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1. 基礎流體動力學和熱力學

基本方程式

李 賢 文

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第一篇 基礎流體動力學

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第一章 緒 論

§ 1.0 前言

海洋運動過程之描述，乃是根據物理學之各種守恒定律；即質量不滅定律 (the law of mass conservation)，溶解物不滅定律 (the conservation law for the dissolved substances)，能量不滅定律 (the law of energy conservation)，和動量不滅定律 (the law of momentum conservation)。由這些定律依次可以演化成下列方程式，即連續方程式 (the equation of continuity)，鹽度輻散方程式 (the equation of diffusion for salinity)，熱傳導方程式 (the equation of heat conduction)，和運動方程式 (the equation of motion)。其中運動方程式為一個向量方程式，含有三個分量方程式，加上其他三個方程式，總共有六個方程式。但是描述海水運動之物理變量，却有七個，即密度 (ρ)，鹽度 (S)，溫度 (T)，壓力 (p) 和三個速度分量 (u , v , w)。因此我們需要狀態方程式 (the equation of state)，使得熱力學之變量能以鹽度、溫度和

壓力來表示。海水之狀態方程式，是一個複雜的經驗方程式。如此我們有七個方程式而能組成一個封閉的系統 (a closed system of equations) 。

上述之方程式都是依據海水連續體之假設 (continuum hypothesis) 而導來。而在海面和海底，分別為空氣和海水之交界面，以及海水和沉積物之交界面，對於海水而言，它們都形成了不連續面。因此在海面和海底，上述之方程式就無法適用，而必須另以邊界條件代替。

§ 1 . 1 場 (field) ; 無向量場和向量場

由於我們將海水視為連續體，其性質在此連續體中，隨時隨地都會改變。因此在海洋空間內的任一點，我們都可以定出顯示其性質的物理量 (譬如，密度、溫度、速度等) 。如此在海洋中，對於此物理量，就有一個場被定義出來 (譬如，密度場、溫度場、速度場等) 。

表示海洋各種場的物理量，可以分為無向量場 (scalar field) ，向量場 (vector field) 和張量場 (tensor field) 。

無向量場在空間上，可以其等值面 (equi-scalar surfaces) 表示。例如壓力場可以等壓面 (isobaric surface) 表示。如果其他場的等值面和等壓面重疊，那麼就稱它為正壓的 (barotropic) ; 如果其他場等值面和等壓面相交，如此則稱它為斜壓的 (baroclinic) 。在海洋學，我們通常是將密度場與壓力場比較 ; 即等密度面與等壓面重疊，則為正壓的 ; 而等密度面與等壓面相交，則為斜壓的。

對於任一向量場 ($\vec{a}$) ，我們可以定義其場線 (field lines

), 其方程式爲:

$$d\vec{r} \times \vec{a} = 0 \quad (1.1.1)$$

亦即在空間上, 場線和向量 $\vec{a}$ 的每一點相切。如果向量 $\vec{a}$ 是速度 $\vec{v}$, 則其場線就稱爲流線 (streamlines)。依據 (1.1.1) 流線之方程式爲:

$$dx : dy : dz = u : v : w \quad (1.1.2)$$

如果一個向量是某一個無向量之梯度, $\vec{a} = \nabla \phi$, 則此向量稱爲位向量 (potential vector)。如果此向量爲速度, 則此種流稱爲位流 (potential flow)。

流速 ($\vec{v}$) 和表面元素 (surface element, $d\vec{\sigma}$) 之乘積爲 $\vec{v} \cdot d\vec{\sigma}$, 其因次爲 $m^3 \text{ sec}^{-1}$; 也就是說, 它代表在單位時間內流經表面元素的流體體積, 我們稱之爲體積通量 (volume flux)。同理 $\rho \vec{v} \cdot d\vec{\sigma}$ [Kg sec^{-1}] 爲單位時間內流經 $d\sigma$ 之質量, 稱爲質量通量 (mass flux)。而它們的面積分:

$$\iint_s \vec{v} \cdot d\vec{\sigma} = \iint_s \vec{v} \cdot \vec{n} d\sigma \quad (1.1.3)$$

$$\text{和 } \iint_s \rho \vec{v} \cdot d\vec{\sigma} = \iint_s \rho \vec{v} \cdot \vec{n} d\sigma$$

就是經過總表面的體積通量和質量通量。任何類似 (1.1.3) 之向量積分, 都可定義一種向量通量 (vector flux)。

在流體動力學中, 我們時常用到 Nabla-operator, 即:

$$\nabla = \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} = \vec{e}_i \frac{\partial}{\partial x_i} \quad (1.1.4)$$

其中 $\vec{i}$, $\vec{j}$, $\vec{k}$ 代表 x , y , z 三個方向之單位向量, 而分別能以 $\vec{e}_i$

， $\vec{e}_2$ ， $\vec{e}_3$ 表示。在 (1.1.4) 右邊，我們採用了和規定 (summation convention)，亦即下註的 (subscript) 文字中，有二個相同者，則表示爲此文字依次變化後之和。對於上註 (superscript) 亦採用同樣的約定。與 Nabla 運算有關，而常使用之定理爲：

$$\iiint_V \nabla \cdot \vec{v} dV = \oiint_S \vec{v} \cdot d\vec{\sigma} \quad (1.1.5)$$

此即爲高氏積分定理 (gaussian integral theorem)。因爲 ∇ 可以應用到任何的場函數 (無向量，向量，張量)，高氏定理可以運用如下：

$$\iiint_V \nabla \phi dV = \oiint_S d\vec{\sigma} \phi, \quad \iiint_V \nabla \cdot \vec{v} dV = \oiint_S d\vec{\sigma} \cdot \vec{v} \quad (1.1.6)$$

$$\iiint_V \nabla \times \vec{v} dV = \oiint_S d\vec{\sigma} \times \vec{v}, \quad \iiint_V \nabla \cdot \Phi dV = \oiint_S d\vec{\sigma} \cdot \Phi$$

其中 ϕ 爲無向量， Φ 爲張量。

另一個重要的關係，爲 stokes 積分定理 (stokes' integral theorem)，即：

$$\iint_S d\vec{\sigma} \cdot \nabla \times \vec{v} = \oint_C d\vec{r} \cdot \vec{v} \quad (1.1.7)$$

表示沿著一個封閉曲線的環流 (circulation)，等於穿過此曲線所圍表面之渦漩度通量 (vorticity flux)。我們把 $\nabla \times \vec{v}$ 稱爲渦漩度 (vorticity)。如同高氏定理，Stokes 定理亦可運用到各種場函數，即：

$$\begin{aligned}\iint d\vec{\sigma} \times \nabla \phi &= \oint d\vec{r} \phi, \quad \iint (d\vec{\sigma} \times \nabla) \cdot \vec{v} = \oint d\vec{r} \cdot \vec{v} \\ \iint (d\vec{\sigma} \times \nabla) \times \vec{v} &= \oint d\vec{r} \times \vec{v}, \quad (1.1.8) \\ \iint (d\vec{\sigma} \times \nabla) \cdot \Phi &= \oint d\vec{r} \cdot \Phi\end{aligned}$$

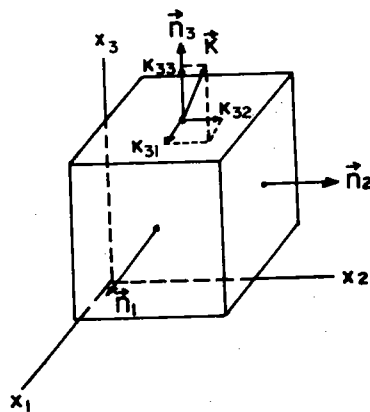
在流體動力學中，渦漩場 (vorticity field) 和流場同樣重要；渦漩度 $\text{curl } \vec{v} = \nabla \times \vec{v}$ ，之場線，稱為渦漩線 (vortex line)，其方程式為：

$$dx : dy : dz = (\nabla \times \vec{v})_x : (\nabla \times \vec{v})_y : (\nabla \times \vec{v})_z, \quad (1.1.9)$$

§ 1.2 張量場 (tensor field)

在一個體積內之變形 (deformation) 和應力 (stress) 通常是以張量來描述。例如二階張量 (second-order tensor)，可用來定義一個流體元素的表面力 (surface forces)，(參考圖 1.1)。

圖 1.1 二階張量 K_{ik} ，第一個腳註表示垂直於此表面之方向，第二個腳註表示作用此表面之力的分量。



在一個液體元素的表面，我們令其單位垂直向量爲 $\vec{n}_1$, $\vec{n}_2$, $\vec{n}_3$ 。

在這些表面上，可能有外力作用。例如，我們考慮，外力 $\vec{K}$ 作用於 $\vec{n}_3$ 所指示之表面。

在這兒，我們需要二個腳註，以便明確的定出它的分量：

K_{31} , K_{32} , K_{33} 。

第一個腳註，表示力所作用之表面（如此例，爲第三個表面）。

第二個腳註，表示此力之分量，所以在此例中：

$$\vec{K} = K_{31} \vec{n}_1 + K_{32} \vec{n}_2 + K_{33} \vec{n}_3$$

如果我們考慮，其他作用於表面 1（即 $\vec{n}_1$ 所示者）和表面 2（即 $\vec{n}_2$ 所示者）之外力，那麼我們總共就需要 9 個分量的力來描述；它們可以矩陣表示，即：

$$(K_{ik}) = \begin{pmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{pmatrix}$$

我們所討論的張量中，最重要的爲：

1) 應力張量 (stress tensor)

2) $\nabla \vec{v}$

一個二階張量 (second-order tensor)，又稱 dyadic；它的形式爲：

$$\Pi = P_{ik} \vec{e}_i \vec{e}_k$$

單位向量的序對， $\vec{e}_i \vec{e}_k$ ，就是一種張量。不像無向量積 ($\cdot$) 及向量積 ($\times$)，此種 $\vec{e}_i \vec{e}_k$ 的積，就稱爲 dyadic product，即二個量之間，不用任何符號。

以後，我們將採用希臘字母大寫，表示張量；而以拉丁字母大寫表示它們的分量。例如：

$$\Pi = P_{ik} \vec{e}_i \vec{e}_k, \quad \text{二階張量 (2 - order tensor)}$$

如果 $P_{ik} = a_i b_k$ ，並且 $\vec{a} = a_i e_i$ ， $\vec{b} = b_k e_k$

$$\text{則 } \Pi = a_i b_k \vec{e}_i \vec{e}_k = \vec{a} \vec{b}$$

$\Sigma = S_{ikl} \vec{n}_i \vec{n}_k \vec{n}_l$ ，表示三階張量 (3 - order tensor)。

張量代數 (tensor algebra)：

1) $\Pi = \Sigma$ ，當 $P_{ik} = S_{ik}$ ，二個張量相等，即當它們的分量相等時。

2) 當二個張量為同階時，我們就可求它們的和及差。

$$T = \Pi + \Sigma$$

$$\text{即 } T_{ik} = P_{ik} + S_{ik}$$

3) 加法運算可交換，即

$$\Pi + \Sigma = \Sigma + \Pi$$

4) 乘法 (multiplication) 有四種情形：

(a) 乘純量 (multiplication by a scalar) λ ：

$$\lambda \Pi = \lambda P_{ik} \vec{e}_i \vec{e}_k$$

即所有的分量，都乘以 λ ：

$$(\lambda P_{ik}) = \begin{pmatrix} \lambda P_{11} & \lambda P_{12} & \lambda P_{13} \\ \lambda P_{21} & \lambda P_{22} & \lambda P_{23} \\ \lambda P_{31} & \lambda P_{32} & \lambda P_{33} \end{pmatrix}$$

(b) Scalar multiplication：

此種乘法根據：

$$\vec{e}_i \cdot \vec{e}_k = \delta_{ik} = \begin{cases} 0 & \text{若 } i \neq k \\ 1 & \text{若 } i = k \end{cases}$$

例如：

$$\begin{aligned} \vec{a} \cdot \Pi &= a_i \vec{e}_i \cdot P_{kl} \vec{e}_k \vec{e}_l = a_i P_{kl} \delta_{ik} \vec{e}_l \\ &= a_k P_{kl} \vec{e}_l \end{aligned}$$

$$\begin{aligned} \Pi \cdot \vec{a} &= P_{kl} \vec{e}_k \vec{e}_l \cdot a_i \vec{e}_i = P_{kl} a_i \delta_{li} \vec{e}_k \\ &= a_l P_{kl} \vec{e}_k = P_{lk} a_l \vec{e}_i \vec{e}_k \cdot \vec{e}_l \\ &= P_{lk} a_l \vec{e}_i \delta_{kl} = P_{lk} a_k \vec{e}_i \\ &= a_k P_{lk} \vec{e}_i \quad (\text{將 index } i, \text{換爲 } l) \end{aligned}$$

因此， $\vec{a} \cdot \Pi \neq \Pi \cdot \vec{a}$ ($\because P_{lk} \neq P_{kl}$)

$$\begin{aligned} \Sigma \cdot \Pi &= S_{ik} \vec{e}_i \vec{e}_k \cdot P_{lm} \vec{e}_l \vec{e}_m = S_{ik} P_{lm} \vec{e}_i \underbrace{\vec{e}_k \cdot \vec{e}_l}_{\delta_{kl}} \vec{e}_m \\ &= S_{ik} P_{lm} \vec{e}_i \delta_{kl} \vec{e}_m = S_{ik} P_{km} \vec{e}_i \vec{e}_m \\ &= T_{im} \vec{e}_i \vec{e}_m \end{aligned}$$

where $T_{im} = S_{ik} P_{km}$

$$\begin{aligned} \text{而 } \Pi \cdot \Sigma &= P_{ik} \vec{e}_i \vec{e}_k \cdot S_{lm} \vec{e}_l \vec{e}_m = P_{ik} S_{lm} \vec{e}_i \underbrace{\vec{e}_k \cdot \vec{e}_l}_{\delta_{kl}} \vec{e}_m \\ &= P_{ik} S_{lm} \vec{e}_i \delta_{kl} \vec{e}_m = P_{ik} S_{km} \vec{e}_i \vec{e}_m \end{aligned}$$

因此 $\Pi \cdot \Sigma \neq \Sigma \cdot \Pi$

(c) Vector multiplication :

根據 $\vec{e}_i \times \vec{e}_k = \epsilon_{ikl} \vec{e}_l$

$$\text{在此 } \varepsilon_{ikl} = \begin{cases} +1 & (\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = +1) \\ -1 & (\varepsilon_{213} = \varepsilon_{132} = \varepsilon_{321} = -1) \\ 0 & \text{二個以上之下標相同} \end{cases}$$

我們得到：

$$\begin{aligned} \vec{a} \times \Pi &= a_i \vec{e}_i \times P_{kl} \vec{e}_k \vec{e}_l = a_i P_{kl} \vec{e}_i \times \vec{e}_k \vec{e}_l \\ &= a_i P_{kl} \varepsilon_{ikm} \vec{e}_m \vec{e}_l \end{aligned}$$

$$\begin{aligned} \text{而 } \Sigma \times \Pi &= S_{ik} \vec{e}_i \vec{e}_k \times P_{lm} \vec{e}_l \vec{e}_m \\ &= S_{ik} P_{lm} \vec{e}_i \vec{e}_k \times \vec{e}_l \vec{e}_m \\ &= S_{ik} P_{lm} \vec{e}_i \varepsilon_{klm} \vec{e}_n \vec{e}_m \\ &= S_{ik} P_{lm} \varepsilon_{klm} \vec{e}_i \vec{e}_n \vec{e}_m \end{aligned}$$

(d) Tensor (dyadic) product :

$$\vec{a} \Pi = a_i \vec{e}_i P_{kl} \vec{e}_k \vec{e}_l = a_i P_{kl} \vec{e}_i \vec{e}_k \vec{e}_l$$

5) Transposed tensor :

$$\text{已知 } \Pi = P_{ik} \vec{e}_i \vec{e}_k$$

則它的 transposed tensor :

$$\widetilde{\Pi} = P_{ki} \vec{e}_i \vec{e}_k$$

即分量的矩陣中，行列的分量，互換位置。

$$\text{一般說來 } \Pi \cdot \vec{a} \neq \vec{a} \cdot \Pi$$

$$\text{但 } \Pi \cdot \vec{a} = \vec{a} \cdot \widetilde{\Pi}$$

$$\text{同時 } \widetilde{\widetilde{\Pi}} = \Pi$$

6) Characteristic vectors :

一個張量和一個基礎向量 (base vector) $\vec{e}_i$ 之 scalar

multiplication, 就可得 characteristic vector, 即:

$$\pi \cdot \vec{e}_i = P_{im} \vec{e}_i \vec{e}_m \cdot \vec{e}_i = P_{ii} \vec{e}_i = \vec{P}_i$$

7) 張量之分量 (components of a tensor) :

一個張量的兩邊, 利用 scalar multiplication, 各乘單位向量, 即可得:

$$\begin{aligned} \vec{e}_i \cdot \pi \cdot \vec{e}_k &= \vec{e}_i \cdot P_{lm} \vec{e}_l \vec{e}_m \cdot \vec{e}_k \\ &= P_{lm} \delta_{il} \delta_{mk} = P_{ik} \end{aligned}$$

8) 特殊張量:

(a) 單位張量 (unit tensor), 其定義為:

$$I = \delta_{ik} \vec{e}_i \vec{e}_k = \vec{e}_i \vec{e}_i$$

任意一個向量, 乘於單位張量, 則得原來自己的向量, 即:

$$\begin{aligned} I \cdot \vec{a} &= \delta_{ik} \vec{e}_i \vec{e}_k \cdot a_l \vec{e}_l = a_l \delta_{ik} \delta_{kl} \vec{e}_i \\ &= a_i \vec{e}_i = \vec{a} \end{aligned}$$

(b) 零張量 (null tensor) :

$$\phi = 0 \vec{e}_i \vec{e}_k$$

9) 任何一個向量可被分解為一個對稱張量 (symmetrical tensor), Σ 與一個反對稱張量 (antisymmetrical tensor) A 之和。

對稱張量: $P_{ik} = P_{ki}$, 即 $\pi = \widetilde{\pi}$

反對稱張量: $P_{ik} = -P_{ki}$, 即 $\pi = -\widetilde{\pi}$; 關於反對

稱張量 A , 其矩陣對角線位置之分量皆為零, 即:

$$P_{11} = P_{22} = P_{33} = 0$$

$$\therefore A = \begin{pmatrix} 0 & P_{12} & P_{13} \\ -P_{12} & 0 & P_{23} \\ -P_{13} & -P_{23} & 0 \end{pmatrix} = (P_{ik})$$

即一個反對稱張量，僅有三個基本分量 P_{12} , P_{13} , P_{23} 。而其行列式 = 0 ,

$$\begin{aligned} |A| &= \begin{vmatrix} 0 & P_{12} & P_{13} \\ -P_{12} & 0 & P_{23} \\ -P_{13} & -P_{23} & 0 \end{vmatrix} \\ &= -P_{12} \begin{vmatrix} -P_{12} & P_{23} \\ -P_{13} & 0 \end{vmatrix} + P_{13} \begin{vmatrix} -P_{12} & 0 \\ -P_{13} & -P_{23} \end{vmatrix} \\ &= -P_{12} P_{23} P_{13} + P_{13} P_{13} P_{23} = 0 \end{aligned}$$

而對稱張量爲：

$$S = \begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{12} & P_{22} & P_{23} \\ P_{13} & P_{23} & P_{33} \end{pmatrix}$$

將一個張量 π 分解爲對稱部分及不對稱部分，就要利用 transposed tensor $\widetilde{\pi}$ ，即

$$\pi = \frac{1}{2}(\pi + \widetilde{\pi}) + \frac{1}{2}(\pi - \widetilde{\pi}) = \Sigma + A$$

$$\therefore \Sigma = \frac{1}{2}(\widetilde{\pi} + \widetilde{\widetilde{\pi}}) = \frac{1}{2}(\widetilde{\pi} + \pi) = \Sigma$$

$$\widetilde{A} = \frac{1}{2}(\widetilde{\pi} - \widetilde{\widetilde{\pi}}) = \frac{1}{2}(\widetilde{\pi} - \pi) = -\frac{1}{2}(\pi - \widetilde{\pi})$$

$$= -A$$

10) 一個反對稱張量以向量表示之形式：

由於一個反對稱張量僅含三個基本分量，因此可以向量表示；如令 A 代表反對稱張量，則可得：

$$A = \vec{a} \times I$$

其中 $\vec{a} = -A_{23} \vec{e}_1 + A_{13} \vec{e}_2 - A_{12} \vec{e}_3$

(證明) 令上式成立，則

$$\begin{aligned} \therefore A &= A_{lm} \vec{e}_l \vec{e}_m = a_i \vec{e}_i \times \delta_{km} \vec{e}_k \vec{e}_m \\ &= a_i \delta_{km} \epsilon_{ikl} \vec{e}_l \vec{e}_m = a_i \epsilon_{ilm} \vec{e}_l \vec{e}_m \\ \therefore A_{lm} &= a_i \epsilon_{ilm} \end{aligned}$$

$$\begin{aligned} \text{同時 } \tilde{A} &= A_{ml} \vec{e}_l \vec{e}_m = a_i \epsilon_{ilm} \vec{e}_l \vec{e}_m \\ &= -a_i \epsilon_{ilm} \vec{e}_l \vec{e}_m = -A \end{aligned}$$

故 $A_{12} = a_i \epsilon_{121} = -a_3$

$$A_{13} = a_i \epsilon_{131} = a_2$$

$$A_{23} = a_i \epsilon_{132} = -a_1$$

$$\text{即得 } \vec{a} = a_i \vec{e}_i = -A_{23} \vec{e}_1 + A_{13} \vec{e}_2 - A_{12} \vec{e}_3$$

II) Trace of a tensor :

即對角線元素之和；通常以符號 tr 表示。如 π 為張量，則

$$\text{Trace } \pi = \text{tr } \pi = P_{11}$$

但因 $\pi = \Sigma + A$ (Σ 和 A 分別為 π 之對稱及反對稱部分)

$$\text{則 } \text{tr } \pi = \text{tr } \Sigma \quad (\because \text{tr } A = 0)$$

在流體動力學中，最重要之張力為應力張量 Σ 和應變張量 (rate-of-strain tensor) D 。

我們考慮一個張量：

(1) $\nabla \vec{v} = \frac{\partial v_k}{\partial x_i} \vec{e}_i \vec{e}_k$, 此種張量係描述一般的運動場。

$$\frac{\partial v_k}{\partial x_i} = \begin{pmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_2}{\partial x_1} & \frac{\partial v_3}{\partial x_1} \\ \frac{\partial v_1}{\partial x_2} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_3}{\partial x_2} \\ \frac{\partial v_1}{\partial x_3} & \frac{\partial v_2}{\partial x_3} & \frac{\partial v_3}{\partial x_3} \end{pmatrix}$$

$\text{tr}(\nabla \vec{v}) = \nabla \cdot \vec{v} = \text{Div } \vec{v}$; 而其他的分量代表切變 (shear)。

$\nabla \vec{v}$ 之 transpose 爲 :

$$(2) \widetilde{\nabla \vec{v}} = \overleftarrow{\nabla} \vec{v} = \frac{\partial v_i}{\partial x_k} \vec{e}_i \vec{e}_k$$

在 ∇ 符號上箭頭, 表示 ∇ 的微分應用到 $\vec{v}$ 。將 $\nabla \vec{v}$ 分解爲一個對稱部分和一個反對稱部分, 即

$$(3) \nabla \vec{v} = \frac{1}{2} (\nabla \vec{v} + \overleftarrow{\nabla} \vec{v}) + \frac{1}{2} (\nabla \vec{v} - \overleftarrow{\nabla} \vec{v}) = \frac{1}{2} (D + D')$$

對稱部分 :

$$(4) D = \left(\frac{\partial v_k}{\partial x_i} + \frac{\partial v_i}{\partial x_k} \right) \vec{e}_i \vec{e}_k$$

稱爲應變張量 (rate-of-strain tensor), 它描述純應變運動 (pure straining motion)。

$$(5) \text{tr } D = D_{ii} = 2 \frac{\partial v_i}{\partial x_i} = 2 \nabla \cdot \vec{v}$$

代表同向延伸 (isotropic expansion)。

$$(6) D = \text{tr } D \mathbf{I} + \overset{\circ}{D} \quad \text{其中} \quad \overset{\circ}{D} = D - \text{tr } D \mathbf{I}$$

其中 $\overset{\circ}{D}$ 代表應變張量之非發散部分 (non-divergent part of

the rate-of-strain tensor)，它描述純扭曲運動。以後我們將討論到它和摩擦張量 (frictional tensor) 有關。其分量爲：

$$(7) \dot{D}_{ik} = D_{ik} - D_{lm} \delta_{lm} \delta_{ik}$$

反對稱部分：

$$(8) D' = \left(\frac{\partial v_k}{\partial x_i} - \frac{\partial v_i}{\partial x_k} \right) \vec{e}_i \vec{e}_k$$

我們知道任意反對稱張量，能以向量表示。

我們令與 D' 相配之向量爲 $\vec{\Omega}$ ，則 $\vec{\Omega}$ 之分量爲：

$$(9) \Omega_1 = -D_{23} = \frac{\partial v_2}{\partial x_3} - \frac{\partial v_3}{\partial x_2}, \quad \Omega_2 = D_{13} = \frac{\partial v_3}{\partial x_1} - \frac{\partial v_1}{\partial x_3}$$

$$\Omega_3 = -D_{12} = \frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1}$$

即：

$$(10) \vec{\Omega} = -\nabla \times \vec{v}$$

於是反對稱部分 D' 代表無變形之轉動，類似剛體轉動。

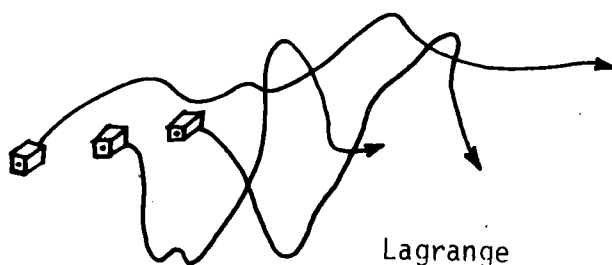
因此由(1)，(6)，(9)，可知 $\nabla \cdot \vec{v}$ 包含所討論之體積元素的發散 (divergence)，無發散扭曲 (non-divergent distortion) 和轉動 (rotation)。

$$(11) \nabla \cdot \vec{v} = \nabla \cdot \vec{v} \cdot \mathbf{I} + \frac{1}{2} \dot{D} - \frac{1}{2} (\nabla \times \vec{v}) \times \mathbf{I}$$

