for the model with source density

Then the solution of this problem is

$$\hat{A}(x; c) = \sin \frac{1}{4}c \frac{\sinh(1 - x)\frac{1}{4}}{\sinh \frac{1}{4}} + \frac{i}{\frac{1}{4}^2} \sin \frac{1}{4}x \sin \frac{1}{4}c;$$

and the computation for the analytical solution is shown in Figure 7.7. We used the numerical method to compute this problem is shown in Figure 7.8.

Compare Figure 7.7 with Figure 7.8, we find that there exists almost no difference between these two solution methods. Thus we can use the numerical method to solve more complicated problems.



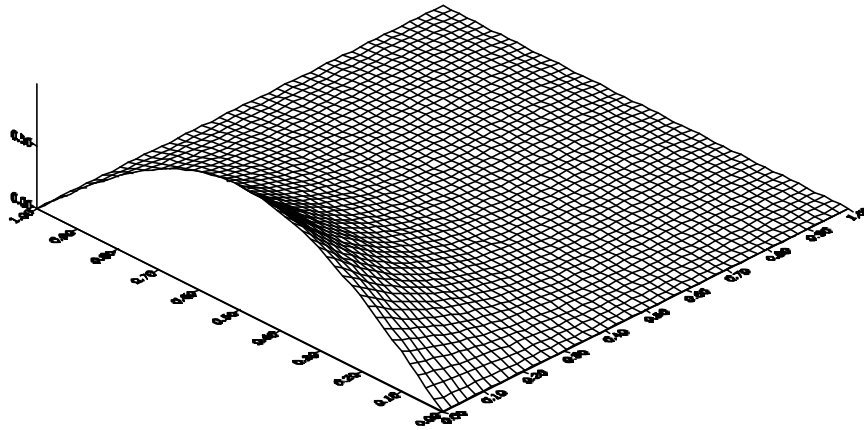


Figure 7.8: Numerical solution in 2D for the model with source density

### 7.3 A 3D-Tra $\pm$ c Model without Source Density

In this section, we also compute the problem using the analytical solution technique. Consider a section, there is heavy flowing tra $\pm$  c on the top boundary, but with no source density in this region.

$$\phi \hat{A}(x; y; z) = 0 \quad \text{in } -; \quad (7.18)$$

$$\hat{A}(0; y; z) = \hat{A}(a; y; z) = 0 \quad 0 \leq x \leq a;$$

$$\hat{A}(x; 0; z) = \hat{A}(x; b; z) = 0 \quad 0 \leq y \leq b;$$

$$\hat{A}(x; y; 0) = 0;$$

$$\hat{A}(x; y; c) = V_0 \quad 0 \leq z \leq c;$$

To find the unknown function  $\hat{A}(x; y; z)$ , we assume that

$$V_{mn}(x; y; z) = X(x)Y(y)Z(z); \quad (7.19)$$

$X(x)$ ;  $Y(y)$ ;  $Z(z)$  are all independent functions.

From (7.18), the boundary conditions were written

$$X(0) = X(a) = 0; \quad (7.20)$$

$$Y(0) = Y(b) = 0; \quad (7.21)$$

$$Z(0) = 0; \quad (7.22)$$

By fundamental methodology in differential equations[[20]], the eigenfunctions of this problem are given by

$$X(x) = A_1 \sin k_1 x + B_1 \cos k_1 x; \quad (7.23)$$

$$Y(y) = A_2 \sin k_2 y + B_2 \cos k_2 y;$$

$$Z(z) = A_3 \sinh k_3 z + B_3 \cosh k_3 z;$$

$$0 = k_1^2 + k_2^2 + k_3^2;$$

The term (7.23) are substituted for (7.20), (7.21), and (7.22), we can obtain

$$X(x) = A_1 \sin \frac{n\pi}{a} x; \quad n = 1; 2; 3; \dots$$

$$Y(y) = A_2 \sin \frac{m}{b} y; \quad m = 1; 2; 3; \dots$$

$$Z(z) = A_3 \sinh \sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2} z;$$

Then combining the  $X(x)$ ;  $Y(y)$  and  $Z(z)$  solutions,  $V_{nm}(x; y; z)$  satisfies

$$\begin{aligned} V_{nm}(x; y; z) &= X(x)Y(y)Z(z) \\ &= A_1 A_2 A_3 \sin \frac{n}{a} x \sin \frac{m}{b} y \cosh \sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2} z \\ &= a_{nm} \sin \frac{n}{a} x \sin \frac{m}{b} y \cosh \sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2} z; \\ a_{nm} &= A_1 A_2 A_3; \end{aligned}$$

By the principle of superposition, the solution is  $V(x; y; z)$  given by

$$V(x; y; z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} V_{nm}(x; y; z); \quad (7.24)$$

The homogeneous boundary conditions of the problem can be written

$$\begin{aligned} V_0 &= V(x; y; c) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} V_{nm}(x; y; z) \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm} \sin \frac{n}{a} x \sin \frac{m}{b} y \cosh \sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2} c; \end{aligned} \quad (7.25)$$

The term (7.25) multiplied by  $\sin \frac{n}{a} x \sin \frac{m}{b} y$ , then double integral

$$\begin{aligned} &\int_0^a \int_0^b V_0 \sin \frac{n}{a} x \sin \frac{m}{b} y dx dy \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \int_0^a \int_0^b a_{nm} \sin \frac{n}{a} x \sin \frac{m}{b} y \left( \sin \frac{n}{a} x \right) \left( \sin \frac{m}{b} y \right) \\ &\quad \cosh \sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2} c dx dy \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm} \int_0^a \sin^2 \frac{n}{a} x dx \int_0^b \sin^2 \frac{m}{b} y dy \\ &\quad \cosh \sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2} c; \end{aligned}$$

the left of term is equal to

$$\begin{aligned}
 & \int_0^a \int_0^b V_0 \sin \frac{n\pi}{a} x \sin \frac{m\pi}{b} y dx dy \\
 &= V_0 \int_0^a \sin \frac{n\pi}{a} x dx \int_0^b \sin \frac{m\pi}{b} y dy \\
 &= \begin{cases} V_0 \frac{4ab}{mn\pi^2} & n, m \text{ are even} \\ 0 & n, m \text{ are odd} \end{cases}
 \end{aligned}$$

the right of term is equal to

$$\begin{aligned}
 & \int_0^a \int_0^b \sin \frac{n\pi}{a} x \sin \frac{m\pi}{b} y dx dy \\
 &= \frac{1}{2} \int_0^a \left[ 1 - \cos \frac{2n\pi}{a} x \right] dx \cdot \frac{1}{2} \int_0^b \left[ 1 - \cos \frac{2m\pi}{b} y \right] dy \\
 &= \frac{1}{4} ab : \\
 & \frac{1}{4} ab a_{nm} \sinh \frac{\pi}{\left( \frac{n}{a} \right)^2 + \left( \frac{m}{b} \right)^2 c} = \begin{cases} V_0 \frac{4ab}{mn\pi^2} & n, m \text{ are even} \\ 0 & n, m \text{ are odd} \end{cases}
 \end{aligned}$$

then

$$a_{nm} = \frac{16V_0}{mn\pi^2 \sinh \frac{\pi}{\left( \frac{n}{a} \right)^2 + \left( \frac{m}{b} \right)^2 c}} \quad (7.26)$$

The term (7.26) is substituted for (7.24), we obtain the solution for problem (7.18)

$$V(x; y; z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{16V_0}{mn\pi^2 \sinh \frac{\pi}{\left( \frac{n}{a} \right)^2 + \left( \frac{m}{b} \right)^2 c}} \sin \frac{n\pi}{a} x \sin \frac{m\pi}{b} y \quad (7.27)$$

$$\sinh \frac{\pi}{\left( \frac{n}{a} \right)^2 + \left( \frac{m}{b} \right)^2 c} z \quad (7.28)$$

In this 3D problem, in order to solve the analytical solution the process is very complicated. We know that the numerical method is a good method to solve more complicated problems in section 7.1 and 7.2. The numerical solution can approximate the analytical solution. It is almost impossible to solve a 3D problem analytically, so a 3D numerical program should be developed for other 3D problems.