§ 1.3 場之 Lagrangian 和 Eulerian 描述

1) Lagrange :

在質點力學中，一個過程的發展，被視為已知；即當所有質點的位置 $\mathbf{r}_a(t)$ ，為時間的函數，能被定出來之後，但如同在 § 1.0 中，我們提到連續假設；即連續質 (continuum) 可由體積元素 (volume elements) 所組成，在所有的體積元素中，質量是連續的散佈在裡面；並且所有的體積元素，都是以它們重心的速度而運動。因此，一種流體動力過程，被視為已知；就是當任意的體積元素的位置，作為時間的函數，能被定出來。這就是 Lagrangian 描述。



我們令體積元素，在時間 $t = 0$ ，所在的位置為 $\vec{r}_0$ 。那麼每一個體積元素，隨時間發展，所定的位置為：

$$(1) \vec{r} = \vec{r}(\vec{r}_0, t) \quad , \quad \text{其中} \quad \vec{r}(\vec{r}_0, 0) = \vec{r}_0$$

反過來，我們能夠由在時間 t 之目前位置 $\vec{r}$ ，由(1)式，解出 $\vec{r}_0$ ，導出起始位置 (initial position)：

$$\vec{r}_0 = \vec{r}_0(\vec{r}, t)$$

由(1)式，我們可以導出體積元素在時間 t 時之速度及加速度，

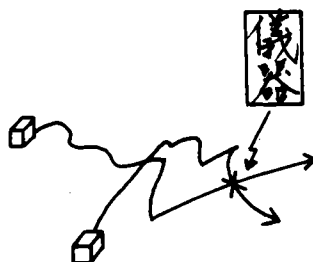
$$(2) \vec{v}_L(\vec{r}_0, t) = \frac{\partial \vec{r}(\vec{r}_0, t)}{\partial t} ,$$

$$\frac{\partial \vec{v}_L(\vec{r}_0, t)}{\partial t} = \frac{\partial^2 \vec{r}(\vec{r}_0, t)}{\partial t^2}$$

此體積元素在時間 $t = 0$ ，所在之位置為 $\vec{r}_0$ 。

2) Euler :

一個場 (field) 之 Eulerian 描述，就是在某一個時間 t ，位置為 $\vec{r}$ 那點之速度 (而在不同的時間 t ，就有不同的體積元素，經過那點 $\vec{r}$)。



在 Eulerian 速度 $\vec{v}$ 和 Lagrangian 速度 $\vec{v}_L$ 間，存有下列等式：

$$(3) \vec{v}(\vec{r}, t) \equiv \vec{v}_L[\vec{r}_0(\vec{r}, t), t]$$

如果 Eulerian 速度已知，則 Lagrangian 速度可由下式求出，

$$(4) \frac{\partial \vec{r}(\vec{r}_0, t)}{\partial t} = \vec{v}_L(\vec{r}_0, t) = \vec{v}[\vec{r}(\vec{r}_0, t), t]$$

已知起始條件 (initial condition) 為：

$$(5) \vec{r}(\vec{r}_0, 0) = \vec{r}_0$$

現在我們考慮一個任意函數：

ϕ 為純量或向量，可代表流體質點 (即 volume element) 之性質。根據 (3) 和 (4)，在 ϕ (Eulerian) 和 ϕ_L (Lagrangian) 間之關係為：

$$(6) \phi(\vec{r}, t) = \phi_L[\vec{r}_0(\vec{r}, t), t],$$

$$\phi_L(\vec{r}_0, t) = \phi[\vec{r}(\vec{r}_0, t), t]$$

對於一個個別的體積元素($\vec{r}_0$ 固定)，它的此種性質 ϕ ，隨時間之變化為：

$$\begin{aligned} & \frac{\partial}{\partial t} \phi_L(\vec{r}_0, t)_{r_0} \\ &= \left[\frac{\partial}{\partial t} \phi[\vec{r}(\vec{r}_0, t), t] \right]_{r_0} \\ &= \left[\frac{\partial}{\partial t} \phi[\vec{r}(\vec{r}_0, t), t]_r \right. \\ & \quad \left. + \frac{\partial}{\partial t} \vec{r}(\vec{r}_0, t) \cdot \nabla \phi[\vec{r}(\vec{r}_0, t), t]_t \right]_{r_0} \\ &= \left[\frac{\partial}{\partial t} \phi[\vec{r}(\vec{r}_0, t), t]_r \right. \\ & \quad \left. + \vec{v}_L(\vec{r}_0, t) \cdot \nabla \phi[\vec{r}(\vec{r}_0, t), t]_t \right]_{r_0} \end{aligned}$$

即：

$$\begin{aligned} (7) \frac{\partial}{\partial t} \phi_L(\vec{r}_0, t)_{r_0} &= \left[\frac{\partial}{\partial t} \phi(\vec{r}, t)_r \right. \\ & \quad \left. + \vec{v}(\vec{r}, t) \cdot \nabla \phi(\vec{r}, t)_t \right]_{r_0} \end{aligned}$$

現在我們定義：

$$(8) \frac{d}{dt} \phi(\vec{r}, t) \equiv \frac{\partial}{\partial t} \phi_L(\vec{r}_0, t)_{r_0}$$

為個體微分(individual derivative)或稱物質微分(material derivative 或 substantial derivative)。
(即流體的性質，隨流體質點運動的時間變化率；也就是所謂的 total time derivatine)

則(7)可寫為：

$$(9) \frac{d\phi}{dt} = \frac{\partial \phi}{\partial t} + \vec{v} \cdot \nabla \phi$$

其中 $\frac{d\phi}{dt}$: 在時間 t 和位置 $\vec{r}$ 之體積元素，其所含之性質 ϕ 的全微分 (total derivative) 。

$\frac{\partial \phi}{\partial t}$: 在位置為 $\vec{r}$ 之當地微分 (local derivative) 。

$\vec{v} \cdot \nabla \phi$: 平流項 (convective term , advection) 。

$\nabla \phi$: 在時間 t 流體性質 ϕ 之空間變化。

同樣地，我們可以導出下列重要的關係：

$$(10) \frac{d}{dt} \iiint \rho \phi dV = \iiint \rho \frac{d\phi}{dt} dV$$

它可應用到一個任意量 ϕ (per unit mass) 。

§ 1.4 場函數不滅定律之一般形式：

海洋動力學的“不滅原理” (conservation theorem)，是 7 個基本流體動力方程式當中 6 個方程式之基礎。根據這個原理，一個個別的流體原素 (fluid element) 的某些性質，保持不變；或者當它受到外界的影響時，它的變化根據一定的方式。這些影響，可能是進入流體中之外力；或分子過程引起流體性質，經過體積元素的表面之交換以及輻射等。

對於在流動中，個別流體之體積元素的不滅定律，就等於在固定位置上，一個體積的平衡方程式 (balance equation)。此二種形式，指稱為不滅定律；即使一般而言，在此種體積內，所考慮的性質

，並不會保持不變。

下面，我們將利用下列定理，以平衡方程式之形式，來導出所有的不滅定律 (conservation laws)。

此定理為：如果 $f(\vec{r}, t)$ 是 V 內之一個連續函數，而且滿足：

$$\iiint_V f(\vec{r}, t) dV = 0, \quad \forall V \subset V$$

$$\text{則 } f(\vec{r}, t) \equiv 0, \quad \text{in } V$$

(證明) $\because$ 如果 $f(\vec{r}, t) \neq 0$ ，對於 $\vec{r} = \vec{r}_0$ 。

則由於 $f(\vec{r}, t)$ 在 V 裏面是連續的，因此，可以找出一個小體積 V' ，包含 $\vec{r}_0$ ，即

$$\vec{r}_0 \in V'$$

而使得 $\iiint_{V'} f(\vec{r}, t) dV \neq 0$ ；則與原題假設矛盾。

$$\therefore f(\vec{r}, t) \equiv 0 \quad \text{in } V$$

一般而言，此定理被用到任意體積，而導出積分形式之方程式，然後根據此定理，則 (被積分函數) = 0。

例如：令 ϕ = 流體單位質量之某一種性質，(因次 = $[\frac{\phi}{g}]$)

則 在固定位置，某一體積 (V) 內，所含此性質 ϕ 之總量

$$= \iiint_V \phi \rho dV$$

$$(\text{因次} = [\frac{\phi}{g}] [\frac{g}{\text{cm}^3}] [\text{cm}^3] = [\phi])$$

這些 ϕ 的總量，會發生變化的情形，為

①穿過體積之表面 σ ，有此種性質 ϕ 的輸送 $-\oint \phi \rho \vec{v} \cdot d\vec{a}$ 發生。

②外力之影響，而在此體積內，創造出一個此性質的源泉密度 (source density) q_ϕ ；對整個體積，此性質之總量為：

$$\iiint q_\phi dV, \quad (q_\phi = [\frac{\phi}{\text{cm}^3 \text{ sec}}])$$

因此，對於 ϕ 之不變原理為：

$$(1) \iiint \frac{\partial \phi \rho}{\partial t} dV = -\oint \phi \rho \vec{v} \cdot d\vec{\sigma} + \iiint q_\phi dV$$

也就是說“在一個固定點位置之體積內，每單位時間，此性質 ϕ 之變化率，等於流體流經其表面，單位時間內 ϕ 的輸送量 (transport) 加上在此體積內，每單位時間 ϕ 的生產量”。

利用 Gauss 定理，可將 (右邊第一項) 面積分改寫為體積分，即

$$(2) \iiint \frac{\partial \phi \rho}{\partial t} dV = -\iiint \nabla \cdot \phi \rho \vec{v} dV + \iiint q_\phi dV$$

因為，此體積為任意大小、形式，故由上述原理 (移項，整理 integrand = 0)，得：

$$(3) \frac{\partial \phi \rho}{\partial t} = -\nabla \cdot \phi \rho \vec{v} + q_\phi$$

此為“微分形式”。

$$\therefore \nabla \cdot \phi \rho \vec{v} = \phi \rho \nabla \cdot \vec{v} + \vec{v} \cdot \nabla \phi \rho$$

$$\therefore \frac{\partial \phi \rho}{\partial t} = -\phi \rho \nabla \cdot \vec{v} - \vec{v} \cdot \nabla \phi \rho + q_\phi$$

移項得： $\frac{\partial \phi \rho}{\partial t} + \vec{v} \cdot \nabla \phi \rho = -\phi \rho \nabla \cdot \vec{v} + q_\phi$

$$\frac{d\phi \rho}{dt} = -\phi \rho \nabla \cdot \vec{v} + q_\phi$$

上式減去連續方程式（將在 § 1 . 6 導出）。

$$\frac{1}{\rho} \frac{d\rho}{dt} = -\nabla \cdot \vec{v} \quad \text{或} \quad \phi \frac{d\rho}{dt} = -\phi \rho \nabla \cdot \vec{v}$$

得：

$$(4) \rho \frac{d\phi}{dt} = \rho q_\phi$$

由 (1 . 3 . 10) 得：

$$(5) \frac{d}{dt} \iiint \rho \phi dV = \iiint \frac{d\phi}{dt} \rho dV = \iiint q_\phi dV$$

此為“積分形式”。

(2), (3) 和 (4), (5) 都是場函數 (field function) ϕ 之不滅定律的一般形式。對 ϕ 和 q_ϕ 給予特定值，就可得到特定的質量、鹽度、動量和能量之不變定律。

§ 1 . 5 狀態方程式 (Equation of State)

在海洋動力學中，我們使用七個場函數 (field functions)，即前所述之三個速度分量 (u , v , w)，密度 (ρ)，海水壓力 (p)，海水溫度 (T)，和海水鹽度 (S)，來描述海洋。

後面 4 個量是描述熱力學之量。

經驗證明，一種純的均勻流體 ($S = 0$)，其靜止狀態，能完全由 2 個參數來描述。

不均勻 (inhomogeneity) 和流動，基本上對於熱力關係沒有影響，因此，我們可以將其他的熱力性質，如溫度 T ，熵 (entropy) 等等。分別當做壓力 (p) 和密度 (ρ) 之函數，即

$$(1) f(p, \rho, T) = 0$$

此種方程式，就稱為狀態方程式 (equation of state)。而(1)式是為純水之狀態方程式。

而海水，當然不是一種純的液體，它裡面含有一些溶解的塩類。這種液體的狀態，也要依賴其溶解物之濃度而定。

在海洋內，所溶解之各種塩的濃度比例，實際上看來，是一定的，如同在1884年W. Dittmar所決定的。根據此種基礎，我們就引進塩度 (salinity) 來代替海洋中個別塩類成分之濃度。

塩度之單位為‰；此即表示一千克海水中，所溶解物質之克數，

$$\left[\frac{g}{Kg} \right]$$

因此，海水之狀態方程式為：

$$(2) f(p, \rho, S, T) = 0$$

而 p ， ρ ， T ，一般就被用來描述海水狀態之3個參數，如同(1)，(2)式中可以其他任意熱力性質來代替 ρ ，而將 ρ 表示為 p ， T 和 S 之函數，即：

$$\rho = \rho(p, T, S)$$

最近由聯合國教科文組織 (UNESCO) 之海洋學表格及標準聯合審查委員會 (Joint Panel on Oceanographic Tables and Standards) 在1981年定義出一個狀態方程式適合於海洋學之用。在這個方程式中，密度 (ρ) 之單位為 Kg/m^3 ($= 10^{-3} g/cm^3$)，壓力 (p) 之單位為巴 (bar)，溫度為 $^{\circ}C$ ，塩度是

實用鹽度單位 (practical salinity unit)。實用鹽度單位與常用鹽度單位 (‰, 即千分之幾) 所表示之鹽度數值, 相差無幾。

利用此種狀態方程式, 以求出海水密度, 需要一連串的步驟。首先利用下式決定純水之密度 (ρ_w):

$$\begin{aligned}(3) \rho_w = & 999.842594 + 6.793952 \times 10^{-2} t \\ & - 9.095290 \times 10^{-3} t^2 + 1.001685 \times 10^{-4} t^3 \\ & - 1.120083 \times 10^{-6} t^4 + 6.536332 \times 10^{-9} t^5\end{aligned}$$

其次, 用下式決定在一大氣壓 (即海水壓力 $p=0$) 之海水密度:

$$\begin{aligned}(4) \rho(S, t, 0) = & \rho_w + S(0.824493 - 4.0899 \times 10^{-3} t \\ & + 7.6438 \times 10^{-5} t^2 - 8.2467 \times 10^{-7} t^3 \\ & + 5.3875 \times 10^{-9} t^4) \\ & + S^{\frac{3}{2}}(-5.72466 \times 10^{-3} \\ & + 1.0227 \times 10^{-4} t - 1.6546 \times 10^{-6} t^2 \\ &) + 4.8314 \times 10^{-4} S^2\end{aligned}$$

最後再由下式決定在海水壓力為 p 時之密度:

$$(5) \rho(S, t, p) = \frac{\rho(S, t, 0)}{1 - \frac{p}{K(S, t, p)}}$$

其中 K 為正割全容係數 (secant bulk modulus)。這個 K 值也是由三個步驟得來。純水之 K 值為:

$$\begin{aligned}(6) K_w = & 19652.21 + 148.4206t - 2.327105t^2 \\ & + 1.360477 \times 10^{-2} t^3 - 5.155288 \times 10^{-5} t^4\end{aligned}$$

在一大氣壓 (海水壓力 $p=0$) 之 K 值為:

$$\begin{aligned}(7) K(S, t, 0) = & K_w + S(54.6746 - 0.603459t \\ & + 1.09987 \times 10^{-2} t^2 - 6.1670 \times 10^{-5} t^3)\end{aligned}$$

$$) + S^{\frac{3}{2}} (7.944 \times 10^{-2} \\ + 1.6483 \times 10^{-2} t - 5.3009 \times 10^{-4} t^2)$$

最後在海水壓力為 p 時之 K 值為：

$$(8) K(S, t, p) = K(S, t, 0) + p (3.239908 \\ + 1.43713 \times 10^{-3} t + 1.16092 \times 10^{-4} t^2 \\ - 5.77905 \times 10^{-7} t^3) \\ + p S (2.2838 \times 10^{-3} \\ - 1.0981 \times 10^{-5} t - 1.6078 \times 10^{-6} t^2) \\ + 1.91075 \times 10^{-4} p S^{\frac{3}{2}} \\ + p^2 (8.50935 \times 10^{-5} \\ - 6.12293 \times 10^{-6} t + 5.2787 \times 10^{-8} t^2) \\ + p^2 S (- 9.9348 \times 10^{-7} \\ + 2.0816 \times 10^{-8} t + 9.1697 \times 10^{-10} t^2)$$

§ 1.6 連續方程式 (質量不減定律)

由 § 1.4 不減定律的一般形式，我們就能直接導出連續方程式。令 ϕ 為比質量 (specific mass, 即 mass per mass)，當質量不減時，外界的影響，並不會在體積內創造質量，故 $q_1 = 0$ (在 § 1.4 中是以 q_ϕ 表示，而今 $\phi \equiv 1$)，因此 (1.4.3)：

$$\frac{\partial \phi \rho}{\partial t} = - \nabla \cdot \phi \rho \vec{v} + q_\phi$$

立即變成：

$$(1) \frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{v} = 0, \quad \frac{\partial \rho}{\partial t} + \frac{\partial \rho v_i}{\partial x_i} = 0$$

由 $\nabla \cdot \rho \vec{v} = \rho \nabla \cdot \vec{v} + \vec{v} \cdot \nabla \rho$, (1)式可改寫為：

$$(2) \frac{1}{\rho} \frac{d\rho}{dt} + \nabla \cdot \vec{v} = 0, \quad \frac{1}{\rho} \frac{d\rho}{dt} + \frac{\partial v_i}{\partial x_i} = 0$$

①均勻介質 (homogeneous medium), 即 $\rho = \text{const}$, 則

$$\nabla \cdot \vec{v} = 0.$$

②不可壓縮介質 (incompressible), (即 $\frac{d\rho}{dt} = 0$, 對某一個液體塊而言, 其密度隨流體運動不變化; 但空間上, 密度分佈可能不均勻)。其實不可壓縮之條件應為：

$$\left(\frac{\partial \rho}{\partial p} \right)_{T,S} = 0$$

$$\therefore d\rho = \left(\frac{\partial \rho}{\partial p} \right) dp + \left(\frac{\partial \rho}{\partial T} \right) dT + \left(\frac{\partial \rho}{\partial S} \right) dS$$

$$\therefore \frac{d\rho}{dt} = \left(\frac{\partial \rho}{\partial p} \right)_{T,S} \frac{dp}{dt} + \left(\frac{\partial \rho}{\partial T} \right)_{S,p} \frac{dT}{dt} + \left(\frac{\partial \rho}{\partial S} \right)_{p,T} \frac{dS}{dt}$$

依不可壓縮條件, $\left(\frac{\partial \rho}{\partial p} \right)_{T,S} = 0$

但如為純水和無擴散, 則 $\frac{dS}{dt} = 0$

而如無熱傳, 則 $\frac{dT}{dt} = 0$

所以在無熱傳導, 無輻散及不可壓縮的液體, 才有

$$(3) \frac{d\rho}{dt} = 0$$

但一般, 我們都把(3)式, 視為“不可壓縮”之條件, 在此條件下：

$$(4) \nabla \cdot \vec{v} = 0$$

③如爲穩定情況，（即 $\frac{\partial}{\partial t} = 0$ ），則(1)式成爲：

$$(5) \nabla \cdot \rho \vec{v} = 0$$

第(2)式以球面座標（ λ, φ, r ）表示，則爲：

$$(6) \frac{1}{\rho} \frac{d\rho}{dt} + \frac{1}{r \cos \varphi} \frac{\partial u}{\partial \lambda} + \frac{1}{r \cos \varphi} \frac{\partial}{\partial \varphi} (v \cos \varphi) + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 w) = 0$$

其中 u, v, w 分別爲沿著 λ, φ 和 r 之速度分量。

§ 1.7 擴散方程式 (Equation of Diffusion)

連續方程式（1.6.2）是應用一個體積元素的全部質量。對於液體混合物（即海水）中之每一種成分，其類似的關係，也應該成立。而且海水中溶解的物質（鹽度），其成分間之比例關係一定。

令 S = 比鹽度 (specific salinity, 即每克海水中所含鹽之克數，即鹽度 $S = 10^3 S$)。

則 ρS = 每單位體積中溶解鹽之質量

而體積 V 中，所含鹽之部分質量 (partial mass) 爲：

$$\iiint \rho S dV$$

在此體積內質量之會有改變，不僅是由於流速 $\vec{v}$ 穿過體積表面之質量流量率所造成；而且亦由於不規則的分子運動所造成。當濃度是位置之函數時，則如此之質量通量 (mass flux) 總會發生。

如果所考慮之體積內，與外界的濃度不等，則分子不規則運動，

越過界面，就會使界面二邊之濃度變化。此種形式之質量流 (mass flow)，就稍為擴散 (diffusion)，而此種過程，就會使兩邊之濃度差減少。因此，由於通量 $-\oint \vec{S} \cdot d\vec{\sigma}$ 經過表面，我們在此體

積內，就有部分質量之增加；在此 $\vec{S}$ = 濃度通量密度 (concentration flux density，其因次為 $[\frac{g}{cm^2 sec}]$)，於是由 (1.4.1)，我們就有部分質量 ρS 之不滅定律 (conservation law for the partial mass ρS) 為：

$$(1) \iiint \frac{\partial \rho S}{\partial t} dV = - \oint \rho S \vec{v} \cdot d\vec{\sigma} - \oint \vec{S} \cdot d\vec{\sigma}$$

其右邊 $\oint (\rho S \vec{v} + \vec{S}) \cdot d\vec{\sigma}$ = 經過表面 σ 之所有鹽的分量 (partial mass of salts)，而

$\oint [\rho (1 - S) \vec{v} - \vec{S}] \cdot d\vec{\sigma}$ = 純水之輸送量 (transport of pure water)。所以上二式之和：

$$\oint \rho \vec{v} \cdot d\vec{\sigma}$$

就如同針對全部質量不滅定律 (即連續方程式) 所要求的一樣。

利用 Gauss 定理，則(1)式成為：

$$\begin{aligned} \iiint \frac{\partial \rho S}{\partial t} dV &= - \oint (\rho S \vec{v} + \vec{S}) \cdot d\vec{\sigma} \\ &= - \iiint \nabla \cdot (\rho S \vec{v} + \vec{S}) dV \end{aligned}$$

即可得溶解鹽分量之連續方程式 (equation of continuity for the partial mass of dissolved salts) :

$$(2) \frac{\partial \rho S}{\partial t} + \nabla \cdot \rho S \vec{v} = - \nabla \cdot \vec{S}$$

$$\text{或 } S \frac{\partial \rho}{\partial t} + \rho \frac{\partial S}{\partial t} + S \nabla \cdot \rho \vec{v} + \rho \vec{v} \cdot \nabla S = - \nabla \cdot \vec{S}$$

$$\rho \left(\frac{\partial S}{\partial t} + \vec{v} \cdot \nabla S \right) + S \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{v} \right) = - \nabla \cdot \vec{S}$$

最後可得：

$$(3) \frac{\partial S}{\partial t} + \vec{v} \cdot \nabla S = - \frac{1}{\rho} \nabla \cdot \vec{S}$$

如果介質中，其鹽度和溫度之分佈是不均勻的，則濃度通量密度 $\vec{S}$ ，就依賴鹽度場，壓力場和溫度場等之梯度而定。但通常可令

$$(4) \vec{S} = - k \nabla S$$

即濃度通量密度 $\vec{S}$ ，其方向與濃度梯度相反（即指向濃度較低之地方）。

將(4)代入(3)，得：

$$\frac{\partial S}{\partial t} + \vec{v} \cdot \nabla S = - \frac{1}{\rho} \nabla \cdot (-k \nabla S) = \frac{k}{\rho} \cdot \nabla^2 S$$

令 $K_0 = \frac{k}{\rho}$ = 擴散係數 (diffusion coefficient , [cm^2 / sec])，將上式乘以 10^3 （因鹽度 $S = 10^3 S$ ），則得：

$$(5) \frac{\partial S}{\partial t} + \vec{v} \cdot \nabla S = k_D \nabla^2 S, \quad k_D = \frac{k}{\rho}$$

這就是鹽度 S 之擴散方程式。

擴散係數 k_D 在 15°C 之值為 $1.1 \times 10^{-5} \text{ cm}^2 / \text{sec}$ 。

第二章 絕對參考座標上 之流體動力方程式

底下各節就要討論流體運動方程式，或稱流體之動量不變定律究竟如何導出。

§ 2 . 1 一般的力場(應力張量)

我們首先研究流體的平衡條件。我們知道剛體 (rigid body) 之平衡條件為：

$$(1) \sum_n \vec{K}_n = 0 \quad \text{及} \quad \sum_n \vec{r}_n \times \vec{K}_n = 0$$

即外力之和為零，及角動量之和為零。這些條件，對於流體中，每一個體積元素也必須適用。

在流體中，我們必須區別體積力 (volume force) 和表面力 (surface force)：

1) 體積力 $\vec{K}_v$ ：其大小與體積成正比。

體積力都是物體力 (body force)，它們的作用，能達到相當遠之距離；並且其隨距離之變化相當和緩。這種力能穿過所有的介質；因為它們隨距離的變化和緩，因此，在一個很小的體積元素中，它們可視為幾乎是常數。

例如重力，引潮力等。如令：

$\vec{k}(r)$ = 力密度 (force density , 單位體積之力)
則正比於體積之力 (如重力) 爲 :

$$(1) d\vec{K}_V = \vec{k}(r) dV$$

2) 表面力 $\vec{K}_F$: 正比於表面積 ; 它們是爲接觸力 (contact force) , 這些力 , 是由於分子間之關係而來 ; 因此 , 它們之特徵是隨距離增加而迅速減少。因此 , 它們僅作用於距離約爲介質分子間之長度的液體元素 ; 即當液體元素有一共同之接觸面 (contact surface) 。

這些接觸力能透過之深度 , 實際上僅限於共同表面兩邊之薄層而已。因此 , 由一個液體元素 , 作用於另一個之力 , 就僅依賴共同表面之面積 , 而與流體之體積無關。也就是 , 這些力是正比於表面積。它們包含分子間之力 (inter-molecular force) 穿過表面分子交換所引起之動量輸送 (molecular momentum transport) , 因爲我們所考慮的是一種連續的介質 , 而不是各個分子之運動 ; 因此 , 我們的興趣 , 就僅在於上述兩種過程之效應而已。