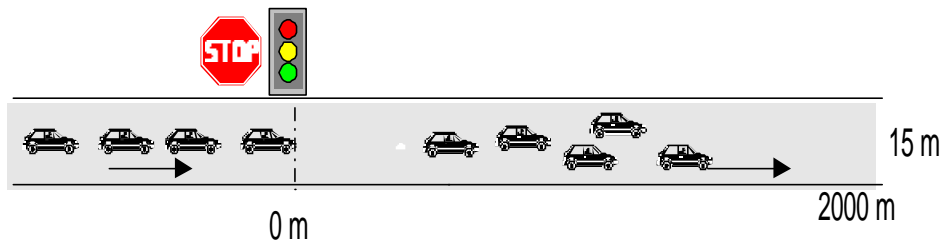


Figure 7.9: Traffic model with source density in 3D problem

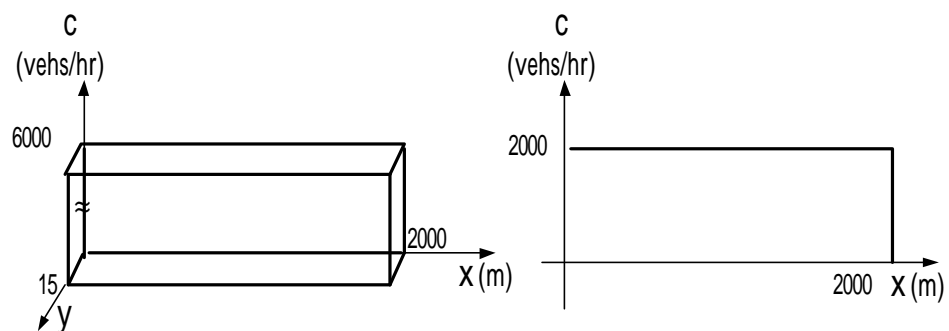


Figure 7.10: Three-dimensions model problem is reduced to two-dimensions model problem.

## 7.4 A Physical Based Traffic Model Problem with Source Density

Consider a long section of roadway, as shown in Figure 7.9, if the roadway length is longer than lane width, then our problem is reduced to a two-dimensional model problem as shown in Figure 7.10.

Without loss of original generality, we let  $\Omega$  denote the unit square  $(0; 2000) \times (2000; 0)$  and consider the problem of finding an unknown function  $\hat{A} = \hat{A}(x; c)$  satis-

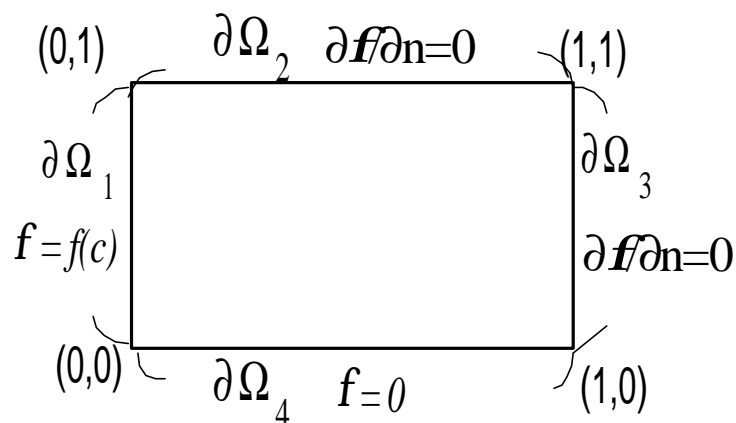


Figure 7.11: Mixed-type boundary conditions

fying

$$\textcircled{c} \quad \hat{A}(x; c) = k(x; c) = 25 \exp \left[ \frac{(x-1000)^2}{200} \right] \sin \frac{\mu c}{4000} \quad \text{in } \Omega; \quad (7.29)$$

$$\hat{A}(0; c) = 1750000 \sin \frac{\mu c}{4000}; \quad \frac{\partial \hat{A}}{\partial n}(2000; c) = 0 \quad 0 \leq c \leq 2000; \quad (7.30)$$

$$\hat{A}(x; 0) = 0; \quad \frac{\partial \hat{A}}{\partial n}(x; 2000) = 0 \quad 0 \leq x \leq 2000;$$

This is a "mixed problem" because specified conditions data are imposed around the entire boundary of  $\Omega$ ; as shown in Figure 7.11:

As shown in Figure 7.11, the potential on boundary  $\partial\Omega_1$  shows the flow is given by  $5f$  and on boundary  $\partial\Omega_4$  the flow variation is specified by zero. On the boundary  $\partial\Omega_2$  and  $\partial\Omega_3$  the longitudinal flow is assumed zero.