當介質受到分子的動量輸送和分子間之力等的作用時 , 此介質就稱爲是處在一種應力的狀態 (in a state of stress) 。

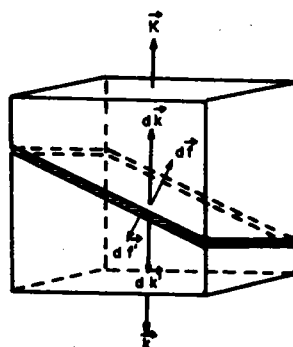
如果流況保持一定 , 那麼分子過程之效應就和靜止的狀態一樣。因此 , 介質的應力狀態 , 在這二種情形中 , 是相同的。但是 , 如果流場隨位置而變 , 則相鄰的體積元素 , 將以不同的速度流動 , 那麼穿過任何假想面之交換 , 就非常地依賴流場差異 (即切

變，shear)而定了。

這也就表示，應力狀態，實質上受到流場之空間分佈的影響 (§ 2 . 4 中，我們將會討論切變力，shearing force)。

現在，我們考慮在一種應力狀態中，所產生之力。如果，我們把一個受到應力作用的液體元素，分成二塊任意形式之部分 (下圖)，那麼，這些作用力，就可以明顯地看出來。

圖 2 . 1 接觸力 (contact force)。輔助的接觸力 $d\vec{K}$ 和 $d\vec{K}'$ 被施用以保持體積不變，如同無力作用之體積一樣。



爲了使這二個部分，保持它們以前原來的形狀 (即分開之前)，我們利用二個輔助 (假想的) 接觸力， $d\vec{K}$ 和 $d\vec{K}'$ ，它們作用在分界面上；它們的大小等於此流體沒有分開之前，作用在分界面上各部分之力，但方向相反。

分界面愈大，這個力就愈大，即此力正比分界面之面積。因此，這個力的 K 分量，可以被假定爲正比於表面積，即使一般而言，此力並不會與此面之垂直向量一致。

我們令

$$(3) dK_k = df_i P_{ik}$$

在此 P_{ik} 爲比例因子。 P_{ik} 也許會隨位置 $\vec{r}$ 而變；即爲位置之函數， $P_{ik} = P_{ik}(\vec{r})$ 。由(3)，我們可以看出， P_{ik} 就是在表面元

素 ($d\vec{f}$) 之 i 分量上 ($\because df_i = d\vec{f} \cdot \vec{e}_i$)，每單位面積作用力之 k 分量。

$$dK_k \vec{e}_k = df_i P_{ik} \vec{e}_k = d\vec{f} \cdot \vec{e}_i P_{ik} \vec{e}_k$$

故得：

$$(4) d\vec{K}_F = d\vec{f} \cdot \Pi(\vec{r}) \quad , \quad \Pi = P_{ik} \vec{e}_i \vec{e}_k$$

其中 Π 爲應力張量 (stress tensor)，其分量爲 P_{ik} 。(1)式之平衡條件，對於一個任意的體積，可以寫成：

$$(5) \iiint \vec{k} dV + \oint d\vec{f} \cdot \Pi = 0$$

$$\iiint dV \vec{r} \times \vec{k} - \oint d\vec{f} \cdot \Pi \times \vec{r} = 0$$

利用 Gauss 定理，可將(5)式之面積分，改爲體積分，即

$$(6) \iiint (\vec{k} + \nabla \cdot \Pi) dV = 0$$

$$\iiint [\vec{r} \times \vec{k} - \nabla \cdot (\Pi \times \vec{r})] dV = 0$$

因爲(6)式之關係，與任何體積之形式無關，故 (被積分函數) $\equiv 0$ 即：

$$(7) \vec{k} + \nabla \cdot \Pi = 0 \quad , \quad \vec{r} \times \vec{k} - \nabla \cdot (\Pi \times \vec{r}) = 0$$

(7)式表示外力與內應力間之平衡。

將 (7a) 之 $\vec{k}$ 代入 (7b)，得：

$$(8) \vec{r} \times (\nabla \cdot \Pi) + \nabla \cdot (\Pi \times \vec{r}) = 0$$

$$\text{因 } \nabla \cdot (\Pi \times \vec{r}) = (\nabla \cdot \overset{\downarrow}{\Pi}) \times \vec{r} + \nabla \cdot (\Pi \times \overset{\downarrow}{\vec{r}})$$

$$= -\vec{r} \times (\nabla \cdot \vec{\pi}) + \nabla \cdot (\pi \times \vec{r})$$

代入(8)式，得：

$$(9) \nabla \cdot (\pi \times \vec{r}) = 0$$

符號 $\downarrow$ 表示運算子 (∇) ，運用到此函數而已。

將(9)式寫成分量，得：

$$\begin{aligned} (10) \frac{\partial}{\partial x_i} \vec{e}_i \cdot (\vec{e}_k P_{kl} \vec{e}_l \times \vec{x}_m \vec{e}_m) \\ = \frac{\partial}{\partial x_i} P_{il} \vec{x}_m \varepsilon_{lmn} \vec{e}_n = 0 \end{aligned}$$

但 $\frac{\partial x_m}{\partial x_i} = \delta_{mi}$ ，因此(10)式變為 $P_{mi} \varepsilon_{lmn} \vec{e}_n = 0$

因 $\varepsilon_{lmn} = -\varepsilon_{mln}$ ，故得：

$$(11) \frac{1}{2} \varepsilon_n \varepsilon_{lmn} (P_{ml} - P_{lm}) = 0$$

$$P_{ml} = P_{lm}, \quad \pi = \widetilde{\pi}$$

上式表示應力張量是對稱的。因此它可被視為同向部分 (isotropic part) 與非同部分 (non-isotropic part) 之和，
即：

$$\begin{aligned} (12) P_{ik} &= \frac{1}{3} P_{ll} \delta_{lm} \delta_{ik} + (P_{ik} - \frac{1}{3} P_{ll} \delta_{lm} \delta_{ik}) \\ &= \frac{1}{3} P_{ll} \delta_{ik} + (P_{ik} - \frac{1}{3} P_{ll} \delta_{ik}) \end{aligned}$$

其中 $\frac{1}{3} P_{ll} = \frac{1}{3} (P_{11} + P_{22} + P_{33})$

$$\delta_x p = g \bar{\rho}^x \delta_x \zeta + \frac{1}{2} \bar{h}^x \delta_x \rho + \delta_x p_a \quad \text{at } i, j + \frac{1}{2}, l, n+1 \quad (32)$$

$$\delta_x (\delta_x p) = g \delta_x \bar{\rho}^x \quad \text{at } i + \frac{1}{2}, j, k + \frac{1}{2}, n+1 \quad (33)$$

$$\delta_x (\delta_x p) = g \delta_x \bar{\rho}^x \quad \text{at } i, j + \frac{1}{2}, k + \frac{1}{2}, n+1 \quad (34)$$

然後，計算水平方向的流速變化；

$$\begin{aligned} \overline{\delta_x (\bar{h}^x u)}' = & -\delta_x (\bar{h}^x u^x \bar{u}^x) - \delta_x (\bar{h}^x v^x \bar{u}^x) - \bar{h}^x \delta_x (\bar{u}^x \bar{w}^x) \\ & + f \bar{h}^x \bar{v}^{xy} - \frac{1}{\bar{\rho}^x} \bar{h}^x \delta_x p + \frac{1}{\bar{\rho}^x} [\theta \bar{\rho}_a w_a^2 \sin \phi \\ & - (E_x \delta_x \bar{u}^{2x})_{k=3/2} + \delta_x (h A_x \delta_x u) - \\ & + \delta_x (\bar{h}^x \bar{A}_x^x \delta_x u) -] \quad \text{at } i + \frac{1}{2}, j, l, n \quad (35) \\ \overline{\delta_x (\bar{h}^x v)}' = & -\delta_x (\bar{h}^x u^x \bar{v}^x) - \delta_x (\bar{h}^x v^x \bar{v}^x) - \bar{h}^x \delta_x (\bar{v}^x \bar{w}^x) \\ & - f \bar{h}^x \bar{u}^{xy} - \frac{1}{\bar{\rho}^y} \bar{h}^y \delta_y p + \frac{1}{\bar{\rho}^y} [\theta \bar{\rho}_a w_a^2 \cos \phi \\ & - (E_y \delta_y \bar{v}^{2y})_{k=3/2} + \delta_x (\bar{h}^y \bar{A}_x^y \delta_x v) - \\ & + \delta_y (h A_y \delta_y v) -] \quad \text{at } i, j + \frac{1}{2}, l, n \quad (36) \end{aligned}$$

以上兩式為表面層的動力平衡式，其中風的剪力係數（ θ ），為當地、當時海面溫度（模式計算）與空氣溫度的函數，由表一及表二中可以看出，當空氣溫度高於海面溫度時，水面的摩擦係數與波浪高度稍低於無溫差時。反之，如果空氣的溫度低於水面的溫度，其摩擦係數反較無溫差時為大。因此同樣的風速可產生不一樣高度的暴潮與氣象潮。

這個基本原理，非但在大氣中如此，在垂直方向有「溫差」的海

$$(15) \vec{k} + \nabla \cdot \pi = \vec{k} - \nabla P = 0$$

在理想流體中，應力張量（ π ）之發散（divergence）等於負的壓力梯度。而(15)式之平衡條件，即是要求力密度與壓力梯度相互平衡。

(二)黏滯性流體：

在黏滯性流體中，應力張量含有非同向部分（non-isotropic）和同向部分（isotropic）。習慣上，同向部分被視為具有如同流體靜止中應力張量之形式，即 $P_{ii} = -3p$ 。而非同向部分，即對於 $i \neq k$ ， $\vec{v} \neq \text{const}$ 時， $P_{ik} \neq 0$ ，同時 $P_{ii} \neq -3p$ 。也就是說，非同向部分對於對角線之元素仍有貢獻。但是，它們的和是為零，切應力代表一種動量的輸送，它相當於內摩擦。

由(12)式，可得：

$$(16) P_{ik} = -P\delta_{ik} + (P_{ik} + P\delta_{ik}) \equiv -P\delta_{ik} + \overset{\circ}{P}_{ik}$$

$$\pi = -PI + \overset{\circ}{\pi}$$

$$\text{或 } \pi = -pI + \overset{\circ}{\pi}$$

其中 $\overset{\circ}{\pi}$ 是摩擦張量（friction tensor），代表黏滯性流體之特徵。其分量為：

$$(17) \overset{\circ}{P}_{ik} = P_{ik} + p\delta_{ik}$$

$\overset{\circ}{\pi}$ 含有正應力和切應力。但正應力之和為零，即

$$\text{tr } \overset{\circ}{\pi} = 0$$

最後，我們結論如下：

- 1) 應力張量 π 是對稱的。
- 2) 一個流體質點是處於平衡中，當

$$(18) \vec{k} + \nabla \cdot \pi = 0$$

$$(19) \vec{k} - \nabla p = 0 \quad \text{代表理想流體之力平衡。}$$

$$(20) \vec{k} - \nabla p + \nabla \cdot \overset{\circ}{\pi} = 0 \quad \text{代表黏滯性流體之力平衡。}$$

§ 2 . 2 一般的運動方程式 (動量不減定律)

在 (1 . 4 . 5) 中：

$$\iiint \rho \frac{d\phi}{dt} dV = \iiint q_{\phi} dV$$

$$\text{如令 } \phi = \frac{\text{動量}}{\text{單位質量}} = \vec{v}$$

則我們可得動量不變定律 (conservation law for momentum) 。

$$\text{而 } q_{\phi} = \frac{\text{動量}}{(\text{單位時間})} \frac{\text{量}}{(\text{單位體積})}$$

可由物體力或表面力得出。根據 § 2 . 1 , 這二種力分別爲：

$$\iiint \vec{k} dV \quad \text{和} \quad \oint \pi \cdot d\vec{\sigma}$$

故由上式 (1 . 4 . 5) 得：

$$\begin{aligned} \iiint \frac{d\vec{v}}{dt} \rho dV &= \iiint \vec{k} dV + \oint \pi \cdot d\vec{\sigma} \\ &= \iiint \vec{k} dV + \iiint \nabla \cdot \pi dV \end{aligned}$$

$$\text{故 (1) } \iiint \frac{d\vec{v}}{dt} \rho dV = \iiint (\vec{k} + \nabla \cdot \Pi) dV$$

$$\text{或 (2) } \rho \frac{d\vec{v}}{dt} = \vec{k} + \nabla \cdot \Pi, \quad \rho \frac{dv_j}{dt} = k_j + \frac{\partial P_{ij}}{\partial x_i}$$

$$\text{由 } \frac{d\phi}{dt} = \frac{\partial \phi}{\partial t} + \vec{v} \cdot \nabla \phi = \frac{\partial \phi}{\partial t} + v_k \frac{\partial \phi}{\partial x_k}$$

$$\text{得 (3) } \rho \left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} \right) = \vec{k} + \nabla \cdot \Pi$$

$$\rho \left(\frac{\partial v_j}{\partial t} + v_i \frac{\partial v_j}{\partial x_i} \right) = k_j + \frac{\partial P_{ij}}{\partial x_i}$$

(2)和(3)爲一般的流體運動方程式。

$$\text{由 (2.1.16) } \Pi = -pI + \overset{\circ}{\Pi}$$

$$\text{得 (4) } \rho \left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} \right) = \vec{k} - \nabla p + \nabla \cdot \overset{\circ}{\Pi}$$

$$\rho \left(\frac{\partial v_j}{\partial t} + v_i \frac{\partial v_j}{\partial x_i} \right) = k_j - \frac{\partial p}{\partial x_j} + \frac{\partial \overset{\circ}{P}_{ij}}{\partial x_i}$$

由 (1.4.2)

$$\iiint \frac{\partial \phi \rho}{\partial t} = - \iiint \nabla \cdot \phi \rho \vec{v} dV + \iiint q_\phi dV$$

可得 (此處令 $\phi \equiv \vec{v}$)

$$\begin{aligned} \text{(5) } \iiint \frac{\partial \rho \vec{v}}{\partial t} dV &= - \iiint \nabla \cdot \rho \vec{v} \vec{v} dV \\ &+ \iiint (\vec{k} + \nabla \cdot \Pi) dV \end{aligned}$$

$$\text{或 (6) } \frac{\partial \rho \vec{v}}{\partial t} + \nabla \cdot \rho \vec{v} \vec{v} = \vec{k} + \nabla \cdot \Pi$$

$$\frac{\partial \rho v_j}{\partial t} + \frac{\partial \rho v_i v_j}{\partial x_i} = k_j + \frac{\partial P_{ij}}{\partial x_i}$$

(6)式為運動方程式之動量通量形式 (momentum flux form of the equation of motion)。

在此我們定義動量通量張量 (momentum flux tensor) 為：

$$(7) \Pi^* = \rho \vec{v} \vec{v} - \Pi$$

在(5)式中，令 $\vec{k} = 0$ ，則由 Gauss 定理，得

$$(8) \iiint \frac{\partial \rho \vec{v}}{\partial t} dV = - \oint (\rho \vec{v} \vec{v} - \Pi) \cdot d\vec{\sigma}$$

由(8)式，可見其右邊含有應力張量 Π 之分子的動量輸送 (molecular momentum transport)。而 $(\rho \vec{v} \vec{v} - \Pi)$ 即為穿過表面 σ 之全部的動量輸送。

§ 2.3 理想流體的運動方程式 (Euler 運動方程式)

在理想流體中，摩擦張量 $\Pi = 0$ ，則 (2.2.4) 成為：

$$(1) \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = \frac{1}{\rho} (\vec{k} - \nabla p)$$

$$\frac{\partial v_j}{\partial t} + v_i \frac{\partial v_j}{\partial x_i} = \frac{1}{\rho} (k_j - \frac{\partial p}{\partial x_j})$$

上式為理想流體之運動方程式。在這兒，慣性力、外力及壓力梯度力互相平衡。

§ 2.4 分子的黏滯係數 (Molecular Viscosity Coefficient)

現在我們要考慮在一般的運動方程式中 $\nabla \cdot \overset{\circ}{\Pi}$ 這一項。在 (2.1.17) 中之摩擦張量的分量， $\overset{\circ}{P}_{ik} = P_{ik} + p\delta_{ik}$ ，與 (1.2.7) 應變張量之非發散部分， $\overset{\circ}{D}_{ik} = D_{ik} - D_{lm}\delta_{lm}\delta_{ik}$ ，兩者之間，我們可以假定有個一般性之關係存在。

因為應力 $\overset{\circ}{P}_{ik}$ 是由於不同速度之流體質點，在黏滯的流體中，互相作用之剪力 (shearing force) 所造成的。

此種剪力是由於流體元素間分子的動量輸送所引起的。因此應變張量中之剪變項 (shearing term)，應該可以用以描述切應力 (tangential stress)。而此剪變項即 $\overset{\circ}{D}$ 的部分。我們知道 (§ 1.2)

$$\begin{aligned}\nabla \cdot \vec{v} &= \frac{1}{2}(\nabla \cdot \vec{v} + \vec{v} \cdot \nabla) + \frac{1}{2}(\nabla \cdot \vec{v} - \vec{v} \cdot \nabla) \\ &= \frac{1}{2}(D + D')\end{aligned}$$

$$D = \left(\frac{\partial v_k}{\partial x_i} + \frac{\partial v_i}{\partial x_k} \right) \vec{e}_i \vec{e}_k$$

$$\text{tr } D = D_{ii} = 2 \frac{\partial v_i}{\partial x_i} = 2 \nabla \cdot \vec{v}$$

$$D = \text{tr } D I + \overset{\circ}{D}, \quad \overset{\circ}{D} = D - \text{tr } D I$$

剪變項與切應力間最簡單之關係為正比例，即

$$\overset{\circ}{\Pi} \sim \overset{\circ}{D}, \quad \text{或以分量表示 } \overset{\circ}{P}_{ik} \sim \overset{\circ}{D}_{ik}$$

假定有二個不同之比例因子，則我們可寫成：

$$(1) \overset{\circ}{P}_{ik} \equiv P_{ik} + p \delta_{ik} = \mu D_{ik} + \frac{\mu'}{2} D_{ii} \delta_{ik}$$

由上式可知：

$$(2) P_{ik} = \mu D_{ik} \quad \text{對於 } i \neq k$$

即切應力正比於剪變。

$$\therefore \frac{1}{2} D_{ii} = \nabla \cdot \vec{v} \quad (\delta_{ii} = 3)$$

$$(3) P_{ii} + 3p = \mu D_{ii} + \frac{3}{2} \mu' D_{ii}$$

$$\text{而 } P_{ii} = -3p$$

即上式左邊 = 0，可得：

$$(4) \mu' = -\frac{2}{3} \mu$$

因此，最後可得，（將(4)式代入(1)式）

$$\overset{\circ}{P}_{ik} = \mu D_{ik} + \left(-\frac{2}{3} \mu \right) \cdot \frac{1}{2} D_{ii} \delta_{ik}$$

亦即

$$(5) \overset{\circ}{P}_{ik} = \mu \left(D_{ik} - \frac{1}{3} D_{ii} \delta_{ik} \right)$$

$$\text{或 } \overset{\circ}{\Pi} = \mu \left(D - \frac{1}{3} \text{tr } D I \right)$$

其中 μ 即是黏滯係數，其因次為：

$$[\mu] = [\overset{\circ}{\Pi}] / [D] = [\text{g cm}^{-1} \text{sec}^{-1}]$$

$$\therefore [\overset{\circ}{\Pi}] = [\text{force} / \text{surface}] = \text{g cm}^{-1} \text{sec}^{-2}$$

$$[D] = [\text{velocity} / \text{lenght}] = \text{sec}^{-1}$$

有時，我們常使用運動黏滯係數 (coefficient of kinematic viscosity) 。

$$\nu = \frac{\mu}{\rho} \quad , \text{其因次爲 } [\text{cm}^2 \text{sec}^{-1}]$$

在不同的溫度之 μ 和 ν 之值如下表：

表 2 . 1 空氣和水之黏滯係數

Air			
T [°C]	ρ [g cm ⁻³]	μ [g cm ⁻¹ sec ⁻¹]	ν [cm ² sec ⁻¹]
0	1.293×10^{-3}	1.71×10^{-4}	0.132
10	1.247	1.76	0.141
20	1.205	1.81	0.150

Water			
T [°C]	ρ [g cm ⁻³]	μ [g cm ⁻¹ sec ⁻¹]	ν [cm ² sec ⁻¹]
0	0.999	1.787×10^{-2}	1.787×10^{-2}
10	0.999	1.304	1.304
20	0.998	1.002	1.004

由(5)和 (2 . 1 . 16) 應力張量之形式爲：

$$(6) \Pi = -pI + \mu \left(D - \frac{1}{3} \text{tr} D I \right)$$

$$P_{ik} = -p \delta_{ik} + \mu \left(D_{ik} - \frac{1}{3} D_{uu} \delta_{ik} \right)$$

§ 2.5 Navier-Stokes 運動方程式

如今，在一般的方程式 (2.2.4) 中， $\nabla \cdot \overset{\circ}{\Pi}$ 此項可由 (2.4.5) 代替，就得黏滯性流體之運動方程式：

$$(1) \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = \frac{1}{\rho} (\vec{k} - \nabla p + \nabla \cdot \mu D - \nabla \cdot \frac{\mu}{3} \text{tr} D I)$$

或以分量表示：

$$(2) \frac{\partial v_j}{\partial t} + v_i \frac{\partial v_j}{\partial x_i} = \frac{1}{\rho} (k_j - \frac{\partial p}{\partial x_j} + \frac{\partial \mu D_{ij}}{\partial x_i} - \frac{1}{3} \frac{\partial \mu D_{ii}}{\partial x_j})$$

在(2)式中

$$(3) D_{ik} = \frac{\partial v_k}{\partial x_i} + \frac{\partial v_i}{\partial x_k}$$

如 $\mu = \text{const}$

因爲 $\nabla \cdot D = \nabla \cdot (\nabla v + v \overleftarrow{\nabla}) = \nabla^2 v + \nabla \nabla \cdot v$

$$\nabla \cdot (\text{tr} D I) = \nabla \cdot (2 \nabla \cdot v I) = 2 \nabla \nabla \cdot v$$

故(1)成爲：

$$(4) \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = \frac{1}{\rho} (\vec{k} - \nabla p + \mu \nabla^2 \vec{v} + \frac{\mu}{3} \nabla \nabla \cdot \vec{v})$$

這就是 Navier-Stokes 運動方程式。而

$$(5) \frac{\mu}{\rho} \nabla^2 v + \frac{1}{3} \frac{\mu}{\rho} \nabla \nabla \cdot v$$

就是 Navier-Stokes 方程式中之摩擦項。

$$\because \nabla^2 v = \nabla \nabla \cdot v - \nabla \times (\nabla \times v)$$

右邊第一項代表壓縮之效應，第二項代表轉動之效應。因此，發散項

僅在可壓縮的流體中出現。這是因為 ($\nabla \cdot \mathbf{v} \neq 0$) 之緣故。而一種不轉動又不可壓縮的流體，就是無黏滯性。因為不可壓縮之條件，使得：

$$\frac{d\rho}{dt} = -\rho \nabla \cdot \mathbf{v} = 0$$

故上式右邊第一項與第二項都為零，自然(5)式亦成為零。

§ 2.6 引力位 (Gravitational Potential)

在絕對坐標系統中，體積力 $\vec{K}(\vec{r})$ 之力密度為 $\vec{k}(\vec{r})$ ；此種體積力僅由引力位產生。

根據牛頓定律，在位置 $\vec{r}=0$ 之質量 M ，能對位置為 $\vec{r}$ 之質量 m ，施以引力 $\vec{K}(\vec{r})$ ，其大小為：

$$(1) K(r) = \frac{\gamma mM}{r^2}$$

或以向量表示，則為：

$$\vec{K}(r) = -K(r) \frac{\vec{r}}{r} = -\frac{\gamma mM}{r^2} \frac{\vec{r}}{r}$$

其中萬有引力常數：

$$\gamma = 6.674 \times 10^{-8} \text{ g}^{-1} \text{ cm}^3 \text{ sec}^{-2}$$

因 $\vec{K}(\vec{r})$ 為中心力 (central force)，根據 (1b)，它指向中心 $\vec{r}=0$ ，因此可以定義一個位能 (potential) 為：

$$(2) \Phi(r) = - \int_{r_0}^r \vec{K}(r) \cdot d\vec{r} = \int_{r_0}^r K(r) \frac{\vec{r}}{r} \cdot d\vec{r}$$

$$= \int_{r_0}^r K(r) dr = \Phi(r) = -\frac{\gamma m M}{r}$$

其中 $\Phi(r_0 = \infty) = 0$

反過來，由(2)及下式可求出 $\vec{K}$ ，即

$$(3) \nabla \Phi = -\vec{K}(\vec{r})$$

這個向量 $\nabla \phi$ 使我們能在任意點上，定出場線 (field lines) 之方向。

如果質量 m ，受到一組質量 M_i (其位置為 $\vec{r}_i$) 之作用，則

$$(4) \Phi(r) = \sum_i \Phi_i = - \sum_i \frac{\gamma m M_i}{|\vec{r} - \vec{r}_i|}$$

$$\vec{K}(r) = \sum_i \frac{\gamma m M_i}{|\vec{r} - \vec{r}_i|^2} \frac{\vec{r} - \vec{r}_i}{|\vec{r} - \vec{r}_i|}$$

位能 $\phi(r)$ 和引力 $\vec{K}(r)$ 主要是由這些質量 M_i 來決定。當然，這些質量必須足夠大，且其距 m 之距離要很近，因為它們的影響，分別依據 $\frac{1}{r}$ 和 $\frac{1}{r^2}$ ，隨距離增加而減少。

對於海洋中之流體元素，較重要的一些質量 M_i 如下：

- 1) 環繞質量 m (即此流體元素) 之地球的全部質點 (它們是在固定的距離)。

對於海平面上之某一點，它們的全部質量 $E = 5.977 \times 10^{27} \text{ g}$ ，可以被假定集中在地心 (即重心)。根據位能理論 (potential theory)，一個球體的外殼，對於球內一點，並無引力作用。因此，對於海內，深度為 z 之一點，其作用力為：

$$\begin{aligned}
\vec{K} &= -\gamma \frac{m(E - \Delta E)}{(R - z)^2} \frac{\vec{r}}{r} \\
&= -\gamma \frac{m(E - \Delta E)}{R^2 (1 - \frac{z}{R})^2} \frac{\vec{r}}{r} \\
&= -\gamma \frac{mE}{R^2 (1 - \frac{z}{R})^2} \frac{\vec{r}}{r} + \gamma \frac{m\Delta E}{R^2 (1 - \frac{z}{R})^2} \frac{\vec{r}}{r} \\
&= -\gamma \frac{mE}{R^2} (1 - \frac{z}{R})^{-2} \frac{\vec{r}}{r} \\
&\quad + \gamma \frac{m\Delta E}{R^2} (1 - \frac{z}{R})^{-2} \frac{\vec{r}}{r} \\
&\cong -\gamma \frac{mE}{R^2} (1 + 2\frac{z}{R} + \dots) \frac{\vec{r}}{r} \\
&\quad + \gamma \frac{mE}{R^2} \frac{\Delta E}{E} (1 + 2\frac{z}{R}) \frac{\vec{r}}{r} \\
&\cong -\gamma \frac{mE}{R^2} \frac{\vec{r}}{r} - \gamma \frac{mE}{R^2} (2\frac{z}{R} - \frac{\Delta E}{E}) \frac{\vec{r}}{r} \\
&= -\gamma \frac{mE}{R^2} \frac{\vec{r}}{r} - \gamma \frac{mE}{R^2} \frac{\vec{r}}{r} 2\frac{z}{R} (1 - \frac{\Delta E}{E} \frac{R}{2z})
\end{aligned}$$

最後可得：

$$\begin{aligned}
(5) \vec{K} &= -\gamma \frac{m(E - \Delta E)}{(R - z)^2} \frac{\vec{r}}{r} \\
&\approx \vec{K}_0 + 2\vec{K}_0 \frac{z}{R} (1 - \frac{\Delta E}{2E} \frac{R}{z})
\end{aligned}$$

$$\approx \vec{K}_0 + 2 \vec{K}_0 \frac{z}{R} \left(1 - \frac{3}{2} \frac{\rho}{\rho_E} \right)$$

而 $\vec{K}_0 = -\gamma \frac{m_E}{R^2} \frac{\vec{r}}{r}$, 爲海面上之引力。

其中 R 爲地球半徑 ($\approx 6371 \text{ Km}$) , $\triangle E$ 爲深度 z 所在外圍地球外殼之質量。

$$\begin{aligned} \therefore \triangle E &= \frac{4}{3} \pi [R^3 - (R - z)^3] \rho \\ &= \frac{4}{3} \pi [R^3 - R^3 \left(1 - \frac{z}{R} \right)^3] \rho \\ &= \frac{4}{3} \pi [R^3 - R^3 \left(1 - 3 \frac{z}{R} + \dots \right)] \rho \\ &\approx 4 \pi R^2 z \rho \end{aligned}$$

如令 ρ_E = 地球的密度 ($\approx 5.51 \text{ g/cm}^3$)

ρ = 地球外殼之密度 (≈ 1.00 , 爲海水所蓋之地球)

則 $\frac{\triangle E}{E} \approx \frac{4 \pi R^2 z \rho}{\frac{4}{3} \pi R^3 \rho_E} = \frac{3 z \rho}{R \rho_E}$

即(5)式可成：

$$\begin{aligned} \vec{K} &\approx \vec{K}_0 + 2 \vec{K}_0 \frac{z}{R} \left(1 - \frac{3 z \rho}{R \rho_E} \frac{R}{2 z} \right) \\ &= \vec{K}_0 + 2 \vec{K}_0 \frac{z}{R} \left(1 - \frac{3}{2} \frac{\rho}{\rho_E} \right) \end{aligned}$$

如果我們定義引力加速度 $\vec{g}^*$ 爲：

$$g^* = |\vec{g}^*| = \frac{\text{力}}{\text{質量}} = \frac{|\vec{K}|}{m}$$

$$\text{則 (6) } g^* = \gamma \frac{E - \triangle E}{(R - z)^2} \approx g_0^* + 2.244 \times 10^{-6} z$$

$$g_0^* \approx 982.1 \quad \vec{K} = m \vec{g}^*$$

$$\vec{g}^* = -\nabla \Phi^* = -g^* \frac{\vec{r}}{r}$$

$g^* (982 \text{ cmsec}^{-2})$ 在整個海洋裏，近似地可視為常數。如果，由於地球質量分佈不均或地球形狀之緣故，使得地心與重心不重合，則此量 g^* ，須再考慮水平方向之額外變化量。

在此，我們不考慮此種情形。因此，在 (2.5.4) Navier-Stokes 運動方程式中，單位質量的力，就含有此種引力。

$$(7) \frac{1}{\rho} \vec{k} = \frac{d\vec{K}}{\rho dV} = \frac{d m \vec{g}^*}{\rho dV} = \vec{g}^* = -\nabla \Phi^*$$

即僅地球的質量構造，產生引力加速度 (gravitational acceleration)。在下一章裡，我們將考慮此種引力與離心力

$-\vec{\Omega} \times (\vec{\Omega} \times \vec{r})$ 結合，而造成實際的重力，

$-\nabla \Phi = -\nabla \Phi^* - \vec{\Omega} \times (\vec{\Omega} \times \vec{r})$ 。

2) 月球和太陽之質量：

由月球和太陽對於 m 作用之引力，可造成引潮力，我們在下節討論。

§ 2.7 潮位 (Tidal Potential)

地球質量 (E)，月球和太陽都會對於海洋的某一個質量元素施以引力。上一節我們已經討論了地球質量所作用之引力。

月球引力和太陽引力中之大部分，都為地球～月球系統，及地球～太陽系統，分別沿它們共同的重心之公轉的離心力所抵消。所餘下之力，即為引潮力 (tide-generating force) 。

我們先介紹一些符號 (看下圖) ：

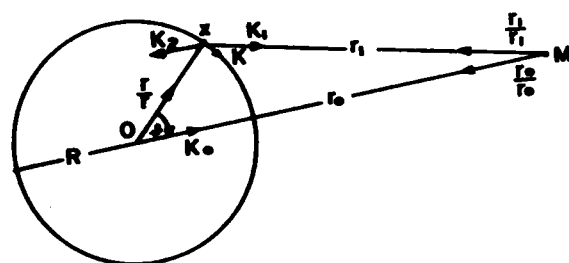


圖 2 . 2 地球上 X 點之引潮力 $\vec{K}$ ， $\vec{K}_1$ 是引力， $\vec{K}_2$ 是此二物體沿共同重心公轉所形成之離心力，而 $\vec{K}$ 就是 $\vec{K}_1$ 與 $\vec{K}_2$ 之合力。

r_0 = 地心與月球中心之距離 (約為 384.4×10^3 Km)

$\frac{\vec{r}_0}{r_0}$: 指向地球。

r_1 = 地球上一點 X，離月球中心之距離

$\frac{\vec{r}_1}{r_1}$: 指向地球。

R = 地球半徑 (6371 Km)

E = 地球質量

M = 月球質量

a = 地心與 (地球～月球系統之) 共同重心之距離

$\frac{\vec{a}}{a}$: 指向地心。

W = 地球和月球環繞共同重心旋轉之角速度

ϕ_i = 潮位 ($i = M$ 代表月球， $i = S$ 代表太陽)

地球沿本身的軸自轉，我們先加以忽略。地球～月球系統之穩定，係因它們以一定的角速度環繞其共同重心。近似看來，每一個天體都在進行公轉（revolution）之圓周運動。

由於 $\frac{E}{M} = 81.53$ 之關係，此地～月系統之共同重心，其距地心的距離比距月心的距離近了 81.53 倍，但月心距離地心大約 60.26 R，故共同重心離地心之距離為：

$$\frac{60.26 R}{1 + 81.53} = 0.73 R = 0.73 \times 6371 \text{ Km} \cong 4650 \text{ Km}$$

這表示，此共同重心位於地球內。

這二個天體，繞此重心，公轉之周期大約 27.3 天（27 天 7 小時 43 分 11.5 秒）。如下圖所示，對於地球上所有的點，它們的離心力都相同。

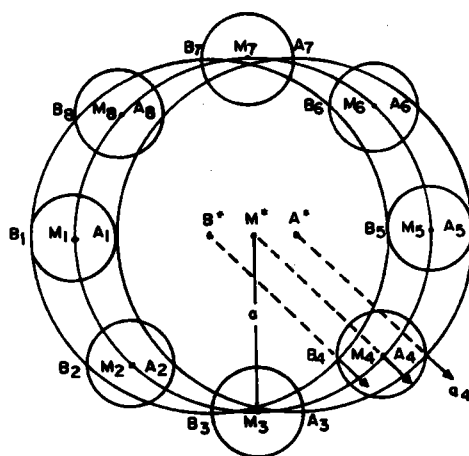


圖 2 . 3

因為僅考慮公轉，地球上每一點，其軌跡都是半徑 a 之圓；所有

的 m_a 點都位於圓心為 M^* 之圓上，所有的 A_a 點都位於圓心為 A^* 之圓上，所有的 B_a 點都位於圓心為 B^* 之圓上。

在地球上的每一個點，其單位質量之離心力，大小為：

$$Z = \omega^2 a$$

而方向為 $\vec{a}$ ，於是地球公轉（在地月系統中）全部的離心力為：

$$Z_E = E \omega^2 a$$

爲了使這個系統（即地月系統）平衡，此離心力 Z_E 必須等於月球施以地球之全部引力。因爲我們可以設想全部地球質量集中於地心，故

$$(1) - \frac{\gamma E M \vec{r}_0}{r_0^2 r_0} + E \omega^2 a \frac{\vec{a}}{a} = 0$$

對此系統必須成立。

因爲離心力對於整個地球，都是一定；但是引力却隨 $\frac{1}{r_1^2}$ 而變，因此，(1)式之平衡，在地心之外的任何點，就不能滿足了。對於這些點，就有一個單位質量（ $m=1$ ）的合成向量了（即剩餘之力）它是由當地的引力：

$$\vec{K}_1 = - \left(\gamma M / r_1^2 \right) \left(\vec{r}_1 / r_1 \right)$$

及一定的離心力：

$$\vec{K}_2 = \omega^2 a \left(\vec{a} / a \right) = - \vec{K}_0 = \left(\gamma M / r_0^2 \right) \left(\vec{r}_0 / r_0 \right)$$

兩者之和造成的。因此，我們得到：

$$(2) \vec{K} = \vec{K}_1 + \vec{K}_2 = \vec{K}_1 - \vec{K}_0 = - \gamma M \left(\frac{\vec{r}_1}{r_1^3} - \frac{\vec{r}_0}{r_0^3} \right)$$

對於 $\vec{K}_1$ ，我們定義一個位能，它依賴距離 r_1 而定，而 $\phi_1 = 0$ 對於

$r_1 = \infty$ ，即

$$\begin{aligned}(3) \Phi_1 &= - \int_{\infty}^{r_1} \vec{K}_1(\vec{r}_1) \cdot d\vec{r}_1 \\ &= \int_{\infty}^{r_1} \frac{\gamma M}{r_1^2} dr_1 = - \frac{\gamma M}{r_1}\end{aligned}$$

而每一地方， $\vec{K}_2$ 之方向為 $\vec{r}_0$ ，大小為 $\frac{\gamma M}{r_0^2}$ 。我們將它的位能，當做 x 點離地心距離之函數，（而 $\phi_2 = 0$ ，對於 $r = 0$ ）則得：

$$\begin{aligned}(4) \Phi_2 &= - \int_0^r \frac{\gamma M \vec{r}_0}{r_0^2 r_0} \cdot d\vec{r} = \int_0^r \frac{\gamma M}{r_0^2} \cos \theta dr \\ &= \frac{\gamma M r \cos \theta}{r_0^2}\end{aligned}$$

因此 $(5) \Phi_M = \Phi_1 + \Phi_2 = - \left(\frac{\gamma M}{r_1} - \frac{\gamma M r \cos \theta}{r_0^2} \right)$

∴ 我們可以加一個任意常數到一個位能上。

我們加 $\frac{+\gamma M}{r_0} = \text{const}$ 到 ϕ_M 上，如此對地心而言，（即 $r = 0$ 時， $r_1 = r_0$ ）我們就有 $\phi_M = 0$ 。

故得：

$$(6) \phi_M(r, \theta) = - \gamma M \left(\frac{1}{r_1} - \frac{1}{r_0} - \frac{r \cos \theta}{r_0^2} \right)$$

ϕ_M 就是月球對於無自轉地球之引潮力的位能。

對於太陽的引潮位，完全類似，即

$$r_1, r_0, \theta \rightarrow r_{1s}, r_{0s}, \theta_s$$

來代替。利用三角學公式：

$$r_1^2 = r_0^2 + r^2 - 2rr_0 \cos \theta$$

$$\text{得 } r_1 = r_0 \left(1 - \frac{2r}{r_0} \cos \theta + \frac{r^2}{r_0^2} \right)^{\frac{1}{2}}$$

$$\text{或 (7) } \frac{1}{r_1} = \frac{1}{r_0} \left(1 - \frac{2r}{r_0} \cos \theta + \frac{r^2}{r_0^2} \right)^{-\frac{1}{2}}$$

右邊可以展開為 $\frac{r}{r_0}$ 之冪級數 (power series)，展開的結果為

Legendre 函數：

$$(8) \frac{1}{r_1} = \frac{1}{r_0} \sum_{n=0}^{\infty} \left(\frac{r}{r_0} \right)^n P_n (\cos \theta)$$

其中：

$$(9) P_0 (\cos \theta) = 1, \quad P_1 (\cos \theta) = \cos \theta$$

$$P_2 (\cos \theta) = \frac{1}{4} (3 \cos 2\theta + 1)$$

$$\begin{aligned} P_3 (\cos \theta) &= \frac{1}{2} (5 \cos^3 \theta - 3 \cos \theta) \\ &= \frac{1}{8} (5 \cos 3\theta + 3 \cos \theta) \end{aligned}$$

$$P_4 (\cos \theta) = \frac{1}{8} (35 \cos^4 \theta - 3 \cos^2 \theta + 3)$$

因此，月球的潮位可以分解為一些位能之和，即

$$\begin{aligned} (10) \Phi_M (r, \theta) &= - \gamma M \frac{r^2}{r_0^3} \left(P_2 + \frac{r}{r_0} P_3 + \frac{r^2}{r_0^2} P_4 + \dots \right) \\ &= \Phi_{M2} + \Phi_{M3} + \Phi_{M4} + \dots \end{aligned}$$

因為 $\frac{r}{r_0} \approx 0.017$ ，因此上式級數之展式，利用前面幾項就夠了，如果我們只考慮 Φ_{M2} ，則

$$(11) \Phi_{M2}(r, \theta) = -\frac{3}{4} \gamma M \frac{r^2}{r_0^3} \left(\cos 2\theta + \frac{1}{3} \right)$$

因此，引潮力（每單位質量）之垂直分量爲：

$$K_r = -\frac{\partial \Phi_{M2}}{\partial r} = \frac{3}{2} \frac{\gamma M r}{r_0^3} \left(\cos 2\theta + \frac{1}{3} \right) \\ \cong m \left(\cos 2\theta + \frac{1}{3} \right)$$

$$(12) \text{ 其中 } m = \frac{3}{2} \frac{\gamma M R}{r_0^3}$$

而水平分量爲：

$$K_\theta = \frac{1}{r} \frac{\partial \Phi_{M2}}{\partial \theta} = \frac{3}{2} \frac{\gamma M r}{r_0^3} \sin 2\theta \approx m \sin 2\theta$$

此處垂直方向是定義在 $+\vec{r}$ 方向，而 θ 分量指向月球時爲正。垂直分量 K_r 之方向與地球之引力相反。

我們現在看看 $\frac{K_r}{g^*}$ 之比例爲多少？在海洋表面， $r = R$ ， $Z = 0$

，則由 (2.6.6)，得：

$$g^* = \frac{\gamma E}{R^2}$$

$$\text{故 } (13) \frac{m}{g^*} = \frac{\frac{3}{2} \frac{\gamma M R}{r_0^3}}{\frac{\gamma E}{R^2}} = \frac{3}{2} \frac{M R^3}{E r_0^3} \approx 8.75 \times 10^{-8}$$

$$m = \frac{3}{2} \frac{\gamma M R}{r_0^3} \quad \left(\text{以 } \frac{m}{g^*} \text{ 代替 } \frac{K_r}{g^*} \right)$$

如果 M 和 r_0 ，分別對於太陽之相當值代入，其值爲 3.78×10^{-8} 。這些量太小了，因此，垂直分量可以被忽略（與地心引力比較之關係），而水平分量之大小，雖然也與垂直分量相似，但在流體運動方

程式中，其他力量之大小與此水平分量之大小接近，因此，水平分量不可忽略。因為，海洋的深度，比起地球半徑（ R ）太小了，因此，通常我們可令 $r = R$ ，即 K_0 與深度無關。

如此，由地心向外之天頂角度（geocentric zenith distance） θ ，就成為唯一的變數。

於是由(11)式，可得：

$$(14) \Phi_i = -\frac{m_i R}{2} \left(\cos 2\theta + \frac{1}{3} \right)$$

$$K_{\theta, i} = \frac{1}{R} \frac{\partial \Phi_i}{\partial \theta} = m_i \sin 2\theta$$

其中 $i = M$ 或 S （分別為月球或太陽）

$$\text{而 } m_M = \frac{3}{2} \frac{\gamma M R}{r_0^3} = 8.75 \times 10^{-8} g^*$$

$$m_s = \frac{3}{2} \frac{\gamma S R}{r_{0s}^3} = 3.78 \times 10^{-8} g^*$$

如果地球不轉動（不自轉），則地球上每一點，一年內有二次太陽潮位引起之高潮，而在 27.3 天內有二次月球潮位引起之高潮。它們之發生是只要月球（太陽）位於它的天頂點或天底點（相對地表觀測點時）。但實際上地球會自轉，而稍微改變這些引潮力。我們現在再考慮地月系統，而忽略地月系統繞太陽之運動。

在轉動的地球上，一個觀測點 X ，月球天頂角以周期 24 小時，而改變，然而當月球本身轉動方向與地球相同時，月球經過一特定子午線二次之間的時間要遲 50.47 分。因此，一個月球日是 24 小時 50.47 分。我們現在，要將天頂角 θ ，表示為地理上之坐標 φ （緯度）， λ （經度）， t （時間）和 δ （月球之赤緯，declination of the moon）之函數。即 $\theta = \theta(\varphi, \lambda, t, \delta)$

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則(14)式之月球潮位，可成爲：

$$(16) \Phi_M = -\frac{m_M R}{2} \left[\frac{1}{3} (1 - 3 \sin^2 \varphi) (1 - 3 \sin^2 \delta) \right. \\ \left. + \sin 2\varphi \sin 2\delta \cos H + \cos^2 \varphi \cos^2 \delta \cos 2H \right]$$

∴ 如果令 $\lambda =$ 觀測點 X 之經點

$\sigma =$ 月球之角速度 ($^\circ / h$)

$\alpha =$ 相位

則得 $H = \sigma t + \lambda + \alpha$

因此，(16)式中之月球（或太陽）潮位，可以變成：

$$(17) \left\{ \begin{aligned} \Phi_i (\varphi, \lambda, t) &= -\frac{m_i R}{2} \left[\frac{1}{3} (1 - 3 \sin^2 \varphi) (1 - 3 \sin^2 \delta_i) \right. \\ &\quad \left. + \sin 2\varphi \sin 2\delta_i \cos (\sigma_i t + \lambda + \alpha_i) \right. \\ &\quad \left. + \cos^2 \varphi \cos^2 \delta_i \cos (2\sigma_i t + 2\lambda + 2\alpha_i) \right] \\ \Phi_G &= \sum \Phi_i \end{aligned} \right.$$

其中 $i = M$ 或 S

1) 第一項， $\frac{1}{3} (1 - 3 \sin^2 \varphi) (1 - 3 \sin^2 \delta_i)$ 與緯度 (λ) 無關。

節線： $1 - 3 \sin^2 \varphi = 0$ ，即 $\sin^2 \varphi = \frac{1}{3}$ ， $\sin \varphi = \pm \sqrt{\frac{1}{3}}$

或 $\varphi = \pm 35^\circ 16'$

極（大，小）值： $\because 0 \leq \sin^2 \varphi \leq 1$

$\therefore \sin^2 \varphi = 0$ 或 $\sin^2 \varphi = 1$

即 $\varphi = 0^\circ$ ， $\varphi = \pm 90^\circ$

這一項，似乎與時間無關。它和時間之關係，是含在 δ_i 內，即黃道和月球軌道之赤緯。

平均說來，月球軌道相對於黃道的交角為 $5^{\circ}8.7'$ ；而其變動約在 $4^{\circ}59'$ 與 $5^{\circ}18'$ 之間。如忽略其變動，而將月球軌道與赤道間之傾斜角取為 $\epsilon_M = 28^{\circ}30'$ （即 $23^{\circ}17' + 5^{\circ}8.7' \approx 28^{\circ}30'$ ）。

對於球面三角形 ABM ，

可得：

$$\sin \delta_M = \sin \epsilon_M \sin b$$

令 h_M = 月球現在位置相對於地球之平均經度

而 $b \approx h_M$

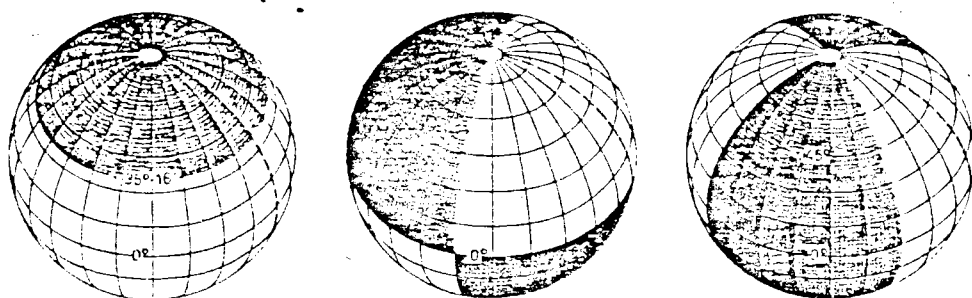
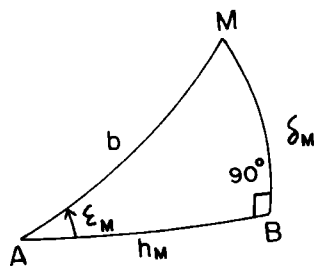


圖 2 . 5 潮位之地理分佈。左：長周期潮位。中：一日周期潮位。右：半日周期潮位。虛線：月球所在之子午線。陰影點區域： $\phi < 0^{\circ}$ 。

故 (18) $\sin \delta_M = \sin \epsilon_M \sin h_M$

而且

$$\begin{aligned} (19) \quad 1 - 3 \sin^2 \delta_M &= 1 - 3 \sin^2 \epsilon_M \sin^2 h_M \\ &= 1 - \frac{3}{2} \sin^2 \epsilon_M + \frac{3}{2} \sin^2 \epsilon_M \cos 2h_M \end{aligned}$$

$$= 0.658 + 0.342 \cos 2h_M$$

所以（月球潮位）第一項含有與時間無關的項（ $1 - 3 \sin^2 \varphi$ ）以及一項長周期的部分。（即18式），此周期係由於 $2h_M$ 之關係。

令 周期為 T ，（ $T = \frac{2\pi}{2h_M}$ ）

則 $2T =$ 月繞其軌道一周之時間 $\cong 27.3$ 天

即 $T \approx 14$ 天

即 第一項 = 一定的月球潮 M_0 + 半月潮（lunar fortnightly tide）

同理，對於太陽潮位，亦可得：

$$\begin{aligned} (20) \quad 1 - 3 \sin^2 \delta_s &= 1 - \frac{3}{2} \sin^2 \epsilon_s + \frac{3}{2} \sin^2 \epsilon_s \cos 2h_s \\ &= 0.762 + 0.237 \cos 2h_s \end{aligned}$$

即 它含有一定的太陽潮 S_0 及半年潮（solar semi-annual tide）

除了緯度 $\varphi = \pm 35^\circ 16'$ 之外，這些潮汐之振幅，在地表上，沒有一個地方會等於零；因為一般而言， $\sin^2 \delta_i \approx 1/3$ ， M_0 和 S_0 ，因此，可以加到地球引力位上，當做與時間無關之位能。在極地，它們使海平面下降約 20 公分，而在赤道使海平面上升約 10 公分。

2) 第二項： $\sin 2\varphi \sin 2\delta_i \cos(\sigma_i t + \lambda + \alpha_i)$ 含有 $\sin 2\varphi$ ，故其：

節線： $\sin 2\varphi = 0$

即 $\varphi = 0^\circ$ ， $\varphi = \pm 90^\circ$

極值： $\because -1 \leq \sin 2\varphi \leq +1$

$$\therefore \sin 2\varphi = -1 \Rightarrow \varphi = -45^\circ$$

$$\sin 2\varphi = +1 \Rightarrow \varphi = 45^\circ$$

對某一固定時間 t ，此位能之分佈，如圖 2.5 (中)。這一項對時間依賴最大者為 $\cos(\sigma_M t + \lambda + \alpha_M)$ 。

σ_M ：相當的周期為 24 小時 50.47 分

σ_S ：相當的周期為 24 小時

因此，第二項描述全日潮，它們的振幅受到 $\sin 2\delta_i$ 之改變。

3) 第三項： $\cos^2 \varphi \cos^2 \delta_i \cos(2\sigma_i t + 2\lambda + 2\alpha_i)$

同理，也是周期性的。其相當周期為 12 小時 25.23 分 (月球) 及 12 小時 (太陽)。因此，它們為半日潮 (semi-diurnal tides)。

$$\therefore 0 \leq \cos^2 \varphi \leq 1$$

$\therefore \varphi = 0^\circ$ ，即在赤道為極大值。

而 $\varphi = \pm 90^\circ$ ，即在兩極為極小值。

對於固定點 t ，此位能之分佈如圖 2.5 (右)。環繞地球位能值，依次為正負正負四個區域，各包含 90° 的經度，此位能之振幅受到 $\cos^2 \delta_i(t)$ 之改變。(17)式只是根據(5)式之級數展開中之一項而定。而且，即使 $m_i = \text{const}$ ，但由(13)式，我們還將 r_0 (地心與月心 [或日心] 之距離) 視為常數。