Then we draw density function distribution in this region, as shown in Figure 7.12. We know that the density function is a normal distribution. There is substantial flow on the left boundary.

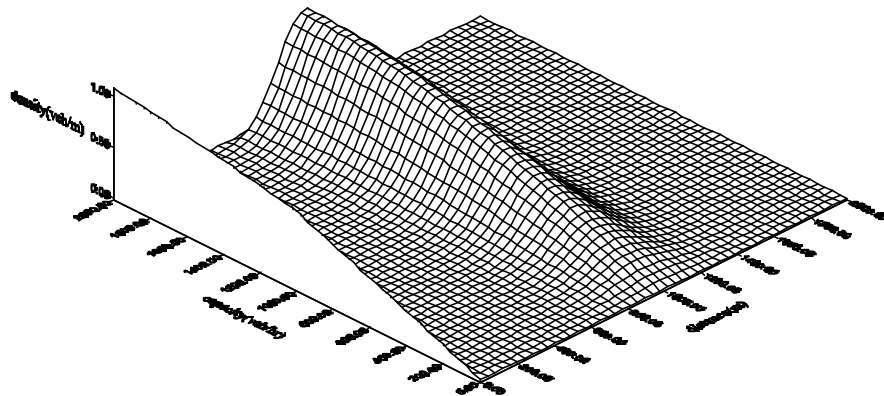


Figure 7.12: The distribution of density function

Using the numerical method computed this problem. The solution is shown in Figure 7.13, on the figure suggesting that the computed potential is a smooth function of  $(x; c)$ . This means that the potential disperses following the distance. A comprehensive comparison of the simulation is also shown in Figure 7.14.

Table 7-1. Potential in various capacity

Capacity(vehs/hr)	Max Potential(vehs/m/hr)	Min Potential(vehs/m/hr)
500	95659.52	40612.55
1000	176759.34	74987.01
1500	230955.79	97837.01
2000	249968.39	105641.40

Referring to Figure 7.9 and Figure 7.11, we find that there exists a peak flow at the initial position, and a normal distribution density source function located in the lane. Under this situation we want to analyze the traffic flow variation for various capacity

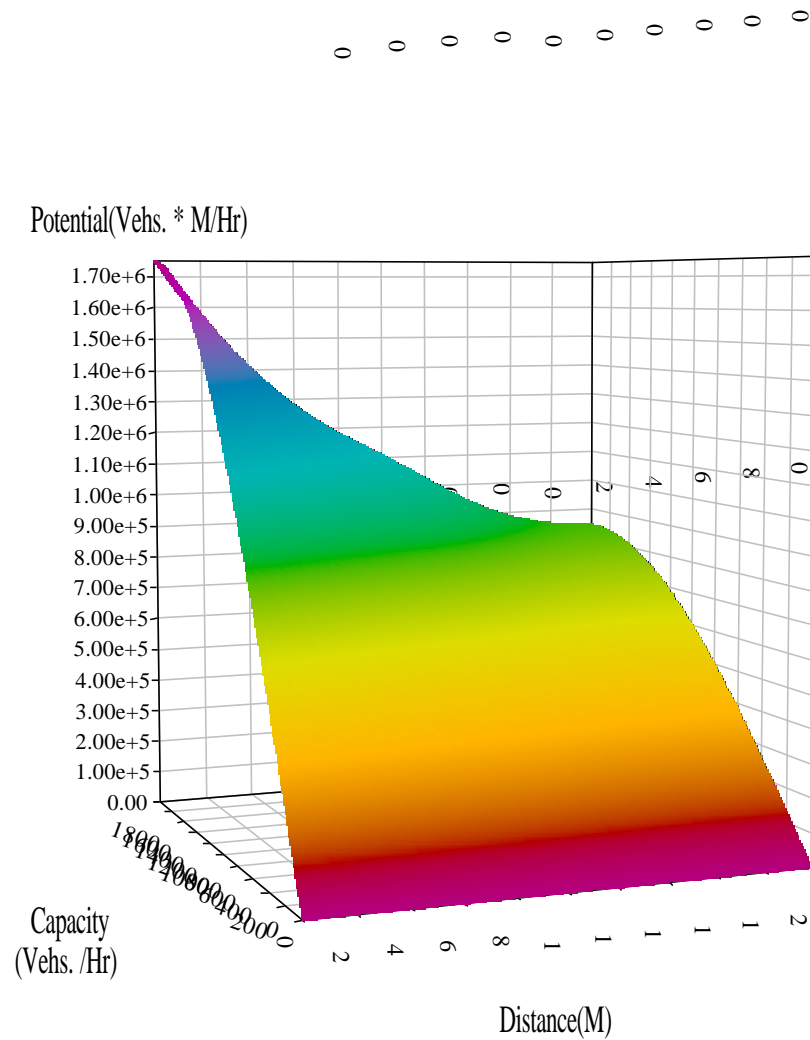


Figure 7.13: Numerical solution



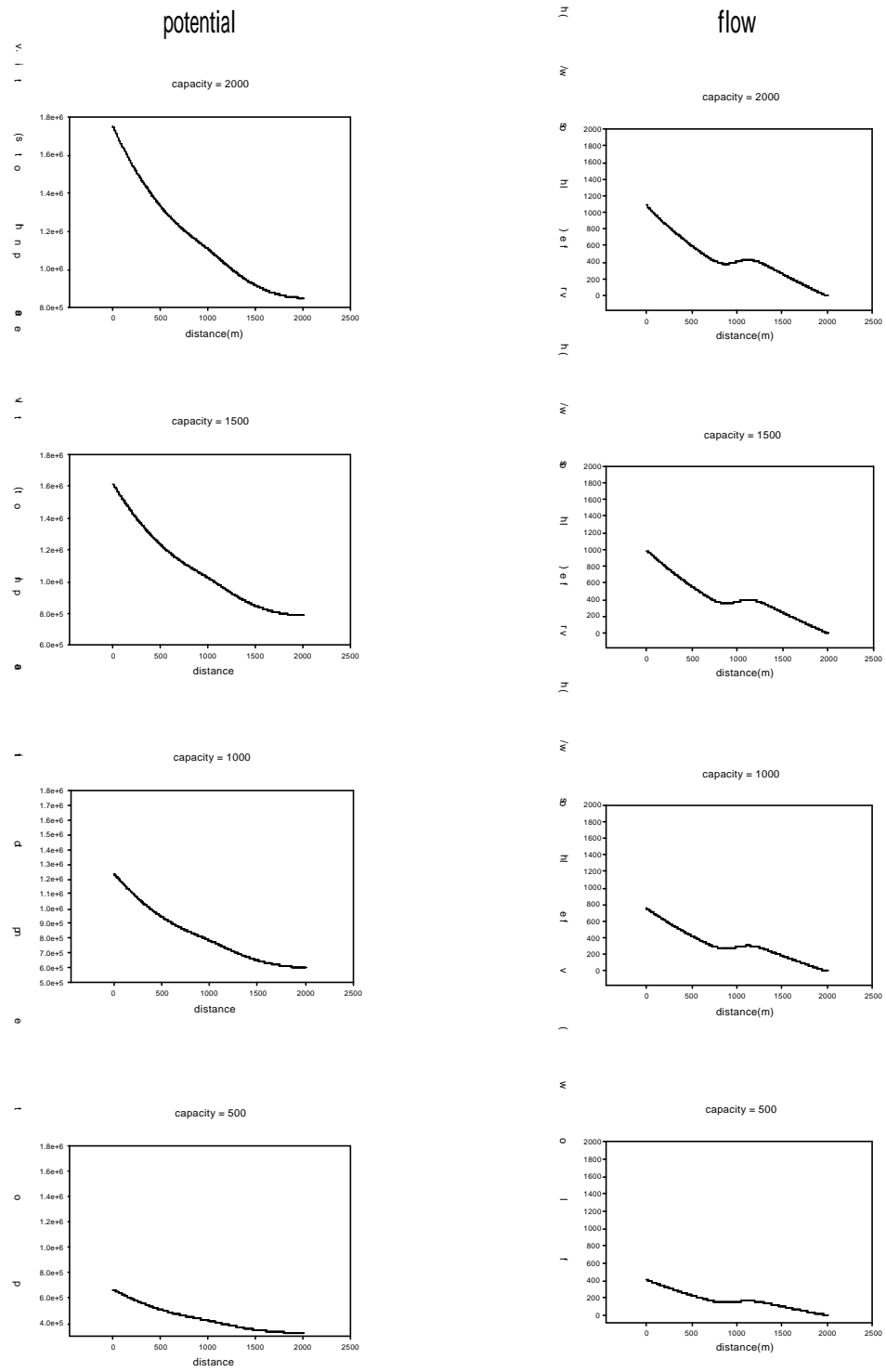


Figure 7.14: Solution of potential and flow in various capacity

quantities. Figure 7.13 shows a good agreement that the flow propagates fast since the capacity is larger than the flow in the lane. By inspecting various capacity quantities the potential and flow variation are examined in detail. The boundary condition is given as Figure 7.13, where the location of maximum potential is the initial position in our simulation. Figure 7.14 and Table 7-1 give a clear indication that larger capacity implies larger potential. The potential function tends to be smooth and slow changing with respect to the position location. Finally, it is almost a constant whenever the position changes.

We have defined the flow  $q = 5\lambda$  in chapter four, using this definition. We find capacity equals 500 is not enough to operate as well as possible. However, as capacity increases to 1200 or 1500, it provides a good flow operation. Thus we can apply the proposed model in traffic engineering and related planning. A simulation tool was developed to model it and provide a way to study the relationship between capacity, flow, and density. This information can be applied to highway traffic flow control and management.

## Chapter 8

### Conclusion and Remarks

A new traffic flow model based on spatial and capacity has been proposed. Using field theory, we have described the traffic flow model as a conservation field. Furthermore the existence and uniqueness of the solution for the model has been also proven using Green's identity. Some general analytical solution formulas were discussed for the proposed model.

Traditionally, traffic flow is only a spatially dependent function. In our study, we extended the relation that the traffic flow is also a function of capacity. The conventional traffic flow model is widely used for studying some macroscopic traffic flow phenomena. The main property of that approach is based on the mass, energy or momentum conservation law. In this thesis, a new traffic flow model, so called the Traffic Dispersion Model, was presented. It provides an efficient tool to study flow variation with traffic capacity. One of the applications is to provide real time traffic flow information in a highway system. The main concept of this new model is that it takes the potential approach for the flow, density, and capacity relationships.