比較精確之分析，應當考慮所有這些量，隨時間之變化。而將 ϕ 以 Fourier 級數表示之。

$$(21) \Phi_G = \sum_n A_n(\varphi) \cos(\sigma_n t - \alpha_n)$$

表 2.2 列出上述方法導出之某些分潮 (partial tides)。

由(17)式，根據 $\vec{K}_h = -\nabla_h \Phi_G$ ，可得引潮力。

表 2 . 2 七個最基本之分潮

Partial tide	Symbol	Hour angle (°/h)	Period (h)	Amplitude ratio $\frac{100 A}{M_2}$
Semi-diurnal Tides				
Principal lunar	M ₂	28.98410	12.42	100
Principal solar	S ₂	30.00000	12.00	46.6
Larger lunar elliptic	N ₂	28.43973	12.66	19.2
Luni-solar	K ₂	30.08214	11.97	12.7
Diurnal Tides				
Luni-solar	K ₁	15.04107	23.93	58.4
Principal lunar	O ₁	13.94304	25.82	41.5
Principal solar	P ₁	14.95893	24.07	19.4

令 K_φ = 向北之分量

K_λ = 向東之分量

則得：

$$(22) K_\varphi = -\frac{1}{R} \frac{\partial \Phi_G}{\partial \varphi}, \quad K_\lambda = -\frac{1}{R \cos \varphi} \frac{\partial \Phi_G}{\partial \lambda}$$

利用上式及 (2 . 6 . 7) $\frac{1}{\rho} \vec{k} = -\nabla \phi^*$ ，則流體運動方程式

(2 . 5 . 4) 成爲：

$$(23) \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\frac{1}{\rho} \nabla p - \nabla \Phi^* - \nabla \Phi_G + \nu \nabla^2 \vec{v} + \frac{\nu}{3} \nabla \nabla \cdot \vec{v}$$

第三章 座標附著在地球上 之流體動力方程式

第二章所討論的運動方程式，僅適用於慣性系統，在相對座標中，有些顯力（ apparent forces ）就會出現。

如果流場是定義在地球上之座標系統 $\vec{e}_i(t)$ ，而不是在經過座標系統 $\vec{e}_i^a$ ，那我們就需要考慮此座標隨地球，以角速度 $\vec{\Omega}$ 轉動之情形
 $(|\vec{\Omega}| = \Omega = 2\pi / \text{star day} = 7.292 \times 10^{-5} \text{ sec}^{-1})$
 因為，此座標之單位向量（即基礎向量）本身是時間的函數。

§ 3、1 柯氏力(Coriolis Force)

在經過座標系中，某一點 P 由位置向量 $\vec{r}^a$ 來決定。
 由下圖（圖 3、01）可知

$$(1) \quad \vec{r}^a = \vec{s} + \vec{r}, \quad x_i^a \vec{e}_i^a = s_i^a \vec{e}_i^a + x_i \vec{e}_i(t)$$

其中 $\vec{s}$ = 相對系統對絕對系統之位移
 $\vec{r}$ = P 點在相對系統之位置

而 $\vec{r}^a$ 之時間變化率為 $\vec{v}^a = \frac{d\vec{r}^a}{dt}$ ，即如圖 3、01

$$(2) \quad \vec{v}^a = \frac{d\vec{s}}{dt} + \frac{dx_i}{dt} \vec{e}_i(t) + x_i \frac{d\vec{e}_i}{dt}$$

但是

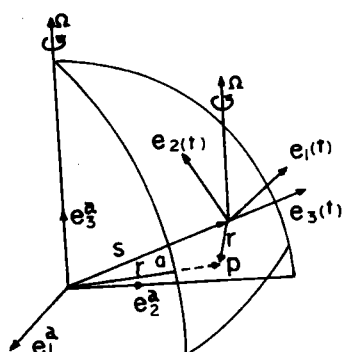


圖 3.01 絕對系統 e_i^a 和相對系統 $e_i(t)$ 相對系統對地球的中心軸 e_3^a 以角速度 $\vec{\Omega}$ 而轉動。

$$(3) \quad \frac{d\vec{s}}{dt} = \vec{\Omega} \times \vec{s}, \quad \frac{d\vec{e}_i}{dt} = \vec{\Omega} \times \vec{e}_i$$

它們僅依賴地球自轉而定

因此，(2)式成爲

$$(4) \quad \vec{v}^a = \vec{\Omega} \times \vec{s} + \frac{dx_1}{dt} \vec{e}_1(t) + x_1 \vec{\Omega} \times \vec{e}_1(t)$$

將(4)式再微分，得到加速度，其形式與上式(4)完全相同，即

$$(5) \quad \frac{d\vec{v}^a}{dt} = \vec{\Omega} \times (\vec{\Omega} \times \vec{s}) + \frac{d^2x_1}{dt^2} \vec{e}_1(t) + 2 \frac{dx_1}{dt} \vec{\Omega} \times \vec{e}_1(t) + x_1 \vec{\Omega} \times (\vec{\Omega} \times \vec{e}_1)$$

將(4)、(5)右邊之第一項和最後項合併，則得

$$\begin{aligned} \vec{\Omega} \times \vec{s} + x_1 \vec{\Omega} \times \vec{e}_1 &= \vec{\Omega} \times \vec{s} + \vec{\Omega} \times \vec{r} = \vec{\Omega} \times (\vec{s} + \vec{r}) \\ \vec{\Omega} \times (\vec{\Omega} \times \vec{s}) + x_1 \vec{\Omega} \times (\vec{\Omega} \times \vec{e}_1) &= \vec{\Omega} \times (\vec{\Omega} \times \vec{s}) + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) \\ &= \vec{\Omega} \times (\vec{\Omega} \times (\vec{s} + \vec{r})) \end{aligned}$$

即

$$(6) \quad \vec{\Omega} \times \vec{r}^* = \vec{\Omega} \times \vec{s} + \vec{\Omega} \times \vec{r}$$

$$(7) \quad \vec{\Omega} \times (\vec{\Omega} \times \vec{r}^*) = \vec{\Omega} \times (\vec{\Omega} \times \vec{s}) + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) \quad \left. \vphantom{\vec{\Omega} \times (\vec{\Omega} \times \vec{r}^*)} \right\} \text{代入(4)(5)}$$

可得絕對座標上速度、加速度 $(\vec{r}^*, \frac{d\vec{v}^*}{dt})$ 與相對座標上加速度

$(\vec{v}, \frac{d\vec{v}}{dt})$ 之關係，即

$$(8) \quad \vec{v}^* = \vec{v} + \vec{\Omega} \times \vec{r}^*, \quad \frac{d\vec{v}^*}{dt} = \frac{d\vec{v}}{dt} + 2\vec{\Omega} \times \vec{v} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}^*)$$

故 Navier-Stokes 運動方程式 (2.7.23)

$$(9) \quad \vec{K} = \rho \frac{d\vec{v}^*}{dt} \text{ 及 } \vec{K} = -\nabla p - \rho(\nabla\Phi^* + \nabla_h\Phi_G) + \mu \nabla^2 \vec{v}^* \\ + \frac{\mu}{3} \nabla \nabla \cdot \vec{v}^*$$

在相對座標系統中，成爲

$$(10) \quad \frac{d\vec{v}}{dt} + 2\vec{\Omega} \times \vec{v} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}^*) = \frac{1}{\rho} [-\nabla p - \rho(\nabla\Phi^* + \nabla_h\Phi_G) \\ + \mu \nabla^2 \vec{v} + \frac{\mu}{3} \nabla \nabla \cdot \vec{v}]$$

$$\therefore \nabla \cdot \vec{v}^* = \nabla \cdot \vec{v}$$

$$\nabla^2 \vec{v}^* = \nabla^2 \vec{v}$$

如以牛頓第二運動定律表示，則上式成爲

$$(10') \quad \frac{d\vec{v}}{dt} = \frac{1}{\rho} [-\nabla p - \rho(\nabla\Phi^* + \nabla_h\Phi_G) - \mu \nabla^2 \vec{v} + \frac{\mu}{3} \nabla \nabla \cdot \vec{v} \\ - 2\rho \vec{\Omega} \times \vec{v} - \rho \vec{\Omega} \times (\vec{\Omega} \times \vec{r}^*)]$$

因此在相對座標系統中，有二種顯力產生，每單位體積此二力為

a) 柯氏力 , $\vec{K}_1 = -2\rho\vec{\Omega} \times \vec{v}$

b) 離心力 , $\vec{K}_2 = -\rho\vec{\Omega} \times (\vec{\Omega} \times \vec{r}^*)$

在直角座標系統 x, y, z 中，柯氏力由下列分量組成（參考圖 3、02）

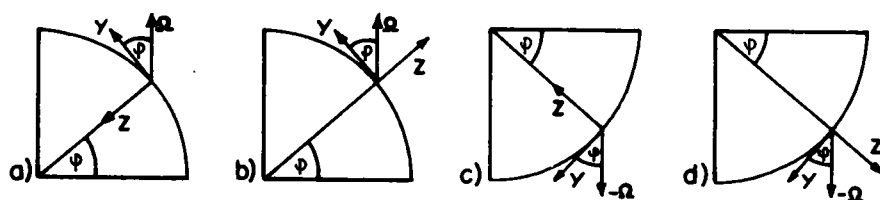


圖 3、02 北半球與南半球之直角座標系統

a) 在北半球：x 向西，y 向北，z 向下

(11) $\vec{K} = -2\Omega\rho[(w\cos\varphi + v\sin\varphi)\vec{i} - u\sin\varphi\vec{j} - u\cos\varphi\vec{k}]$;

b) 在北半球：x 向東，y 向北，z 向上

(12) $\vec{K} = -2\Omega\rho[(w\cos\varphi - v\sin\varphi)\vec{i} + u\sin\varphi\vec{j} - u\cos\varphi\vec{k}]$;

c) 在南半球：x 向東，y 向南，z 向下

(13) $\vec{K} = -2\Omega\rho[-(w\cos\varphi + v\sin\varphi)\vec{i} + u\sin\varphi\vec{j} + u\cos\varphi\vec{k}]$;

d) 在南半球：x 向西，y 向南，z 向上

(14) $\vec{K} = -2\Omega\rho[(-w\cos\varphi + v\sin\varphi)\vec{i} - u\sin\varphi\vec{j} + u\cos\varphi\vec{k}]$

因為，一般說來， w 比 u 和 v 小得多，而且柯氏力之垂直分量，

與地球引力比較，就可忽略，我們通常就將地球自轉之角速度向量，寫成

$$(15) \quad \vec{f} \equiv (0, 0, 2\Omega_z) \quad \text{其中} \quad 2\Omega_z = -2\Omega \sin\varphi = -f$$

在此符號（即負號），係由於 z 軸（在北半球）指向下之緣故，因此，柯氏力可以簡化為

$$(16) \quad \vec{K} = -\rho \vec{f} \times \vec{v}$$

如果在北半球 z 軸朝上，則取正的符號。

§ 3、2 引力與自然座標系統 (Natural Coordinate System)

在運動方程式 (3.1.10') 中，我們有依賴時間而定之引力 $[-\nabla\phi^*]$ 和離心力 $[-\vec{\Omega} \times (\vec{\Omega} \times \vec{r}^a)]$ 。

以下我們拿掉上註 a ，即 $\vec{r} \equiv \vec{r}^a =$ 在地球上某一點之位置向量而 $r = 0 =$ 地心。 $[-\nabla\phi^*]$ 為一個向量，指向地心，但向量 $[-\vec{\Omega} \times (\vec{\Omega} \times \vec{r})]$ 位於 $\vec{\Omega}$ ， $\vec{r}$ 平面上，並且垂直於 $\vec{\Omega}$ 而其大小為 $\Omega^2 r \cos\varphi$ ，即依賴緯度 φ 而定（如圖 3、03）。

將引力和離心力合併，就形成重力 (force of gravity)。

$$(1) \quad \nabla\Phi = \nabla\Phi^* + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}), \quad \Phi = \Phi^* - (\Omega^2 / 2) r^2 \cos^2\varphi$$

Φ 就稱為重力位 (geopotential)。

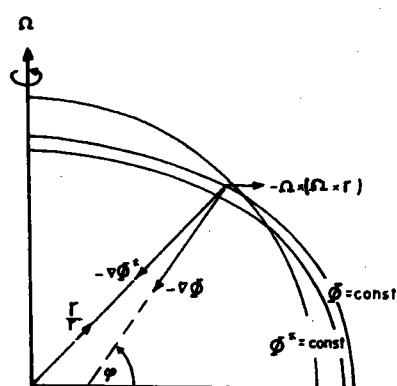


圖 3、03

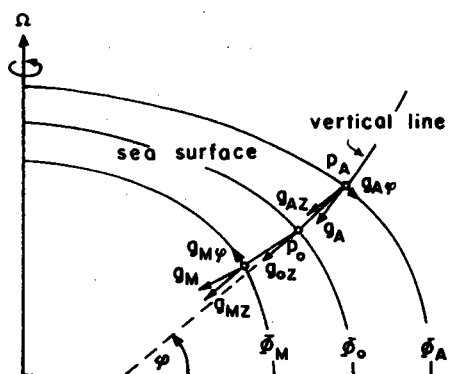


圖 3、04

在不轉動的地球上，靜止海面由 $\phi^* = \text{const}$ 來決定，而轉動使得地球稍微扁平。如同 § 2、6 之說明， $\vec{g}^* = -\nabla\phi^*$

我們稱

$$(2) \quad \vec{g} = -\nabla\Phi$$

為重力加速度， $\vec{g}$ 之大小為 $|\vec{g}| = g$ 。

$$(3) \quad g = 978.049 (1 + 5.2884 \times 10^{-3} \sin^2 \varphi - 5.9 \times 10^{-6} \sin^2 2\varphi) + 2.244 \times 10^{-6} z$$

其中 $z = 0$ 為海面

z (單位為公分) 向下為正

$$\vec{g}^* \text{ 與 } \vec{g} \text{ 之夾角} = \begin{cases} 0, & \text{在赤道與兩極} \\ 0.085, & \text{在緯度 } 60^\circ \end{cases}$$

因此， $\vec{g}$ 基本上可說由分量 $\partial\Phi/\partial r = -g_r$ 構成，此分量指向地心，然而對於大規模運動過程，不僅需考慮等重力位面 (equipotential surfaces) $\Phi = -\gamma E/r - (\Omega^2/2) r^2 \cos^2 \varphi = \text{const}$

之上升（此重力位面，由兩極向赤道升高；如 $r = \text{const}$ ，則升高量 $\approx 21.4 \text{ Km}$ ；赤道半徑 6378.2 Km ，極地半徑 6356.8 Km ）。

同時，也要考慮它們的垂直距離 $|\nabla z| = \frac{|\nabla \phi|}{g}$ ，隨 g 之變化（由(3)），垂直距離在赤道最大，而向兩極減少（如圖 3、04）。如果海平面，是定義為 $\phi_0 = 0$ ，則由(3)，對高度積分，可得

$$(4) \quad \Phi = -[978.049(1 + 5.2884 \times 10^{-3} \sin^2 \varphi - 5.9 \times 10^{-6} \sin^2 2\varphi)z + 1.122 \times 10^{-6} z^2]$$

因此，對 $\Phi = \text{const}$ 之面，其深度（適當的近似值）為

$$(5) \quad z = -[978.049(1 + 5.2884 \times 10^{-3} \sin^2 \varphi - 5.9 \times 10^{-6} \sin^2 2\varphi)]^{-1} \Phi - 1.2 \times 10^{-15} \Phi^2$$

因此，在海面下 1000 m 平均深度之 $\phi = \text{const}$ 的面，由極地向赤道傾斜了 5.29 m 。這些等重力位面，相對於 $\varphi = 45^\circ$ ，而成反對稱（anti-symmetrical）。如無其他力作用，則一個質量元素僅能在等重力位面上靜止。

因轉動的地球上的自然座標是由曲面 $\phi = \text{const}$ 和 $\pm \nabla \phi$ 來決定， $\nabla \phi$ 也是曲線狀（圖 3、04）。因此，利用此種座標系統（即轉動的橢圓體之座標）有很大的困難。在理論上的研究，海面及所有與它平行之表面，一般都定義為水平面；而 $\nabla \phi$ 被認為是直線的 z 座標，而在當地，是垂直於 $\phi = \text{const}$ 之表面，所以我們常應用球面座標以及 f 平面和 β 平面之座標。在這些座標系統，根據(4)，重力的子午線方向之分量 $g_{\Delta \varphi}$ （大氣中）和 $g_{M\varphi}$ （海洋中），除了在 $\varphi = 0^\circ$ 和 $\varphi = 90^\circ$ ，它們都是對 z 而為線性變化，並且在海洋面，是為零（圖 3、04）。此緯向（即子午線方向上）分量，在大氣中指向赤道，而在海洋中指向極地。

我們令此分量爲 g' ; 則由(4), 令 $R = 6371 \text{ Km}$, 可得

$$(6) \quad g' = -\frac{1}{r_E} \frac{\partial \Phi}{\partial \varphi} = (8.12 \times 10^{-9} \sin 2\varphi - 1.8 \times 10^{-11} \sin 4\varphi) z$$

$$(\text{cm sec}^{-2})。$$

由此式, 在 $\varphi = 45^\circ$, 深度爲 1000 m 之處, 單位質量之力

$g' = 8.12 \times 10^{-4} \text{ cm/sec}^2$, 指向極地, 而平行於海平面。
此大小程度與其他水平力接近。

因此, 當我們利用上述座標系統, 來研究深海環流時, 如無其他的條件, g' 不能任意忽略。但是在靜止的海洋中, 此力常由引力加速度之緯度變化所形成之水平壓力梯度所抵消。

在海洋學上, g' 被忽略掉, 就得一個常數 g , 因此, 對於流體動力方程式, 就可大大地簡化。反之, 如 g' 仍被使用, 則在流體動力方程式中, g 也必須是隨緯度和深度變化之加速度。

§ 3、3 以向量表示之運動方程式

引力與離心力合併爲重力之後, 則流體動力方程式(10)成爲

$$(1) \quad \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} + 2\vec{\Omega} \times \vec{v} = -\frac{1}{\rho} \nabla p - \nabla \Phi - \nabla_h \Phi_G + \nu \nabla^2 \vec{v}$$

$$+ \frac{\nu}{3} \nabla \nabla \cdot \vec{v}$$

通常我們可寫成

$$(2) \quad 2\vec{\Omega} \times \vec{v} \approx \vec{f} \times \vec{v}, \quad \vec{f} = (0, 0, -2\Omega \sin \varphi) \quad (\text{北半球, 向下爲正})$$

$$\nabla \Phi \approx -g \vec{k}, \quad g = \text{const}$$

$$\Phi_g \approx \sum A_n(\varphi) e^{i(w_n t + \alpha_n)}$$

在此 A_n 對於各別的分潮必須指定特定的值。

§ 3、4 在直角座標系統 (f 平面) 之運動方程式

在小規模的運動中，地球的曲率以及柯氏參數 (Coriolis parameter)，隨緯度之變化，能被忽略。如此我們可以引進一種直角座標系統 (rectangular coordinate system)；它在位置 $\vec{r}$ 之 x, y 平面中，即切於海平面之平面，其 z 軸指向下， $z = 0$ 為靜止之海面。一般說來，可以使用 (3 . 3 . 2) 之簡化形式。因為 $f = f_0 = \text{const} (= 2\Omega \sin\varphi_0)$ ，以及此種假設之應用的限制，此 x, y plane 就稱為 f 平面 (f - plane)。令 u, v, w 分別代表 x, y, z 方向之速度分量。由 (3 . 3 . 1) 同時利用 (3 . 3 . 2) 之近似值，可得各分量之無向量方程式。

$$\begin{aligned} (1) \quad & \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} + f_0 v \\ & = -\frac{1}{\rho} \frac{\partial p}{\partial x} - \frac{\partial \Phi_g}{\partial x} + \nu \nabla^2 u + \frac{\nu}{3} \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \end{aligned}$$

$$\begin{aligned} (2) \quad & \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} - f_0 u \\ & = -\frac{1}{\rho} \frac{\partial p}{\partial y} - \frac{\partial \Phi_g}{\partial y} + \nu \nabla^2 v + \frac{\nu}{3} \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \end{aligned}$$

$$\begin{aligned} (3) \quad & \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \\ & = -\frac{1}{\rho} \frac{\partial p}{\partial z} + g + \nu \nabla^2 w + \frac{\nu}{3} \frac{\partial}{\partial z} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \end{aligned}$$

以上方程式中，係在北半球上；x 向西，y 向北，z 向下。如同樣在北半球上，x 向東，y 向北，z 向上，則(1)(2)式中柯氏力項，及(3)式中之重力項，它們的符號都需相反。

§ 3、5 球面座標上之運動方程式

取球面座標(λ , φ , r)； r 為離地心距離， λ 為經度向東為正， φ 為緯度向北為正。則運動方程式為

$$\begin{aligned}
 (1) \quad & \frac{\partial u}{\partial t} + \frac{u}{r \cos \varphi} \frac{\partial u}{\partial \lambda} + \frac{v}{r \cos \varphi} \frac{\partial}{\partial \varphi} (u \cos \varphi) + \frac{w}{r} \frac{\partial (r u)}{\partial r} \\
 & - 2 \Omega v \sin \varphi + 2 \Omega w \cos \varphi \\
 & = - \frac{1}{\rho r \cos \varphi} \frac{\partial p}{\partial \lambda} - \frac{1}{r \cos \varphi} \frac{\partial \Phi_g}{\partial \lambda} \\
 (2) \quad & \frac{\partial v}{\partial t} + \frac{u}{r \cos \varphi} \frac{\partial v}{\partial \lambda} + \frac{v}{r} \frac{\partial v}{\partial \varphi} + \frac{w}{r} \frac{\partial (r v)}{\partial r} + \frac{u^2 \tan \varphi}{r} + 2 \Omega u \sin \varphi \\
 & = - \frac{1}{\rho r} \frac{\partial p}{\partial \varphi} - \frac{1}{r} \frac{\partial \Phi}{\partial \varphi} - \frac{1}{r} \frac{\partial \Phi_g}{\partial \varphi} \\
 (3) \quad & \frac{\partial w}{\partial t} + \frac{u}{r \cos \varphi} \frac{\partial w}{\partial \lambda} + \frac{v}{r} \frac{\partial w}{\partial \varphi} + w \frac{\partial w}{\partial r} - \frac{u^2 + v^2}{r} - 2 \Omega u \cos \varphi \\
 & = - \frac{1}{\rho} \frac{\partial p}{\partial r} - \frac{\partial \Phi}{\partial r}
 \end{aligned}$$

在此， u , v , w 為沿著 λ , φ , r 之速度分量，有時候，我們利用餘緯度 (colatitude) θ ，來代替 φ

$$\text{即 } \theta = 90^\circ - \varphi \quad \partial / \partial \varphi \rightarrow - \partial / \partial \theta$$

(1) — (3) 式中，我們忽略了摩擦力項，因為在大規模運動中（如此才用得著球面座標），分子摩擦力與其非線性項所引起之摩擦力比較，就

可以忽略了。

§ 3、6 β 平面上之運動方程式

要用球面座標解流體動力方程式，通常很困難。許多問題大部份使用 § 3、4 之直角座標（即 Cartesian Coordinate System）。然而，此座標僅適用小規模運動，對於柯氏參數隨緯度變化及地球曲率無關緊要時才用。因此，如果引進一種直角座標系統，它至少能部分地含有柯氏參數（ f ）隨緯度之變化，而能描述大規模之運動過程，只要此過程不延伸到整個地球就可。這就是利用 β 平面，它忽略地球曲率，而含有柯氏參數隨緯度之近似變化，這是由 C. G.