Implementation solution approaches and algorithms for the proposed model and method were given in detail. Based on the novel traffic potential behavior of this model, the proposed model has several important properties. First, it has been proven that traffic flow is a conservation field. In other words, there exists a potential function such that the flow and density relationship can be transferred into a potential and density relation model. Secondly, not only spatial dependence but also the capacity as a variation variable was considered in the proposed model. This model describes many more traffic physical phenomena. Thirdly, the completed solution approaches, such as the analytical and numerical techniques, were also given for more physical studies.

# Appendix A

## Mathematical Definition

### A.1 Maximum Principal[[35]]

Let  $\Omega$  be a connected bounded open set in three-dimensional space. Let  $\hat{A}(x; y; c)$  be a harmonic function in  $\Omega$  that is continuous on  $\bar{\Omega} = \Omega \cup \partial\Omega$ . Then the maximum and the minimum values of  $\hat{A}$  are attained on  $\text{bdy } \Omega$  and nowhere inside ( $\hat{A} \neq \text{constant}$ ).

In other words, a harmonic function is its biggest somewhere on the boundary and its smallest somewhere else on the boundary.

To understand the maximum principle, let us use the vector shorthand  $x = (x; y; c)$  in three dimensions. Also, the radial coordinate is written as  $|x| = (x^2 + y^2 + c^2)^{\frac{1}{2}}$ . The maximum principle asserts that there exist points  $x_M$  and  $x_m$  on  $\text{bdy } \Omega$  such that

$$\hat{A}(x_m) \leq \hat{A}(x) \leq \hat{A}(x_M) \quad (\text{A.1})$$

for all  $x \in \bar{\Omega}$  (see Figure ). Also, there are no points inside  $\Omega$  with this property ( $\hat{A} \neq \text{constant}$ ). There could be several such points on the boundary.

The idea of the maximum principle is as follows, in three dimensions, say. At a maximum point inside  $\Omega$ , if there were one, we'd have  $\Delta u = 0$ ;  $u_{xx} = 0$  and  $u_{yy} = 0$ . (This is the second derivative test of calculus.) So  $\Delta u = u_{xx} + u_{yy} + u_{zz} = 0$ . At most maximum points,  $u_{xx} = 0$ ;  $u_{yy} = 0$  and  $u_{zz} = 0$ . So we'd get a contradiction to Laplace's equation. However, since it is possible that  $u_{xx} = u_{yy} = u_{zz} = 0$  at a maximum points, we have to work a little harder to get a proof.

Here we go. Let  $\epsilon > 0$ . Let  $v(x) = u(x) + \epsilon |x|^2$ . Then, still in three dimensions, say,

$$\Delta v = \Delta u + 2\epsilon (x^2 + y^2 + z^2) = 0 + 6\epsilon > 0 \quad \text{in } \Omega.$$

But  $\Delta v = v_{xx} + v_{yy} + v_{zz} = 0$  at an interior maximum point, by the second derivative test in calculus. Therefore,  $v(x)$  has no interior maximum in  $\Omega$ .

Now  $v(x)$ , being a continuous function, has to have maximum somewhere in the closure  $\bar{\Omega} = \Omega \cup \partial\Omega$ . Say that the maximum of  $v(x)$  is attained at  $x_0 \in \partial\Omega$ . Then, for all  $x \in \Omega$ ,

$$u(x) < v(x) \leq v(x_0) = u(x_0) + \epsilon |x_0|^2 \leq \max_{\partial\Omega} u + \epsilon l^2;$$

where  $l$  is the greatest distance from  $\text{bdy } \Omega$  to the origin. Since this is true for any  $\epsilon > 0$ , we have

$$u(x) \leq \max_{\partial\Omega} u \quad \text{for all } x \in \Omega; \quad (\text{A.2})$$

Now this maximum is attained at some point  $x_M \in \partial\Omega$ . So  $u(x) \leq u(x_M)$  for all  $x \in \bar{\Omega}$ , which is the desired conclusion.

The existence of a minimum point  $x_m$  is similarly demonstrated.

**Lemma 10 Maximum Principle:**

The solution of

$$\Delta u = 0 \quad \text{in } \Omega$$

$$u|_{\partial\Omega} = f$$

attains its maximum and minimum on the boundary  $\partial\Omega$ , does not occur inside  $\Omega$ , unless  $u = \text{constant}$  when  $u$  is continuous on  $\bar{\Omega} = \Omega \cup \partial\Omega$ .

## A.2 Divergence Theorem[[41]]

Let  $V$  denote a bounded region in  $\mathbb{R}^3$  having a smooth boundary  $\partial V$  with  $\mathbf{n}$  denoting the unit outward normal vector to  $\partial V$ . Let  $\mathbf{f}(\mathbf{x})$  be any  $C^1$  vector field on  $\bar{V} = V \cup \partial V$ .

Then

$$\iiint_V \nabla \cdot \mathbf{f} \, dV = \iint_{\partial V} \mathbf{f} \cdot \mathbf{n} \, dS; \quad (\text{A.3})$$

where  $\nabla \cdot \mathbf{f}$  is the three-dimensional divergence of  $\mathbf{f}$  and  $dS$  is the element of surface area on  $\partial V$ .

For example, let  $\mathbf{f} = (x; y; z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$  and let  $V$  be the ball with center at the origin and radius  $a$ . Then  $\nabla \cdot \mathbf{f} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 3$ , so that the left side of (A.3) equals

$$\iiint_V 3 \, dV = 3 \int_0^a \int_0^\pi \int_0^{2\pi} r^2 \sin \theta \, d\phi \, d\theta \, dr = 4\pi a^3$$

Furthermore,  $\mathbf{n} = \frac{\mathbf{x}}{a}$  on the spherical surface, so that  $\mathbf{f} \cdot \mathbf{n} = \mathbf{x} \cdot \frac{\mathbf{x}}{a} = \frac{a^2}{a} = a$  and the right side of (A.3) is  $\iint_{\partial V} a \, dS = a(4\pi a^2) = 4\pi a^3$ :

That demonstrates

$$\iiint_V \nabla \cdot \mathbf{f} \, dV = 4\pi a^3 = \iint_{\partial V} \mathbf{f} \cdot \mathbf{n} \, dS:$$



### A.3 Green's Identities[[13]]

Let  $\Omega$  denote a bounded region in  $\mathbb{R}^3$  having a smooth boundary  $\partial\Omega$  with  $\mathbf{n}$  denoting the unit outward normal vector to  $\partial\Omega$ . Let  $\mathbf{f}(\mathbf{x})$  be any  $C^1$  vector field on  $\bar{\Omega} = \Omega \cup \partial\Omega$ .