Rossby (1939) 所提出。

將柯氏參數 $f = 2\Omega \sin\varphi$ ，對於某一個參考緯度 φ_0 展開為 Taylor 級數。

$$(1) \quad f = 2\Omega \sin\varphi \approx 2\Omega [\sin\varphi_0 + (\varphi - \varphi_0) \cos\varphi_0] = f_0 + \beta y$$

其中

$$(2) \quad f_0 = 2\Omega \sin\varphi_0, \quad \beta = \left(\frac{\partial f}{\partial y} \right)_{\varphi_0} = \frac{2\Omega \cos\varphi_0}{R},$$

$$\varphi - \varphi_0 = \frac{y}{R}$$

而柯氏力之其他項，就不考慮了。因此 β 平面之運動方程式為

$$(3) \quad \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} + (f_0 + \beta y) v \\ = -\frac{1}{\rho} \frac{\partial p}{\partial x} - \frac{\partial \Phi_g}{\partial x}$$

$$(4) \quad \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} - (f_0 + \beta y) u$$

$$= -\frac{1}{\rho} \frac{\partial p}{\partial y} - \frac{\partial \Phi}{\partial y} - \frac{\partial \Phi_g}{\partial y}$$

$$(5) \quad \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} - \frac{\partial \Phi}{\partial z}$$

在(3)和(4)中，當對 y 微分時， βy 就能被考慮了；否則與 f_0 比較，就可忽略；因 $\beta y \approx \frac{f_0 y}{R}$ 而且 $y \ll R$ 。

第四章 熱傳導方程式

(Equation of Heat Conditions)

前面我們已經介紹了狀態方程式 (1 . 5 . 3)、連續方程式 (1 . 6 . 1)、塩度擴散方程式 (1 . 7 . 5) 以及運動方程式 (3 . 3 . 1)，總共有 6 個方程式，但我們有 7 個量：

$$u, v, w, p, \rho, s, t$$

因此我們還需要一個方程式；此方程式由能量不滅定律導出，即為熱傳導方程式。

§ 4、1 熱傳導方程式

$$\text{在 (1 . 4 . 2) 中, } \iiint \frac{\partial \varphi \rho}{\partial t} dV = - \iiint \nabla \cdot \varphi \rho \vec{v} dV + \iiint q_{\varphi} dV$$

如令 $\varphi = C_p T$ (總比熱, specific total heat), 在體積 V 內, 如溫度與外界不同, 則可想像分子能越過邊界, 而形成熱移轉; 導致溫度差減少 (如同塩度之輻散一樣), 此種熱之流動就是熱通量 (heat flux, $\vec{H}$); $-\oint \vec{H} \cdot d\vec{\sigma}$ 表示穿越全部表面積進入體積內之熱通量。而此種熱通量, 就使得內部熱量增加, 相當於一種熱源 (heat source), 即 $\iiint q_{\varphi} dV = -\oint \vec{H} \cdot d\vec{\sigma}$

如此可得

$$\iiint \frac{\partial \rho C_p T}{\partial t} dV = - \iiint \nabla \cdot \rho C_p T \vec{v} dV - \oint \vec{H} \cdot d\vec{\sigma}$$

$$= - \iiint \nabla \cdot \rho C_p T \vec{v} dV - \iiint \nabla \cdot \vec{H} dV$$

即得

$$(1) \quad \frac{\partial \rho C_p T}{\partial t} = - \nabla \cdot \rho C_p T \vec{v} - \nabla \cdot \vec{H}$$

熱通量 $\vec{H}$ 主要依賴溫度梯度，其方向朝低溫。在海洋學上，我們可以利用 Fourier 假設

$$(2) \quad \vec{H} = -k \nabla T \quad (k = \text{const})$$

則(1)式成爲

$$(3) \quad \frac{\partial \rho C_p T}{\partial t} = - \nabla \cdot \rho C_p T \vec{v} + \nabla \cdot (k \nabla T) \quad (C_p = \text{const})$$

或

$$C_p T \frac{\partial \rho}{\partial t} + C_p \rho \frac{\partial T}{\partial t} = - C_p T \nabla \cdot \rho \vec{v} - \rho \vec{v} \cdot \nabla C_p T \\ + \nabla \cdot (k \nabla T)$$

$$C_p T \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{v} \right) + C_p \rho \left(\frac{\partial T}{\partial t} + \vec{v} \cdot \nabla T \right) = k \nabla^2 T$$

利用連續方程式，可得

$$\frac{\partial T}{\partial t} + \vec{v} \cdot \nabla T = \frac{k}{\rho C_p} \nabla^2 T$$

令 $K_T = \frac{k}{\rho C_p}$ = 熱擴散係數 (thermo-diffusion coefficient $\approx 1.4 \times 10^{-3} \text{ cm}^2/\text{sec}$)

則最後可得

$$(4) \quad \frac{dT}{dt} = K_T \nabla^2 T$$

此即爲熱傳導方程式。

第五章 特定條件

我們有 7 個未知數：u，v，w，p，ρ，T，s。而也有了 7 個方程式，即

(1) 狀態方程式： $\rho = \rho(p, s, T)$

(2) 連續方程式： $\frac{\partial \rho}{\partial t} + \vec{v} \cdot \nabla \rho = 0$

(3) 鹽度擴散方程式： $\frac{\partial S}{\partial t} + \vec{v} \cdot \nabla S = K_D \nabla^2 S$

(4, 5, 6) 運動方程式：
$$\begin{aligned} \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} + 2 \vec{\Omega} \times \vec{v} \\ = -\frac{1}{\rho} \nabla p - \nabla \phi - \nabla_h \phi_G + \nu \nabla^2 \vec{v} \\ + \frac{\nu}{3} \nabla \nabla \cdot \vec{v} \end{aligned}$$

(7) 熱傳導方程式： $\frac{\partial T}{\partial t} + \vec{v} \cdot \nabla T = K_T \nabla^2 T$

這些方程式中，有許多項是依據分子輸送過程而定，因此，對於海洋學上的大部份過程，它們並不重要。因為另有其他項，所造成之混合效果更為強烈，這些項為平流項 (convective term)，即 $\vec{v} \cdot \nabla \rho$ (2)， $\vec{v} \cdot \nabla s$ (3)， $\vec{v} \cdot \nabla \vec{v}$ (4, 5, 6)， $\vec{v} \cdot \nabla T$ (7)，而且上述的方程式，可以利用適當的特定條件來簡化，以及利用平均的方法或利用攝動方法 (perturbation method)。利用某一種特定方法，所得的解；至少能夠代表某一過程之基本特徵。

§ 5、1 不可壓縮的條件

依據狀態方程式 $\rho = \rho(p, S, T)$ 可得

$$(1) \quad \frac{d\rho}{dt} = \left(\frac{\partial \rho}{\partial p}\right)_{s,T} \frac{dp}{dt} + \left(\frac{\partial \rho}{\partial S}\right)_{p,T} \frac{dS}{dt} + \left(\frac{\partial \rho}{\partial T}\right)_{p,s} \frac{dT}{dt}$$

某一種介質，其密度不隨壓力而變，即

$$(2) \quad \left(\frac{\partial \rho}{\partial p}\right)_{s,T} = 0$$

稱為不可壓縮 (incompressible)，對於水而言，上式幾乎可滿足。如果我們忽略分子之熱傳導及鹽擴散，即 $K_T = K_D = 0$ 。

$$\text{則可得} \quad \frac{dT}{dt} = 0, \quad \frac{dS}{dt} = 0$$

因此，可得

$$(3) \quad \frac{d\rho}{dt} = 0, \text{ 對於不可壓縮的介質，大部分的情形，(3)式就直接稱}$$

為不可壓縮之條件，在此種條件下，連續方程式成為

$$(4) \quad \nabla \cdot \vec{v} = 0$$

§ 5、2 不可壓縮均勻的及正壓的 (Barotropic) 海洋

均勻的海洋 (homogeneous ocean) 之定義為

$$(1) \quad \rho = \rho_0 = \text{const}, \quad \Rightarrow \nabla \cdot \vec{v} = 0$$

$$T = T_0 = \text{const},$$

$$S = S_0 = \text{const}.$$

如此，由 (5 . 1 . 1) 可得 $(\partial \rho / \partial p)_{s,T} = 0$

即不可壓縮。

(1)式之條件，可以用來代替狀態方程式。同時，擴散方程式(1 . 5 . 3)和熱傳導方程式(4 . 1 . 4)一定滿足。如此，整個方程式系統成為

$$(2) \quad \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} + 2 \vec{\Omega} \times \vec{v} = -\frac{1}{\rho_0} \nabla p - \nabla \Phi - \nabla_h \Phi_G + \nu \nabla^2 \vec{v},$$

$$(3) \quad \nabla \cdot \vec{v} = 0$$

在均勻的海洋中，沒有混合過程發生，一塊液體元素能由外力推動到任何位置，並且在那兒再靜止下來。

如果 $\left(\frac{\partial \rho}{\partial p} \right)_{s,T} \neq 0$ ，則此種海洋稱為 *piezotropic*，並且能以一種 *piezotropy* 係數來定義。根據 § 1、1 此種場就是正壓的 (*barotropic*)。

§ 5、3 不可壓縮不均勻的海洋但無混合過程

假定沒有混合發生，那麼不可壓縮條件 $(\partial \rho / \partial p)_{s,T} = 0$ ，就產生 $d\rho/dt = 0$ (由 § 5、1)，即一塊個別流體元素的密度，在不均勻的海洋中，保持一定。如此，除了在它的原來位置之外，它的密度，一般而言，與其周圍環境不同。這樣就有回復力發生。因此，不可壓縮、不均勻的海洋，而無混合時，其基本的運動特徵為振動，即水內波 (*internal waves*)。水內波的運動過程，另有專文討論，它們的方程式為

$$(1) \quad \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} + 2 \vec{\Omega} \times \vec{v} = -\frac{1}{\rho} \nabla p - \nabla \Phi - \nabla_h \Phi_G + \nu \nabla^2 \vec{v}$$

$$(2) \quad \nabla \cdot \vec{v} = 0$$

$$(3) \quad \frac{d\rho}{dt} = 0$$

第六章 邊界條件

(Boundary Conditions)

我們前面所導出的 7 個場函數，都是基於變數 $\vec{v}$, p , ρ , S 和 T 是連續的假設。在不連續的界面，這些場函數，就必須由邊界條件來代替。海洋學上最重要的不連續面為海的邊界，即海面為水與空氣之界面，而海底（包括海岸線）為水與固體地球之界面。為了簡化，對於迅速變化的場函數之層次（例如斜溫層）也通常以不連續面來代表。在這種情形，類似空氣與海洋之界面的條件，亦可應用。

在海洋學上，最重要的，就是要把邊界條件，精確地列出公式。因為所有影響海洋的力量，僅僅引潮力直接透入海洋中，而是隨時間而變之體積力。所有其他的作用力，僅作用於海面，因此，在邊界條件中出現。海洋的邊界，並非恰好由相同的質點所組成的自由表面或堅固表面。越過界面，部份質量會交換，隨著位置與時間，海水的全部質量會變化。降雨、蒸發、結冰和溶冰等，使得純水流過海面，而且河水在海岸附近流入海洋。這對於鹽度有決定性的影響。對於擴散，我們就必須要考慮到這些過程。海底的過程比較不明顯，但對於某一部門的海洋學（例如漂沙，sand transport）同樣很重要。那時沈積物（sediments）不再是堅固界面了。

§ 6、1 不連續面上一般的保守定律 (Kotchine 定理)

在 § 1、4 中，我們考慮了場函數的一般不減定律，讓我們現在

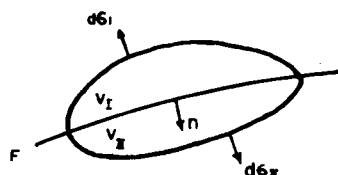
延伸此定律，考慮一個體積元素，由一個單異面 (singular surface) 穿過。令此單異面之方程式為

$$F(\vec{r}, t) = 0$$

而且 $dF = 0$

如此，如同 § 1、3，可得

$$(1) \quad \frac{dF}{dt} = \frac{\partial F}{\partial t} + \vec{c} \cdot \nabla F = 0$$



在此， $\vec{c}$ 為表面 F 移動的速度。 F 把此體積分成兩個區域 V_I 和 V_{II} ，此表面之單位垂直向量 $\vec{n}$ 指向下 (即在正 z 軸向有分量)，沿整個表面，速度分佈為

$$(2) \quad \vec{v} = \vec{v}_I \text{ 對於 } \sigma_I, \quad \vec{v} = \vec{c} \text{ 對於 } F, \quad \vec{v} = \vec{v}_{II} \text{ 對於 } \sigma_{II}$$

依據 (1.4.1) 和 (1.4.5) 對於此物質體積 (material volume) 之一般的不減定律為

$$(3) \quad \frac{d}{dt} \iiint \rho \psi dV = \iiint \frac{\partial \rho \psi}{\partial t} dV + \oint \rho \psi \vec{v} \cdot d\vec{\sigma} = \iiint q_\psi dV$$

我們現在 q_ψ 分為一種真的源強度 (source strength) q 和由分子穿越表面進入體積之輸送所造成之通量 (flux) $\vec{q}_\psi$ ，即

$$(4) \quad \iiint q_\psi dV = \iiint q dV - \oint \vec{q}_\psi \cdot d\vec{\sigma}$$

由(2)及上圖，此通量成為

$$\begin{aligned} \oint \rho \psi \vec{v} \cdot d\vec{\sigma} &= \iint (\rho \psi \vec{v})_I \cdot d\vec{\sigma} + \iint (\rho \psi)_I \vec{c} \cdot d\vec{F} \\ &\quad + \iint (\rho \psi \vec{v})_{II} \cdot d\vec{\sigma} - \iint (\rho \psi)_{II} \vec{c} \cdot d\vec{F} \end{aligned}$$

如此可得

$$(5) \quad \iiint \frac{\partial \rho \psi}{\partial t} dV + \iint \rho \psi \vec{v} \cdot d\vec{\sigma} + \iint [(\rho \psi)_I - (\rho \psi)_{II}] \vec{c} \cdot d\vec{F}$$

$$= \iiint q \, dV - \oint \vec{q}_\phi \cdot d\vec{\sigma}$$

如果 $\partial \rho \phi / \partial t$ 和 q 在 F 附近是有限的，則極限 $V \rightarrow 0$ ，體積分 $\rightarrow 0$

$$\therefore \iint (\rho \phi \vec{v})_I \cdot d\vec{\sigma}_I + \iint (\rho \phi \vec{v})_{II} \cdot d\vec{\sigma}_{II} + \iint [(\rho \phi)_I - (\rho \phi)_{II}] \vec{c} \cdot d\vec{F} + \iint \vec{q}_\phi \cdot d\vec{\sigma}_I + \iint \vec{q}_\phi \cdot d\vec{\sigma}_{II} = 0$$

極限情形：同時令表面 σ_I 和 σ_{II} 收縮到 F ，使得體積 V_I 和 V_{II} 消失，而 F 的面積，依然一定，即

$$d\vec{\sigma}_I \rightarrow -d\vec{F}, \quad d\vec{\sigma}_{II} \rightarrow d\vec{F}$$

則上式成爲

$$(6) \quad \iint \{ (\rho \phi \vec{v})_I - (\rho \phi \vec{v})_{II} - [(\rho \phi)_I - (\rho \phi)_{II}] \vec{c} + (\vec{q}_{\phi I} - \vec{q}_{\phi II}) \} \cdot d\vec{F} = 0$$

因爲表面 F 爲任意的，故得

$$(7) \quad \{ (\rho \phi \vec{v})_I - (\rho \phi \vec{v})_{II} - [(\rho \phi)_I - (\rho \phi)_{II}] \vec{c} + (\vec{q}_{\phi I} - \vec{q}_{\phi II}) \} \cdot \vec{n} = 0$$

這就是 Kottchine 定理。由此，可以導出特別的邊界條件，它們必須用在表面 $F(x, y, z, t) = 0$ 處，一般而言，此表面是在運動情形。爲了避免這些困難，通常將邊界條件，沿 F 的平均位置，展開爲 Taylor 級數。

§ 6、2 運動邊界條件 (Kinematic Boundary Condition)

在一個不連續的表面，運動邊界條件就代替了連續方程式，令 ϕ 代表單位質量之質量，即

$$\phi = \text{質量} / \text{單位質量} = 1$$

如果沒有分子輸送過程， $q_\phi = 0$ ，則(6.1.7)簡化為

$$(1) \quad (\rho v_n)_I - (\rho v_n)_II - (\rho_I - \rho_{II}) c_n = 0$$

在這種情形，所有的海洋邊界是物質面(material surfaces)，即此表面正向速度(normal velocity) c_n 是與其兩側流體元素之正向速度 v_{nI} ， v_{nII} 一致。

亦即

$$(2) \quad v_{nI} = v_{nII} = v_n = c_n$$

這就是運動邊界條件。

將(2)代入(6.1.1)，得

$$(3) \quad \frac{\partial F}{\partial t} + \vec{v} \cdot \nabla F = 0$$

如果，我們對於此表面的兩邊之過程有興趣，則(3)式須分別令 $\vec{v} = \vec{v}_I$ 和 $\vec{v} = \vec{v}_{II}$ 而由兩個方程式表示

即

$$(4) \quad \frac{\partial F}{\partial t} + \vec{v}_I \cdot \nabla F = \frac{\partial F}{\partial t} + \vec{v}_{II} \cdot \nabla F \quad (\vec{v}_I - \vec{v}_{II}) \cdot \nabla F = 0$$

令海面之方程式為

$$(5) \quad F(x, y, z, t) = z + \zeta(x, y, t) = 0$$

則(3)式成為

$$(6) \quad \frac{\partial \zeta}{\partial t} + v_h \cdot \nabla \zeta + w = 0 \quad \text{for} \quad z = -\zeta(x, y, t)$$

而海底之方程式為

$$(7) \quad F(x, y, z, t) = z - H(x, y, t) = 0$$

則由(3)式可得

$$(8) \quad \frac{\partial H}{\partial t} + \vec{v}_h \cdot \nabla H - w = 0 \quad \text{for } z = H(x, y, t)$$

海底隨時間之變化，僅在一些特例中，需加以考慮，一般而言，海底可被視為穩定，故(8)式簡化為

$$(9) \quad w = \vec{v}_h \cdot \nabla H \quad \text{for } z = H(x, y)$$

對於水平面的海底，則

$$(10) \quad w = 0 \quad \text{對於 } z = H_0 = \text{const}$$

而對於一般的堅硬表面

$$(11) \quad v_n = 0$$

對於許多過程，例如熱鹽循環 (thermohaline circulation) 穿透表面之質量輸送不可忽略。蒸發表示部分質量穿過海表面之分子輸送量，而雨量是質量之直接供應。

因為我們總是規定海水的表面為海洋的界面，我們必須考慮降水為穿過表面之輸送量。

在 $\vec{n}$ 方向，質量之輸送量 (Δm) 由降水量 (N) 減去蒸發量 (V) 來決定，在 (6 · 1 · 7) 中 (即 Kotchine 定理)，因此我們可以令

$$(12) \quad (\vec{q}_{\phi I} - \vec{q}_{\phi II}) \cdot \vec{n} = \Delta m, \quad \Delta m \sim N - V$$

而且，根據上述，液體表面應認為是界面，如同(2)式，得

$$(13) \quad v_{nI} = c_n$$

然後由 (6 · 1 · 7) 和 (12) 得

$$(14) \quad v_{nI} - v_{nII} = -\frac{\Delta m}{\rho_{II}}$$

這就是有質量輸送經過表面時之運動邊界條件。在邊界上，垂直速度之差，正比於穿過邊界之質量輸送。

如果 $\Delta m > 0$ 即 $N > V$ (降水大於蒸發)

則 $v_{nII} > v_{nI}$

反之爲蒸發大於降水。

由 (6 · 1 · 1) 可得

$$(15) \quad c_n = v_{nI} = - \frac{\frac{\partial F}{\partial t}}{|\nabla F|}$$

則(14)成爲

$$(16) \quad \frac{\partial F}{\partial t} + v_{nII} |\nabla F| = \frac{\Delta m}{\rho_n} |\nabla F|$$

$$\therefore F = z + \zeta$$

$$(17) \quad |\nabla F| = \sqrt{1 + |\nabla_h \zeta|^2}, \quad \cos(z, n) = \cos \gamma$$

$$= \frac{\partial F}{\partial z} / \sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}$$

$$= 1 / \sqrt{1 + |\nabla_h \zeta|^2}$$

$$\therefore |\nabla F| = \frac{1}{\cos \gamma}$$

如此(16)又化爲

$$(18) \quad \frac{\partial \zeta}{\partial t} + \vec{v}_h \cdot \nabla \zeta + w = (N - V) / \rho \text{ 對於 } z = -\zeta(x, y, t)$$

在此，我們利用下列定義

$$(19) \quad \frac{\Delta m}{\cos \gamma} = N - V, \quad \rho_n = \rho, \quad \vec{v}_n = \vec{v}$$

γ 代表表面法線與 z 軸之夾角，並且 $N - V$ 代表降水量減去蒸發量之差。

§ 6、3 鹽度之邊界條件

在一個表面上，連續方程式需由運動邊界條件 (6 · 2 · 3) 或 (6 · 2 · 18) 來代替。同理，擴散方程式需由相當之部分質量 (partial mass) 的條件來代替。經由蒸發、降水、結冰、溶解等，海水表面，就有純水在海洋與大氣 (或冰) 間之輸送。但鹽依然在海洋中，

$$\text{令 } \phi \equiv s$$

則邊界條件 (6 · 1 · 7) 對於部分質量 S ，也成立。同時 $\vec{q}_s =$ 溶解鹽之濃度通量密度 (concentration flux density) 方程式 (6 · 2 · 13.) 和 (6 · 2 · 14.) 也對於速度 v_{nI} , v_{nII} , c_n 成立，故 (6 · 1 · 7) 成爲

$$(1) \quad (\vec{q}_{sI} - \vec{q}_{sII}) \cdot \vec{n} = s \Delta m$$

因所有的鹽依然存在海洋中， $\therefore \vec{q}_{sI} = 0$

因此，由 (6 · 2 · 19.)， $\frac{\Delta m}{\cos \gamma} = N - V$ ，得

$$(2) \quad \vec{q}_{sII} \cdot \vec{n} = -s(N - V) \cos \gamma$$

我們令 $\vec{q}_{sII} = -k \nabla s$ (參考 § 1 · 7)

而且，如果令 $S = 10^3 s$ ，則得

$$(3) \quad \frac{\partial S}{\partial n} = \frac{S(N - V) \cos \gamma}{k_D \rho}, \quad k_D = \frac{k}{\rho}$$

或

$$k_D \rho (\nabla_h S \cdot \nabla \zeta + \frac{\partial S}{\partial z}) = S(N - V)$$

在此 $\rho = \rho_{II}$ 代表 $z = -\zeta(x, y, t)$ 處，海水的密度

(3)式爲部分質量 S 之邊界條件，在海面上，它將代替擴散方程式，如海面總是保持水平， $\zeta = 0$ ，則

$$(4) \quad \frac{\partial S}{\partial z} = \frac{S(N-V)}{k_D \rho}$$

§ 6、4 動力邊界條件 (Dynamic Boundary Condition)

如果有質量輸送越過邊界而發生，那就必須要利用運動邊界條件 (6·2·18)。而對於其他情形，我們還是由 (6·2·2) 式開始導出來。因此 (6·1·7) 化爲

$$(1) \quad (\vec{q}_{\phi I} - \vec{q}_{\phi II}) \cdot \vec{n} = 0,$$

即在垂直方向上由分子輸送過程之通量在邊界上必須是連續的。現在我們將(1)式應用到動量通量 (momentum flux)，

即令 $\phi =$ 每單位質量之通量 $\equiv \vec{v}$

如此，我們就得到動力邊界條件。它就是在一個不連續面上，用來代替動量不變定律。

如同 § 2、2，分子動量輸送，是由應力在單異面之投影而定，

$$\iint \vec{q}_{\phi} \cdot d\vec{\sigma} \equiv \iint \Pi \cdot d\vec{F}$$

所以(1)式，可以變成

$$(2) \quad (\Pi_I - \Pi_{II}) \cdot \vec{n} = 0$$

而且利用 $\Pi = \vec{e}_i P_{ik} \vec{e}_k$ ， $\therefore \Pi \cdot \vec{n} = P_{ik} \vec{e}_i \vec{e}_k \cdot \vec{n}$

$$\text{故} \quad [(P_{ik})_I - (P_{ik})_{II}] \vec{e}_i \cos(\vec{e}_k, \vec{n}) = 0$$

我們再利用 x, y, z 座標系統之方向餘弦 (direction co-

sines), $\cos \alpha = \cos(x, n)$, $\cos \beta = \cos(y, n)$, $\cos \gamma = \cos(z, n)$, 以 x, y, z 系統,

$$\text{即 } \begin{pmatrix} i = 1, 2, 3 \\ k = 1, 2, 3 \end{pmatrix}$$

則上式化爲

$$(3) \quad \begin{cases} [(P_{xx})_I - (P_{xx})_{II}] \cos \alpha + [(P_{xy})_I - (P_{xy})_{II}] \cos \beta \\ \quad + [(P_{xz})_I - (P_{xz})_{II}] \cos \gamma = 0 \\ [(P_{yx})_I - (P_{yx})_{II}] \cos \alpha + [(P_{yy})_I - (P_{yy})_{II}] \cos \beta \\ \quad + [(P_{yz})_I - (P_{yz})_{II}] \cos \gamma = 0 \\ [(P_{zx})_I - (P_{zx})_{II}] \cos \alpha + [(P_{zy})_I - (P_{zy})_{II}] \cos \beta \\ \quad + [(P_{zz})_I - (P_{zz})_{II}] \cos \gamma = 0 \end{cases}$$

同時, 對於單位垂直向量 $\vec{n}$ 之分量爲

$$(4) \quad \cos \alpha_i = \frac{\frac{\partial F}{\partial x_i}}{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}}$$

其中 $\alpha_i = \alpha, \beta, \gamma$ 並且 $x_i = x, y, z$ 對於 $i = 1, 2, 3$

對於不可壓縮的流體, 應力張量 Π 之分量爲

$$(5) \quad (P_{ik}) = \begin{bmatrix} -P + 2\mu \frac{\partial u}{\partial x} & \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) & \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \\ \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) & -P + 2\mu \frac{\partial v}{\partial y} & \mu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \\ \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) & \mu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) & -P + 2\mu \frac{\partial w}{\partial z} \end{bmatrix}$$

對於表面 $F(x, y, z) = 0$, 我們令

$$(6) \quad F(x, y, z, t) = z + \zeta(x, y, t)$$

如此，(3)式中各項都可決定。最後(3)式可以明白地表示為

$$(7) \quad \left\{ \begin{aligned} & [(-p + 2\mu \frac{\partial u}{\partial x})_I - (-p + 2\mu \frac{\partial u}{\partial x})_N] \frac{\partial \zeta}{\partial x} \\ & + [\mu_I (\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})_I - \mu_N (\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})_N] \frac{\partial \zeta}{\partial y} \\ & + \mu_I (\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})_I - \mu_N (\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})_N = 0 \\ & [\mu_I (\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) - \mu_N (\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})_N] \frac{\partial \zeta}{\partial x} \\ & + [(-p + 2\mu \frac{\partial v}{\partial y})_I - (-p + 2\mu \frac{\partial v}{\partial y})_N] \frac{\partial \zeta}{\partial y} \\ & + \mu_I (\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y})_I - \mu_N (\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y})_N = 0 \\ & [\mu_I (\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})_I - \mu_N (\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})_N] \frac{\partial \zeta}{\partial x} \\ & + [\mu_I (\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y})_I - \mu_N (\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y})_N] \frac{\partial \zeta}{\partial y} \\ & + (-p + 2\mu \frac{\partial w}{\partial z})_I - (-p + 2\mu \frac{\partial w}{\partial z})_N = 0 \end{aligned} \right.$$

特例：

- 1) 平面運動 (plane motions)，只限於在 x, z 平面， $v = 0$ ， $\frac{\partial}{\partial y} = 0$ ：

則(7)式化為

$$(8) \quad \left\{ \begin{array}{l} [(-p + 2\mu \frac{\partial u}{\partial x})_I - (-p + 2\mu \frac{\partial u}{\partial x})_{II}] \frac{\partial \zeta}{\partial x} \\ \quad + \mu_I (\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})_I - \mu_{II} (\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})_{II} = 0 \\ [\mu_I (\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})_I - \mu_{II} (\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})_{II}] \frac{\partial \zeta}{\partial x} \\ \quad + (-p + 2\mu \frac{\partial w}{\partial z})_I - (-p + 2\mu \frac{\partial w}{\partial z})_{II} = 0 \end{array} \right.$$

2) 理想流體(上下兩層都是, $\mu_I = 0$, $\mu_{II} = 0$) $\mu = 0$:

則(7)式化爲

$$(9) \quad p_I = p_{II}$$

3) 水平海面, 沒有垂直運動, 即 $\zeta = 0$, $w = 0$:

則(7)式化爲

$$(\mu \frac{\partial u}{\partial z})_I = (\mu \frac{\partial u}{\partial z})_{II}, \quad (\mu \frac{\partial v}{\partial z})_I = (\mu \frac{\partial v}{\partial z})_{II}$$

上述條件(2)及其特例, 並沒有將在界面上, 所有的分子力 (molecular force) 表示出來。

在不同的介質, 而由 $F(x, y, z, t) = 0$ 來分界時, 由於分子的內聚力造成了表面張力, 它能產生垂直應力之不連續。因此, 它可能被假定包含在平衡條件(2)內之壓力項。這些毛細力 (capillary force) 是依賴界面之曲率而定。它們表示界面上, 正比於曲率之壓力差。即

兩邊之表面力差值 = 毛細力

此種差值爲

$$(10) \quad \sigma \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \cos(\vec{e}_1, \vec{n})$$

此種力指向下面之介質。

其中 R_1, R_2 : 曲率的正半徑

σ : 表面張力係數。

在 20°C 空氣與水之交界面, $\sigma = 72.8 \text{ dyn/cm}$

上面(10)式之來源如下:

假定表面元素 (surface element) $df = ds_1 \cdot ds_2$ 是一個圓筒表面, ds_1 代表直線元素 (straight line element)。

ds_2 代表曲線元素 (curved line element)。

參考下圖。

在 ds_2 之作用力, 位於此表面上; 而在 ds_1 之作用力互相傾斜, 其交角為 $d\psi$ ($d\vec{K}_1'$ 和 $d\vec{K}_1''$), 但它們的合力向凹面方向;

$$\text{即 } d\vec{K}_1 = d\vec{K}_1' + d\vec{K}_1''$$

角度很小時, 則

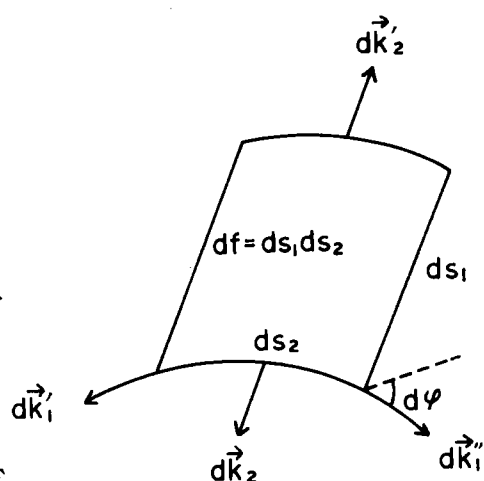
$$dK_1 \vec{n} = \sigma ds_1 d\varphi \vec{n}$$

$$\therefore R = ds_2 / d\varphi$$

$$\therefore dK_1 \vec{n} = \sigma \frac{ds_1 ds_2}{R} \vec{n} = \frac{\sigma}{R} df \vec{n}$$

在此, 曲率半徑之定義為

$$(11) \quad \frac{1}{R} = \lim_{\Delta s \rightarrow 0} \frac{\Delta \varphi}{\Delta s} = - \frac{\frac{\partial^2 F}{\partial x^2}}{[1 + (\frac{\partial F}{\partial x})^2]^{3/2}}$$



而如對於一個任意表面, 則與上述之差異為二個曲率主半徑 (即 R_1 與 R_2) 都需加以考慮, 而出現在方程式中。因此, 作用於表面元素之力為

$$dK\vec{n} = \sigma \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \vec{n} df$$

如將此力投影於基礎向量 $\vec{e}_i$ ，則為(10)式。現在由於表面張力關係，(3)式就可由下式代替了，

$$(12) \quad \left\{ \begin{array}{l} [(P_{xx})_I - (P_{xx})_II] \cos \alpha + [(P_{xy})_I - (P_{xy})_II] \cos \beta \\ \quad + [(P_{xz})_I - (P_{xz})_II] \cos \gamma = \sigma (R_1^{-1} + R_2^{-1}) \cos \alpha \\ [(P_{yx})_I - (P_{yx})_II] \cos \alpha + [(P_{yy})_I - (P_{yy})_II] \cos \beta \\ \quad + [(P_{yz})_I - (P_{yz})_II] \cos \gamma = \sigma (R_1^{-1} + R_2^{-1}) \cos \beta \\ [(P_{zx})_I - (P_{zx})_II] \cos \alpha + [(P_{zy})_I - (P_{zy})_II] \cos \beta \\ \quad + [(P_{zz})_I - (P_{zz})_II] \cos \gamma = \sigma (R_1^{-1} + R_2^{-1}) \cos \gamma \end{array} \right.$$

在此，同樣也有和(7)、(8)、(9)之類似關係成立；只是右邊改變而已，但對於二層理想流體之交界面。

$$(13) \quad P_{II} - P_I = \sigma (R_1^{-1} + R_2^{-1})$$

如將(11)式延伸，則 $\left(\frac{1}{R_1} + \frac{1}{R_2} \right)$ 之方程式如下：

$$(14) \quad \begin{aligned} \frac{1}{R_1} + \frac{1}{R_2} = & - \left\{ \frac{\partial^2 F}{\partial x^2} \left[\left(\frac{\partial F}{\partial y} \right)^2 + \left(\frac{\partial F}{\partial z} \right)^2 \right] \right. \\ & + \frac{\partial^2 F}{\partial y^2} \left[\left(\frac{\partial F}{\partial x} \right)^2 + \left(\frac{\partial F}{\partial z} \right)^2 \right] + \frac{\partial^2 F}{\partial z^2} \left[\left(\frac{\partial F}{\partial x} \right)^2 + \left(\frac{\partial F}{\partial y} \right)^2 \right] \\ & + 2 \left(\frac{\partial^2 F}{\partial x \partial y} \frac{\partial F}{\partial x} \frac{\partial F}{\partial y} + \frac{\partial^2 F}{\partial y \partial z} \frac{\partial F}{\partial y} \frac{\partial F}{\partial z} + \frac{\partial^2 F}{\partial z \partial x} \frac{\partial F}{\partial z} \frac{\partial F}{\partial x} \right) \left. \right\} \\ & \left[\left(\frac{\partial F}{\partial x} \right)^2 + \left(\frac{\partial F}{\partial y} \right)^2 + \left(\frac{\partial F}{\partial z} \right)^2 \right]^{-3/2} \end{aligned}$$

再由(6)式，則上式又化爲

$$(15) \quad \frac{1}{R_1} + \frac{1}{R_2} =$$

$$= - \frac{\frac{\partial^2 \zeta}{\partial x^2} [1 + (\frac{\partial \zeta}{\partial y})^2] + \frac{\partial^2 \zeta}{\partial y^2} [1 + (\frac{\partial \zeta}{\partial x})^2] - 2 \frac{\partial^2 \zeta}{\partial x \partial y} \frac{\partial \zeta}{\partial x} \frac{\partial \zeta}{\partial y}}{[1 + (\frac{\partial \zeta}{\partial x})^2 + (\frac{\partial \zeta}{\partial y})^2]^{-3/2}}$$

對於在 x, z 平面上之二度空間運動，則(15)式又可簡化為

$$(16) \quad \frac{1}{R_1} = - \frac{\partial^2 \zeta}{\partial x^2} / [1 + (\frac{\partial \zeta}{\partial x})^2]^{3/2}$$

如果(8)式是正確的話，則在簡單的情形中（即理想流體），我們可以將運動及動力二種邊界條件，合併為一種混合的邊界條件。因為，那時候 $F(x, y, z, t) = 0$ 就與 $P_I = P_{II}$ 之表面結合在一起。

由(6·2·3)得

$$(17) \quad \frac{\partial p_I}{\partial t} + \vec{v}_I \cdot \nabla p_I = \frac{\partial p_{II}}{\partial t} + \vec{v}_I \cdot \nabla p_{II},$$

$$\frac{\partial p_I}{\partial t} + \vec{v}_{II} \cdot \nabla p_I = \frac{\partial p_{II}}{\partial t} + \vec{v}_{II} \cdot \nabla p_{II}$$

§ 6、5 溫度的邊界條件

在一個不連續之表面，熱傳導方程式(4, 4)， $\frac{dT}{dt} = K_T \nabla^2 T$ ，就需由(6·4·1)來代替了。

在此，我們令 $\vec{q}_\phi \equiv \vec{H}$ ，如果穿過表面，無質量輸送發生，則得

$$(1) \quad (\vec{H}_I - \vec{H}_{II}) \cdot \vec{n} = 0,$$

由 Fourier 假設， $\vec{H} = -k \nabla T$

則得

$$(2) \quad (k \frac{\partial T}{\partial n})_I - (k \frac{\partial T}{\partial n})_{II} = 0$$

$$\text{或} \quad \frac{\partial T}{\partial n} = \frac{Q}{k_T \rho c_P},$$

其中 Q = 由大氣傳來之熱通量 (單位為 $\text{erg cm}^{-2} \text{ sec}^{-1}$)

$$c_P = 4.19 \times 10^7 \text{ cm}^2 \text{ sec}^{-2} (\text{°C})^{-1}$$

$$K_T = 1.4 \times 10^{-3} \text{ cm}^2 \text{ sec}^{-1}$$

§ 6、6 速度 ($\vec{v}$)、鹽度 (S) 和溫度 (T) 之平衡條件

前面 § 6、3 – § 6、5 之邊界條件，一般而言，是描述動量通量、擴散和熱通量之過程。這些過程就是可以減少在界面兩邊之差值。但通常，在邊界之外邊的通量是未知的或根本無法決定的。因此，我們就假定，由於通量之結果，在邊界之兩邊，其平衡早已達到。因此，對於這些通量之平衡條件，就由其場函數之平衡條件來代替；即

$$(1) \quad \vec{v}_I = \vec{v}_{II}, \quad T_I = T_{II}$$

特別對於堅固表面，無滑動條件 (non-slip condition)，即

$$(2) \quad \vec{v} = 0$$

此條件通常假定在海底之情形。

在許多例子中，流體動力實驗已證明(2)為真；然而在海洋中，其應用性尙未知。一般而言，條件(1)在海面上，並無實用；因為緊鄰海面上之風速及大氣溫度皆無法知道。

2. 紊流模式 (Turbulence Modelling)

黃 榮 鑑

I、紊流之發展與回顧 (The Development and Review of Turbulence)

A. 15 ~ 16 世紀 Leonardo da Vinci (1452 ~ 1519) 觀察風雲自然現象並予描述，實為最先提出紊流現象者，惟缺少數學模式描述解析。

B. 17 ~ 18 世紀間 I. Newton (1642 ~ 1729), L. Euler (1707 ~ 1782), D. Bernoulli (1700 ~ 1782), J.D'Alembert (1717 ~ 1782) 等學者相繼研究，乃有非黏性連體流體模式 (continuum inviscid fluid model) 之提出。

$$\rho \frac{DU_1}{Dt} = -\rho G_1 + \frac{\partial P}{\partial x_1} + (??)$$

↑
Viscous fluid model

C. 19 世紀 L.M. Navier (1785 ~ 1836), J.B. Fourier (~1882 ~), B. de Saint Venant (1797 ~ 1886), G.G. Stokes (1819 ~ 1903) 等學者研究導出了黏性傳導連體流體模式 (continuum viscous-conducting fluid model)。

$$\rho \frac{DU_i}{Dt} = -\rho G_i + \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + (??)$$

↑
Turbulent flow model

$$C_p \rho \frac{DT}{Dt} = \tau_{ij} \frac{\partial U_i}{\partial x_j} + \frac{Dp}{Dt} + \frac{\partial q_i}{\partial x_i} + (??)$$

↑
Turbulent heat flux model

D. 20 世紀 O. Reynolds (1842 ~ 1912), T. Von Karman (1881 ~ 1963), L. Prandtl (1875 ~ 1953), Sir. G. I. Taylor (1880 ~ 1975) 等學者研究提出平均黏性傳導連體系流模式 (Averaged continuum viscous-conducting turbulent model)。

$$\rho \frac{DU_i}{Dt} = -\rho G_i + \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \frac{\partial \tau'_{ij}}{\partial x_j} + (??)$$

$$C_p \rho \frac{DT}{Dt} = \tau_{ij} \frac{\partial U_i}{\partial x_j} + \frac{Dp}{Dt} + \frac{\partial q_i}{\partial x_i} + \frac{\partial q'_i}{\partial x_i} + \phi + (??)$$

II、紊流運動之處理及封閉性問題 (The Treatment of Turbulent Flow and the Problem of Closure)

II-1 紊流運動之處理方法 (Methods of the Treatment of Turbulent Flow)

紊流運動學理依其處理方法可分為二類：

A. 現象分析法 (phenomenological analysis)

該法觀察紊流現象並做某種學理假說，分析觀測資料以獲致結果描述紊流。該方法又稱為半經驗學理 (semi-empirical theories of turbulent flow)，因其使用因次及相似推論 (

dimensional and similarity arguments)而得之調混尺度學理 (mixing-length theories)。半世紀末，利用此方法處理紊流運動，雖已被工程界廣泛採用，然其理論基礎並不健全合理，又無法提供吾人進一步瞭解紊流內部之重要性質。

B. 統計法 (statistical analysis)

該法引用統計力學方法，對於隨機函數 (random functions) 之平均值做嚴格的規定，並引入等向性紊流 (isotropic turbulence)、相關函數 (correlation function) 及頻譜函數 (spectral function) 等，分析紊流內部之結構、能量消弱、及擴散原理等。

II - 2 閉合問題 (Problem of Closure)

在描述紊流運動之數學模式裡，總是會碰到未知數的數目多於方程式數目，此即所謂閉合問題。為解決此問題，對於紊流項 (turbulence term) 的模擬即有各種不同的方法，亦即有不同的紊流模式。就模擬紊流項使用方程式數目而言，有所謂零方程式模式 (Zero-equation model)、單方程式模式 (one-equation model)、雙方程式模式 (two-equation model) 等。就模擬紊流項方程式之階數 (Order) 而言，有所謂第一階閉合 (first order closure)、第二階閉合 (second order closure)、高階閉合 (high order closure) 等。此處僅就第一階閉合、第二階閉合之紊流模式概略討論介紹如後。

III、紊流模式 (Turbulence Models)

III-1 紊流本質 (The Nature of Turbulence)

紊流是一種渦流運動 (eddy motion)，該渦流運動為旋性流 (rotational flow) 並可視為許多旋渦元素 (vortex elements) 糾纏在一起，該等旋渦元素之渦度向量 (vorticity vectors) 是各方向的而且相當不定的 (unsteady)。最大渦流伴隨低頻擾動 (low frequency fluctuations) 係由流場的邊界條件來決定，其尺度與流場區域 (flow domain) 同一級序 (order of magnitude)。最小渦流伴隨高頻擾動 (high frequency fluctuations)，且由黏滯力 (viscous force) 來決定。

動量、熱量之傳輸及紊流相關性 (correlation) $\overline{u_i u_j}$, $\overline{u_i \phi}$ 等之貢獻主要靠大尺度紊流運動，因此在紊流模式中大尺度運動必須模擬以決定 $\overline{u_i u_j}$, $\overline{u_i \phi}$ ，而速度、長度尺度 (velocity and length scales) 引入紊流模式中當作決定此運動之參數。

大渦流與平均流 (mean flow) 作用後，由平均流中取得動能轉給大尺度紊流運動。渦流中旋渦元素彼此擴展，故旋渦擴展 (vortex stretching) 為紊流運動之基本現象，藉此旋渦擴展將能量傳遞至更小的渦流，直到黏滯力明顯重要，而能量則消散。

雖然能量消散是一種黏滯過程 (viscous process) 且發生在最小渦流，然能量消散率仍由大尺度運動決定。黏滯係數 (viscosity) 並不決定能量消散量，僅決定發生能量消散之尺度。

III-2 平均紊流方程式 (Averaged Turbulence Equations)

III-2-1 Navier-Stokes 方程式

$$u_i^* = U_i + u_i \quad p^* = P + p \quad T^* = T + \theta$$

A. 連續方程式 (Continuity equation)

$$\frac{\partial U_i}{\partial x_i} = 0 \quad (\bar{c})$$

B. 動量方程式 (momentum equation)

$$\rho \frac{DU_i}{Dt} = -\rho G_i + \frac{\partial P}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \frac{\partial \tau_{ij}^t}{\partial x_j} \quad (\bar{M}_i)$$

$$\tau_{ij} = \mu \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \quad \text{由 viscous fluid model 決定}$$

$$\tau_{ij}^t = -\overline{\rho u_i u_j} \quad \text{由紊流模式決定}$$

C. 能量方程式 (Energy equation)

$$C_p \rho \frac{DT}{Dt} = \tau_{ij} \frac{\partial U_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i} + \frac{\partial q_i'}{\partial x_i} + \phi \quad (T)$$

$$q_i = -k \frac{\partial T}{\partial x_i} \quad \text{由 Fourier 假說決定}$$

$$\left. \begin{aligned} q_i' &= -C_p \rho \overline{u_i \theta} \\ \phi &= \overline{\tau_{ij}' \frac{\partial u_i}{\partial x_j}} \end{aligned} \right\} \text{由紊流模式決定}$$

D. 擾動方程式 (fluctuation equation)

$$[\text{動量方程式}] - [\overline{\text{動量方程式}}] = u_i \text{ 方程式} \quad (m_i)$$

[能量方程式] - [能量方程式] = θ 方程式 (e)

III-2-2 紊流傳輸方程式 (Turbulence Transport Equations)

A. $\overline{u_i u_j}$ 方程式

利用上節求得之擾動方程式 m_i ，運算 $\overline{m_i u_j + m_j u_i}$ 整理後得

$$\begin{aligned} \frac{D \overline{u_i u_j}}{D t} = \frac{\partial}{\partial x_i} \left[- \overline{u_i u_j u_i} - \frac{p}{\rho} (\delta_{ji} u_i + \delta_{ii} u_j) \right. \\ \left. + \nu \frac{\partial \overline{u_i u_j}}{\partial x_i} \right] - (\overline{u_i u_i} \frac{\partial U_j}{\partial x_i} \\ + \overline{u_j u_i} \frac{\partial U_i}{\partial x_i} - 2\nu \frac{\partial \overline{u_i}}{\partial x_i} \frac{\partial \overline{u_j}}{\partial x_i} \\ + \frac{p}{\rho} (\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i})) \end{aligned}$$

式中等號右邊第一項為擴散項 (Diffusion term)，其中含 ν 之項為分子擴散 (molecular diffusion)，其餘為紊流擴散 (turbulent diffusion)；第二項為產生項 (production term)；第三項為消散項 (Dissipation term)；第四項為壓力應變項 (pressure-strain-term)。

B. k 方程式

令 $i = j$ 則 $k = \overline{u_i u_i} / 2$ ， $K' = u_i u_i / 2$ ；上式改寫成：

$$\begin{aligned} \frac{D k}{D t} = \frac{\partial}{\partial x_i} \left[- \overline{k' u_i} - \frac{p u_i}{\rho} + \nu \frac{\partial k}{\partial x_i} \right] \\ - \overline{u_i u_i} \frac{\partial U_i}{\partial x_i} - \epsilon \end{aligned}$$

C. ε 方程式

$$\text{令 } \varepsilon = \nu \overline{\frac{\partial u_i}{\partial x_i} \frac{\partial u_i}{\partial x_i}}, \quad \varepsilon' = \nu \frac{\partial u_i}{\partial x_i} \frac{\partial u_i}{\partial x_i}$$

經由運算 $2 \overline{\frac{\partial m_i}{\partial x_i} \frac{\partial u_i}{\partial x_i}}$ 整理後得

$$\begin{aligned} \frac{D\varepsilon}{Dt} = \frac{\partial}{\partial x_i} \left[-\overline{\varepsilon' u_i} - \frac{2\nu}{\rho} \overline{\frac{\partial u_i}{\partial x_j} \frac{\partial p}{\partial x_j}} + \nu \frac{\partial \varepsilon}{\partial x_i} \right] \\ - 2\nu u_i \overline{\frac{\partial u_i}{\partial x_j} \frac{\partial^2 U_i}{\partial x_i \partial x_j}} - 2\nu \frac{\partial U_i}{\partial x_j} \left(\overline{\frac{\partial u_i}{\partial x_i} \frac{\partial u_i}{\partial x_j}} \right. \\ \left. + \overline{\frac{\partial u_i}{\partial x_i} \frac{\partial u_j}{\partial x_i}} \right) - 2\nu \overline{\frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_i} \frac{\partial u_j}{\partial x_i}} \\ - 2 \left(\nu \overline{\frac{\partial^2 u_i}{\partial x_i \partial x_j}} \right)^2 \end{aligned}$$

式中等號右邊第一項為擴散項，第二、三項為產生項，第四、五項為消毀項 (Destruction term)。

D. $\overline{u_i \theta}$ 方程式

令 $\phi' = \tau' u_i \frac{\partial u_i}{\partial x_j}$ ，利用上節求得之擾動方程式 m_i ， e ，

運算 $\overline{m_i \theta + e u_i}$ ，整理得

$$\begin{aligned} \frac{D\overline{u_i \theta}}{Dt} = \frac{\partial}{\partial x_i} \left[-\overline{u_i u_i \theta} - \delta_{ij} \frac{\overline{p \theta}}{\rho} + \alpha u_i \frac{\partial \overline{\theta}}{\partial x_i} + \nu \theta \frac{\partial \overline{u_i}}{\partial x_i} \right] \\ - \left(\overline{u_i u_i} \frac{\partial T}{\partial x_i} + \overline{u_i \theta} \frac{\partial U_i}{\partial x_i} \right) \\ - (\alpha + \nu) \overline{\frac{\partial u_i}{\partial x_i} \frac{\partial \theta}{\partial x_i}} + \frac{\overline{p}}{\rho} \frac{\partial \overline{\theta}}{\partial x_i} + \overline{\phi' u_i} \end{aligned}$$

式中等號右邊第一項爲擴散項，其中含 α 及 ν 之項爲分子擴散，其餘爲紊流擴散；第二項爲產生項；第三項爲消散項；第四項爲壓力溫度項 (pressure-Temperature term)；第五項爲摩擦項 (Friction term)。

III-3 一階閉合紊流模式 (First Order Closure Turbulence Model)

$$\rho \frac{DU_i}{Dt} = -\rho G_i + \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \frac{\partial \tau_{ij}^t}{\partial x_j}$$

$$\tau_{ij} = \mu \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \quad \tau_{ij}^t = -\overline{\rho u_i u_j}$$

一階閉合模式又稱爲零方程式模式，亦即對紊流項 $\overline{u_i u_j}$ 直接模式化 (modelling)，例如：

A. Boussinesq 渦黏性模式 (Boussinesq eddy viscosity Model)

Boussinesq 氏認爲 $-\overline{u v}$ 爲平均速度梯度與渦黏性係數 (或紊流黏性係數 turbulent viscosity) 之乘積，即

$$-\overline{u v} = \nu_t \frac{\partial U}{\partial Y}$$

ν_t 不是流體之性質 (property of the fluid)，隨所在點之紊流結構而變化。

B. Prandtl 混調尺度模式 (Prandtl mixing length Model)

Prandtl 氏認為 ν_t 為速度尺度 (velocity scale) 與長度尺度 (length scale) 之乘積；而速度尺度為混調尺度與平均流速梯度之乘積；是故 ν_t 可表為：

$$\nu_t = \ell^2 \left| \frac{\partial U}{\partial y} \right| \quad \text{亦即 } -\overline{uv} \text{ 可寫成}$$

$$-\overline{uv} = \ell^2 \left| \frac{\partial U}{\partial y} \right| \frac{\partial U}{\partial y}$$

此即所謂 Prandtl 混調尺度模式。

C. Karman 混調尺度模式 (Karman mixing length Model)

Karman 氏認為混調尺度 ℓ 與 $\frac{\partial U}{\partial y} / \frac{\partial^2 U}{\partial y^2}$ 成比例，故 $-\overline{uv}$ 表為：

$$-\overline{uv} = k \left| \frac{\partial U}{\partial y} / \frac{\partial^2 U}{\partial y^2} \right| \frac{\partial U}{\partial y}$$

以上所舉一階閉合紊流模式優點是簡易，適用於簡單流況 (simple flow)，缺點是流況變化較複雜則不適用，而且將紊流複雜情況全歸於一經驗係數，亦即解決問題之預測性 (predictability) 甚低。

III - 4 二階閉合紊流模式 (Second Order Closure Turbulence Model)

III - 4 - 1 紊流閉合假設 (Turbulence Closure Postulations)

為模式化不可壓縮紊流 (incompressible turbulence)，

做如下之假設：

1. 平均 Navier - Stokes 方程式 (Averaged Navier - Stokes Equations) 可以應用，亦即與紊流之渦流結構無關。
2. $\overline{\rho u_i u_j}$ 之紊流擴散與 $\overline{u_i u_j}$ 之梯度成正比例。
3. $\overline{\rho \phi}$ 之紊流擴散與 ϕ 之梯度成正比例。
4. 小渦流 (small eddies) 爲等向性 (isotropic)。
5. 所有紊流量 (turbulent quantities) 均表示爲 $f(\overline{u_i u_j}, k, \varepsilon, \overline{u_i \theta}, U_i, P, T, \rho, \nu, \alpha)$ 。
6. 模式化之紊流現象在對稱 (symmetry)，不變性 (invariant)，排列 (permutation)，觀測 (observation) 等必須一致的。
7. 紊流尺度假說：即紊流尺度是 k, ε 之函數， $u = \sqrt{k}$ ， $\ell = k^{3/2} / \varepsilon$ ， $t = k / \varepsilon$ 。
8. 紊流常數 $C_\varepsilon, C_{\varepsilon_1}, C_{\varepsilon_2}, C_k, C_1, C_2, C_T, C_{T_1}, C_{T_2}$ 由實驗決定。

III - 4 - 2 $\overline{u_i u_j}$ 方程式之模式化 (Modelling of $\overline{u_i u_j}$ Equations)

$$\begin{aligned} \frac{D \overline{u_i u_j}}{D t} = & \frac{\partial}{\partial x_l} \left\{ - \overline{u_i u_j u_l} - \frac{p}{\rho} (\delta_{jl} u_i + \delta_{il} u_j) \right. \\ & \left. + \nu \frac{\partial \overline{u_i u_j}}{\partial x_l} \right\} \\ & - \left(\overline{u_j u_l} \frac{\partial U_i}{\partial x_l} + \overline{u_i u_l} \frac{\partial U_j}{\partial x_l} \right) \\ & - 2\nu \frac{\partial \overline{u_i}}{\partial x_l} \frac{\partial \overline{u_j}}{\partial x_l} + \frac{p}{\rho} \left(\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i} \right) \end{aligned}$$