Then

$$\iiint_{\Omega} \nabla \cdot \mathbf{f} \, dV = \iint_{\partial\Omega} \mathbf{f} \cdot \mathbf{n} \, dS; \quad (\text{A.4})$$

where  $\nabla \cdot \mathbf{f}$  is the three-dimensional divergence of  $\mathbf{f}$  and  $dS$  is the element of surface area on  $\partial\Omega$ .

If we apply the divergence theorem (see appendix A.2) in the special case that

$$\mathbf{f} = \phi \nabla v$$

for smooth, scalar-valued functions  $\phi$  and  $v$ , then

$$\text{div} [\phi \nabla v] = \phi \nabla^2 v + \nabla \phi \cdot \nabla v \quad (\text{A.5})$$

This identity (A.5) follows from many applications of the usual product rule (A.6) for derivatives.

$$(\phi v_x)_x = \phi_x v_x + \phi v_{xx} \quad (\text{A.6})$$

Then (A.4) becomes

$$\iint_{\partial\Omega} \phi \frac{\partial v}{\partial n} dS = \iiint_{\Omega} \left[ \phi \nabla^2 v + \nabla \phi \cdot \nabla v \right] dV \quad (\text{A.7})$$

where  $\frac{\partial v}{\partial n} = \nabla v \cdot \mathbf{n}$  = outward normal derivative

= directional derivative of  $v$  in direction of  $\mathbf{n}$

The identity (A.7) is known as Green's first identity.

The middle term in (A.7) does not change if  $\tilde{A}$  and  $v$  are switched. So if we write (A.7) for the pair  $\tilde{A}$  and  $v$ , and again for the pair  $v$  and  $\tilde{A}$ , and then subtract, we get

$$\iiint_{\Omega} (\tilde{A} \Delta v - v \Delta \tilde{A}) dx = \iint_{\partial \Omega} \tilde{A} \frac{\partial v}{\partial n} - v \frac{\partial \tilde{A}}{\partial n} dS: \quad (\text{A.8})$$

This is Green's second identity. Just like (A.8), it is valid for any pair of functions  $\tilde{A}$  and  $v$ .

It leads to the following natural definition. A boundary condition is called symmetric for the operator  $\Delta$  if the right side of (A.8) vanishes for all pairs of functions  $\tilde{A}, v$  that satisfy the boundary condition. Each of the three classical boundary conditions (Dirichlet, Neumann, and Robin) is symmetric. In other words, since  $\iint_{\partial \Omega} \tilde{A} \frac{\partial v}{\partial n} - v \frac{\partial \tilde{A}}{\partial n} dS = 0$ . By (A.8) we know

$$\iiint_{\Omega} \tilde{A} \Delta v dx = \iiint_{\Omega} v \Delta \tilde{A} dx: \quad (\text{A.9})$$

If  $v$  is a solution of (A.10)

$$\Delta \tilde{A} = k = P \tilde{A} \quad (\text{A.10})$$

$$P(x) \geq 0$$

and  $\tilde{A} = v$ , then in (A.7), we have

$$\iiint_{\Omega} \frac{1}{4} \left( \frac{\partial \tilde{A}}{\partial x_1} \right)^2 + \dots + \left( \frac{\partial \tilde{A}}{\partial x_n} \right)^2 + P \tilde{A}^2 dx = \iint_{\partial \Omega} \tilde{A} \frac{\partial \tilde{A}}{\partial n} dS \quad (\text{A.11})$$

We call the term on the left the Dirichlet integral associated with (A.10), and we introduce for it the notation

$$k \tilde{A}^2 = \iiint_{\Omega} \frac{1}{4} \left( \frac{\partial \tilde{A}}{\partial x_1} \right)^2 + \dots + \left( \frac{\partial \tilde{A}}{\partial x_n} \right)^2 + P \tilde{A}^2 dx: \quad (\text{A.12})$$

The importance of the norm  $\|A\|$  for uniqueness proofs lies in the fact that it is positive when  $A \notin \text{const.}$ , or even when  $A$  differs from zero and  $P \neq 0$ , in view of the assumption  $P \geq 0$ .

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