A. 擴散項之模式化

$$-\overline{u_i u_j u_i} - \frac{p}{\rho} (\delta_{ji} u_i + \delta_{ii} u_j) = C_k \left[\frac{\ell^2}{t} \right] \frac{\partial \overline{u_i u_j}}{\partial x_i}$$

由因次分析得 $u = \sqrt{K}$, $\ell = k^{3/2}/\varepsilon$, $t = k/\varepsilon$

即 $[\ell^2/t] = k^2/\varepsilon$, 故

$$-\overline{u_i u_j u_i} = C_k \frac{k^2}{\varepsilon} \frac{\partial \overline{u_i u_j}}{\partial x_i}$$

B. 消散項之模式化

$$2\nu \frac{\partial u_i}{\partial x_i} \frac{\partial u_j}{\partial x_i} = \frac{2}{3} \delta_{ij} \varepsilon \quad \varepsilon = \nu \frac{\partial u_i}{\partial x_i} \frac{\partial u_i}{\partial x_i}$$

由假設 4. 知當 $i \neq j$ 時 $\delta_{ij} = 0$ 即此項爲零。

C. 壓力應變項之模式化

由擾動方程式 m_i 得

$$\frac{Du_i}{Dt} + u_i \frac{\partial U_i}{\partial x_i} + u_i \frac{\partial u_i}{\partial x_i} - \frac{\partial \overline{u_i u_i} \ell}{\partial x_i} = \frac{-1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_i \partial x_i}$$

由 $\nabla \cdot (m_i)$ 得

$$\nabla^2 \frac{p}{\rho} = - \left[\frac{\partial^2 (u_i u_m - \overline{u_i u_m})}{\partial x_i \partial x_m} + 2 \frac{\partial U_i}{\partial x_i} \frac{\partial U_m}{\partial x_i} \right]$$

積分得

$$\frac{p}{\rho} = \frac{1}{4\pi} \int_{vol} \left[\frac{\partial^2 (u_i u_m - \overline{u_i u_m})}{\partial x_i \partial x_m} + 2 \frac{\partial U_i}{\partial x_i} \frac{\partial u_m}{\partial x_i} \right] \frac{dVol}{Y}$$

故壓力應變項取平均值得：

$$\overline{\frac{p}{\rho} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)} = \frac{1}{4\pi} \int_{\text{Vol}} \left[\overline{\left(\frac{\partial^2 u_l u_m}{\partial x_l \partial x_m} \right) \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)} + \right. \\ \left. 2 \frac{\partial U_l}{\partial x_m} \frac{\partial u_m}{\partial x_l} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] \frac{d \text{Vol}}{Y}$$

上式右邊第一項令爲 $\Phi_{ij,1}$ ，當 Vol 趨近於很小時

$$\Phi_{ij,1} = \text{Constant} \left[\frac{1}{t} \right] \left(\overline{u_i u_j} - \frac{2}{3} \delta_{ij} k \right) \\ = -C_1 \frac{\varepsilon}{k} \left(\overline{u_i u_j} - \frac{2}{3} \delta_{ij} k \right)$$

上式右邊第二項令爲 $\Phi_{ij,2}$ ，當 Vol 趨近於很小

$$\Phi_{ij,2} = \text{Constant} \left(\frac{\partial U_l}{\partial x_m} \overline{u_m u_i} + \frac{\partial U_l}{\partial x_m} \overline{u_m u_j} \right. \\ \left. - \frac{2}{3} \delta_{ij} \frac{\partial U_n}{\partial x_m} \overline{u_m u_n} \right) \\ = -C_2 \left(p_{ij} - \frac{2}{3} \delta_{ij} p_k \right) \\ p_{ij} = - \left(\overline{u_i u_m} \frac{\partial U_j}{\partial x_m} + \overline{u_j u_m} \frac{\partial U_i}{\partial x_m} \right) \\ p_k = - \overline{u_n u_m} \frac{\partial U_n}{\partial x_m}$$

D. 綜合上述各項模式化結果整理得

$$\frac{D \overline{u_i u_j}}{Dt} = \frac{\partial}{\partial x_l} \left[C_k \frac{k^2}{\varepsilon} \frac{\partial \overline{u_i u_j}}{\partial x_l} + \nu \frac{\partial \overline{u_i u_j}}{\partial x_l} \right] + p_{ij} \\ - \frac{2}{3} \delta_{ij} \varepsilon - C_1 \frac{\varepsilon}{k} \left(\overline{u_i u_j} - \frac{2}{3} \delta_{ij} k \right) \\ - C_2 \left(p_{ij} - \frac{2}{3} \delta_{ij} p_k \right)$$

式中係數可由實驗決定之， $C_k \doteq 0.225 \sim 0.25$ ， $C_1 \doteq 1.5 \sim 2.2$ ， $C_2 \doteq 0.4 \sim 0.5$

III-4-3 k 方程式之模式化 (Modelling of k Equation)

因 $\overline{u_i u_j}$ 方程式已如上述模式化，k 方程式無須再模式化，即令 $i=j$ ， $i = 1, 2, 3$ 予以累加，即得模式化後之 k 方程式如下：

$$\frac{Dk}{Dt} = \frac{\partial}{\partial x_i} \left[C_k \frac{k^2}{\varepsilon} \frac{\partial k}{\partial x_i} + \nu \frac{\partial k}{\partial x_i} \right] - \overline{u_i u_i} \frac{\partial U_i}{\partial x_i} - \varepsilon$$

III-4-4 ε 方程式之模式化 (Modelling of ε Equation)

$$\begin{aligned} \frac{D\varepsilon}{Dt} = & \frac{\partial}{\partial x_i} \left[-\overline{\varepsilon' u_i} - \frac{2\nu}{\rho} \overline{\frac{\partial u_i}{\partial x_j} \frac{\partial p}{\partial x_j}} + \nu \frac{\partial \varepsilon}{\partial x_i} \right] \\ & - 2\nu \frac{\partial U_i}{\partial x_j} \left(\overline{\frac{\partial u_i}{\partial x_i} \frac{\partial u_i}{\partial x_j}} + \overline{\frac{\partial u_i}{\partial x_i} \frac{\partial u_j}{\partial x_i}} \right) \\ & - 2\nu \overline{u_i \frac{\partial u_i}{\partial x_j} \frac{\partial^2 U_i}{\partial x_i \partial x_j}} - 2\nu \overline{\frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_i} \frac{\partial u_j}{\partial x_i}} \\ & - 2 \left(\nu \overline{\frac{\partial^2 u_i}{\partial x_i \partial x_j}} \right)^2 \end{aligned}$$

A. 紊流擴散項之模式化

$$-\overline{\varepsilon' u_i} - \frac{2\nu}{\rho} \overline{\frac{\partial u_i}{\partial x_j} \frac{\partial p}{\partial x_j}} = C_\varepsilon \left[\frac{\ell^2}{t} \right] \frac{\partial \varepsilon}{\partial x_i}$$

由 (k, ε) 尺度分析， $\ell = k^{3/2} / \varepsilon$ ， $t = k / \varepsilon$ ，故

$$-\overline{\varepsilon' u_i} - \frac{2\nu}{\rho} \overline{\frac{\partial u_i}{\partial x_j} \frac{\partial p}{\partial x_j}} = C_\varepsilon \frac{k^2}{\varepsilon} \frac{\partial \varepsilon}{\partial x_i}$$

B. 產生項之模式化

由級序大小 (order of magnitude) 分析，得

$$2\nu \overline{u_i} \frac{\partial u_i}{\partial x_j} \frac{\partial^2 U_i}{\partial x_i \partial x_j} \doteq 0$$

又 $\nu \frac{\partial U_i}{\partial x_j} \left(\overline{\frac{\partial u_i}{\partial x_i} \frac{\partial u_i}{\partial x_j}} + \overline{\frac{\partial u_i}{\partial x_i} \frac{\partial u_j}{\partial x_i}} \right)$ 分析如下：

當 $i = j$ 時，因流體之不可壓縮性 $\nu \frac{\partial U_i}{\partial x_i} = 0$ ，故上式為 0；

當 $i \neq j$ 時，由於等向性能量消散， $\left(\overline{\frac{\partial u_i}{\partial x_i} \frac{\partial u_i}{\partial x_j}} + \overline{\frac{\partial u_i}{\partial x_i} \frac{\partial u_j}{\partial x_i}} \right) = 0$

C. 消毀項之模式化

$$\begin{aligned} & -2\nu \overline{\frac{\partial x_i}{\partial x_j} \frac{\partial u_i}{\partial x_i} \frac{\partial u_j}{\partial x_i}} - 2 \left(\nu \overline{\frac{\partial^2 u_i}{\partial x_i \partial x_j}} \right)^2 \\ & = C_{\epsilon_1} \left[\frac{1}{t} \right] \left(-\overline{u_i u_i} \frac{\partial U_i}{\partial x_i} \right) - C_{\epsilon_2} \left[\frac{\epsilon}{t} \right] \end{aligned}$$

若由 (k, ϵ) 尺度分析，則 $\left[\frac{1}{t} \right] = \frac{\epsilon}{k}$

上式變為：

$$-C_{\epsilon_1} \frac{\epsilon}{k} \overline{u_i u_i} \frac{\partial U_i}{\partial x_i} - C_{\epsilon_2} \frac{\epsilon^2}{k}$$

若由 (ν, ϵ) 尺度分析，則 $\left[\frac{1}{t} \right] = \sqrt{\frac{\epsilon}{\nu}}$

上式變為

$$-C_{\epsilon_1} \sqrt{\frac{\epsilon}{\nu}} \overline{u_i u_i} \frac{\partial U_i}{\partial x_i} - C_{\epsilon_2} \sqrt{\frac{\epsilon}{\nu}} \epsilon$$

D. 綜合以上所述各項模式化結果，得

$$\frac{D\varepsilon}{Dt} = \frac{\partial}{\partial x_i} \left[C_\varepsilon \frac{k^2}{\varepsilon} \frac{\partial \varepsilon}{\partial x_i} + \nu \frac{\partial \varepsilon}{\partial x_i} \right] - C_{\varepsilon 1} \left[\frac{1}{t} \right] \overline{u_i u_i} \frac{\partial U_i}{\partial x_i} \\ - C_{\varepsilon 2} \left[\frac{1}{t} \right] \varepsilon$$

式中各係數 $C_\varepsilon = 0.13 \sim 0.17$, $C_{\varepsilon 1} = 1.41 \sim 1.45$, $C_{\varepsilon 2} = 1.9 \sim 1.92$; 若採用 (k, ε) 尺度分析, 則 $\left[\frac{1}{t} \right] = \frac{\varepsilon}{k}$; 若採用 (ν, ε) 尺度分析, 則 $\left[\frac{1}{t} \right] = \sqrt{\frac{\varepsilon}{\nu}}$ 。

III-4-5 $\overline{u_i \theta}$ 方程式之模式化 (Modelling of $\overline{u_i \theta}$ equations)

$$\frac{D \overline{u_i \theta}}{Dt} = \frac{\partial}{\partial x_i} \left[- \overline{u_i u_i \theta} - \delta_{ij} \frac{\overline{p \theta}}{\rho} + \alpha \overline{u_i} \frac{\partial \theta}{\partial x_j} \right. \\ \left. + \nu \theta \frac{\partial u_i}{\partial x_j} \right] - \left(\overline{u_i u_i} \frac{\partial T}{\partial x_i} + \overline{u_i \theta} \frac{\partial U_i}{\partial x_i} \right) \\ - (\alpha + \nu) \frac{\partial u_i}{\partial x_i} \frac{\partial \theta}{\partial x_i} + \frac{\overline{p}}{\rho} \frac{\partial \theta}{\partial x_i} + \overline{\phi' u_i}$$

A. 擴散項之模式化

$$- \overline{u_i u_i \theta} - \delta_{ij} \frac{\overline{p \theta}}{\rho} = C_T \left[\frac{1}{t} \right] \frac{\partial \overline{u_i \theta}}{\partial x_i}$$

由 (k, ε) 尺度分析 $\left[\frac{1}{t} \right] = \left[\frac{k^2}{\varepsilon} \right]$, 故上式為 $C_T \frac{k^2}{\varepsilon} \frac{\partial \overline{u_i \theta}}{\partial x_i}$ 。

$\alpha \overline{u_i} \frac{\partial \theta}{\partial x_i} + \nu \theta \frac{\partial U_i}{\partial x_i}$, 當 $\alpha = \nu$ 時, 上式可寫成

$$\alpha \overline{u_i} \frac{\partial \theta}{\partial x_i} + \nu \theta \frac{\partial u_i}{\partial x_i} \div \alpha \frac{\partial \overline{u_i \theta}}{\partial x_i}$$

B. 消散項之模式化

由假設 4，即等向性消散 (isotropic dissipation) 得

$$-(\alpha + \nu) \overline{\frac{\partial u_i}{\partial x_i} \frac{\partial \theta}{\partial x_i}} \doteq 0$$

C. 壓力溫度 (p - \theta) 項之模式化

$$\begin{aligned} \frac{p}{\rho} \frac{\partial \theta}{\partial x_i} &= \frac{1}{4\pi} \int_{\text{Vol}} \left[\overline{\frac{\partial^2 u_i u_m}{\partial x_i \partial x_m} \frac{\partial \theta}{\partial x_i}} \right. \\ &\quad \left. + 2 \left(\frac{\partial U_i}{\partial x_m} \right) \overline{\left(\frac{\partial u_m}{\partial x_i} \right) \frac{\partial \theta}{\partial x_i}} \right] \frac{d \text{Vol}}{r} \end{aligned}$$

當 Vol 趨近於很小時，上式寫成

$$\text{constant} \left[\overline{\frac{\partial^2 u_i u_m}{\partial x_i \partial x_m} \frac{\partial \theta}{\partial x_i}} + 2 \frac{\partial U_i}{\partial x_m} \overline{\frac{\partial u_m}{\partial x_i} \frac{\partial \theta}{\partial x_i}} \right] \ell^2$$

由假設 5、6，上式再寫成

$$-C_{T1} \left[\frac{1}{t} \right] \overline{u_i \theta} + C_{T2} \frac{\partial U_i}{\partial x_m} \overline{u_m \theta}$$

由 (k, \epsilon) 尺度分析，\left[\frac{1}{t} \right] = \left[\frac{\epsilon}{k} \right]，上式為

$$C_{T1} \frac{\epsilon}{k} \overline{u_i \theta} + C_{T2} \frac{\partial U_i}{\partial x_m} \overline{u_m \theta}$$

D. 摩擦項之模式化

該項之級序大小較之他項為小，故

$$\overline{\phi' u_i} \doteq 0$$

E. 綜合上述各項之模式化結果，得

$$\begin{aligned} \frac{D\overline{u_i\theta}}{Dt} = \frac{\partial}{\partial x_i} \left[C_T \frac{k^2}{\varepsilon} \frac{\partial \overline{u_i\theta}}{\partial x_i} + \alpha \frac{\partial \overline{u_i\theta}}{\partial x_i} \right] - \left(\overline{u_i u_i} \frac{\partial T}{\partial x_i} \right. \\ \left. + \overline{u_i\theta} \frac{\partial U_i}{\partial x_i} \right) - C_{T1} \frac{\varepsilon}{k} \overline{u_i\theta} + C_{T2} \frac{\partial U_i}{\partial x_m} \overline{u_m\theta} \end{aligned}$$

式中各係數可由實驗決定得 $C_T = 0.07$, $C_{T1} = 3.2$,

$$C_{T2} = 0.5 \circ$$

III - 4 - 6 微分模式或雷諾茲應力模式 (Differential Model or Reynolds Stress Model)

該模式簡稱為R SM模式。因使用 $\overline{u_i u_j}$, $\overline{u_i \theta}$ 之微分式，故稱為微分模式。

該紊流模式使用紊流傳輸方程式

$$\frac{D\overline{u_i u_j}}{Dt} = D_{ij} + p_{ij} - \frac{2}{3} \delta_{ij} k + p_s$$

$$\frac{Dk}{Dt} = D_k + p_k - \varepsilon \quad \varepsilon = \mu \frac{\partial u_i}{\partial x_i} \frac{\partial u_i}{\partial x_i}$$

$$\frac{D\varepsilon}{Dt} = D\varepsilon + p\varepsilon - \phi\varepsilon$$

$$\frac{D\overline{u_i \theta}}{Dt} = D\overline{u_i \theta} + P\overline{u_i \theta} + p_T$$

配合平均紊流方程式

$$\frac{DU_i}{Dt} = -\rho G_i + \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \frac{D\tau_{ij}^t}{Dt}$$

$$\tau_{ij} = \mu \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \quad \tau_{ij}^t = -\rho \overline{u_i u_j}$$

$$\frac{DT}{Dt} = \tau_{ij} \frac{\partial U_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i} + \frac{\partial q_i^t}{\partial x_i} + \phi$$

$$q_i = -k \frac{\partial T}{\partial x_i} \quad q_i^t = -\rho c_p \overline{u_i \theta}$$

$$\phi = \tau_{ij} \frac{\partial u_i}{\partial x_j} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \frac{\partial u_i}{\partial x_j} \div \varepsilon$$

$$\frac{\partial U_i}{\partial x_i} = 0$$

合計紊流傳輸方程式 11 個，平均紊流方程式 5 個，共計方程式 16 個，解 U_i ， p ， T ， $\overline{u_i u_j}$ ， k ， ε ， $\overline{u_i \theta}$ 16 個未知數，配合適當的邊界條件及起始條件，解出紊流流場。

III-4-7 $k-\varepsilon-A$ (ASM) 模式 ($k-\varepsilon-A$ (Algebraic Stress Model))

RSM 模式須解 11 個紊流傳輸方程式，過於耗費電腦時間，對於工程上來說不經濟且不必要。因此透過適當的近似 (approximation) 可以簡化紊流傳輸方程式，減少電腦運算時間。

對流體來說當紊流對流及擴散項很小時 (即高剪力 (high shear flows)) 或紊流對流及擴散項近似相等 (即局部平衡 (local equilibrium)) 時， $\overline{u_i u_j}$ 方程式可簡化成

$$(1 - C_2) p_{ij} - C_1 \frac{\varepsilon}{k} \left(\overline{u_i u_j} - \frac{2}{3} \delta_{ij} k \right)$$

$$- \frac{2}{3} \delta_{ij} (\varepsilon - C_2 p_k) = 0$$

同理在高剪力流且高溫度梯度時或局部紊流平衡時， $\overline{u_i \theta}$ 方程式可簡化成

$$\overline{u_i u_i} \frac{\partial T}{\partial x_i} + C_{T1} \frac{\epsilon}{k} \overline{u_i \theta} + (1 - C_{T2}) \overline{u_m \theta} \frac{\partial U_i}{\partial x_m} = 0$$

由上述知 $\overline{u_i u_j}$, $\overline{u_i \theta}$ 紊流傳輸方程式已由微分方程形式轉成代數方程式 (algebraic equations) 形成。

該等代數方程式形式之 $\overline{u_i u_j}$, $\overline{u_i \theta}$ 配合 k 方程式, ϵ 方程式, 即構成 $k - \epsilon - A$ 模式, 聯合平均紊流方程式, 加上邊界條件、起始條件, 即可求解紊流流場。

III-4-8 $k - \epsilon - E$ 模式 ($k - \epsilon - Eddy$ viscosity Model)

該模式中紊流傳輸方程式 $\overline{u_i u_j}$, $\overline{u_i \theta}$ 採用一般化之 Boussinesq 渦黏性模式 (generalized Boussinesq eddy viscosity model), 故 $\overline{u_i u_j}$ 與 $\overline{u_i \theta}$ 分別可表為

$$-\overline{u_i u_j} = \nu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3} \delta_{ij} k$$

$$-\overline{u_i \theta} = \alpha_t \frac{\partial T}{\partial x_j}$$

由因次分析, ν_t , α_t 可表為

$$\nu_t = C_\mu \frac{k^2}{\epsilon}, \quad \alpha_t = C_\alpha \frac{k^2}{\epsilon} = \frac{C_\mu}{Pr_t} \frac{k^2}{\epsilon}$$

k , ϵ 由 k 與 ϵ 方程式求得, $C_\mu \doteq 0.09$, $Pr_t \doteq 0.8 \sim 1.3$

上述之 $\overline{u_i u_j}$, $\overline{u_i \theta}$ 配合 k , ϵ 構成 $k - \epsilon - E$ 模式。

一般說來 $k - \epsilon - E$ 模式不如 $k - \epsilon - A$ 模式好。

III-4-9 K (單方程式) 模式 (K (one-equation) Model)

$$\frac{Dk}{Dt} = \frac{\partial}{\partial x_i} \left[\left(C_k \frac{k^2}{\varepsilon} + \nu \right) \frac{\partial k}{\partial x_i} \right] - \overline{u_i u_i} \frac{\partial U_i}{\partial x_i} - \varepsilon$$

由 (k, ε) 尺度分析, $\ell = k^{3/2} / \varepsilon$, 故 $\varepsilon = \frac{k^{3/2}}{\ell}$, 即

$$\frac{Dk}{Dt} = \frac{\partial}{\partial x_i} \left[(C_k \sqrt{k} \ell + \nu) \frac{\partial k}{\partial x_i} \right] - \overline{u_i u_i} \frac{\partial U_i}{\partial x_i} - \frac{k^{3/2}}{\ell}$$

而 $\overline{u_i u_j}$, $\overline{u_i \theta}$ 分別可表為

$$\begin{aligned} -\overline{u_i u_j} &= \nu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3} \delta_{ij} K \\ &= C_\mu \sqrt{k} \ell \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3} \delta_{ij} K \end{aligned}$$

$$-\overline{u_i \theta} = \frac{\nu_t}{Pr_t} \frac{\partial T}{\partial x_i} = C_\mu \sqrt{k} \ell \frac{\partial T}{\partial x_i} \frac{1}{Pr_t}$$

式中 ℓ 為混調尺度

Pr_t 為紊流 Prandtl 數, 介於 0.8 ~ 1.3

該等紊流傳輸方程式 $\overline{u_i u_j}$, $\overline{u_i \theta}$ 只蘊含 K 方程式, 故此模式稱為 K 模式或單方程式模式。

單方程式或稱 K 方程式模式雖是較簡單, 然不如雙方程式 $(k - \varepsilon)$ 模式準確。

III-4-10 綜論 (Summary)

A. 以上所述 RSM 模式, $k - \varepsilon - A$ 模式, $k - \varepsilon - E$ 模式三者應用於非分離流 (non separated flow), 例如自由剪力流 (free shear flows)、壁面剪力流 (Wall shear flows), 結果滿意。對於複雜流 (Complex flows) 例

如有分離 (separation) 及不規則周界 (irregular geometry) 的流況上, RSM 模式應用結果比較好, 而 $k - \epsilon - A$ 模式依次比 $k - \epsilon - E$ 模式好。

B. 各紊流傳輸方程式之模式化中, 以 ϵ 方程式最為脆弱, 亦即該方程式之模式化尚有許多疑義, 以待繼續探討。

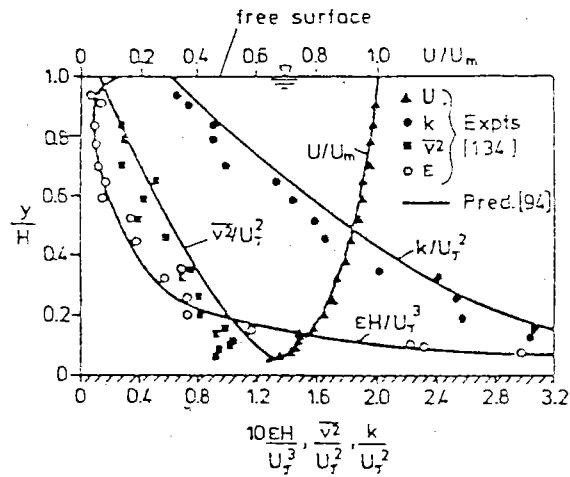
C. $\overline{u_i u_j}$ 紊流傳輸方程式中之壓力應變項之模式化尚可改進; $\overline{u_i \theta}$ 紊流傳輸方程式中之等向性消散及渦黏性亦可重新另行模式化。

IV、紊流模式之應用範例 (Examples of Application of Turbulence Models)

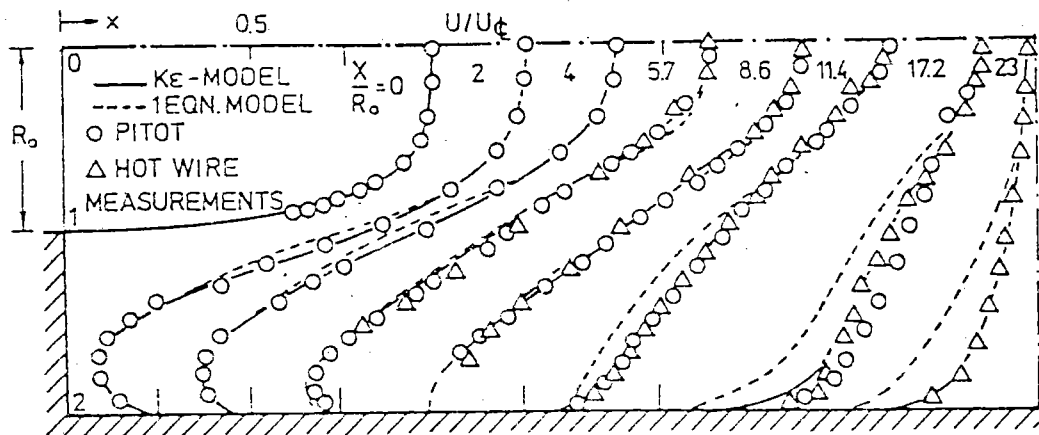
以上所述各紊流模式在水力學方面的應用有渠流 (channel flows)、射流 (jets)、跡流 (wakes) 及河川、湖泊或海洋放流 (discharges into streams, lakes or ocean) 等。茲舉部份學者應用該等紊流模式計算結果, 簡述如下, 詳細部份可參考原論文。

IV-1 渠流與管流 (channel flow and pipe flow)

A. Hossain (1980) 利用 $k - \epsilon - A$ 模式計算二維渠流 (Developed two-dimensional channel flow) 平均速度分佈及各紊流量 (turbulent quantities) k, ϵ, v^2 等分佈, 計算結果與實驗值比較, 結果滿意。

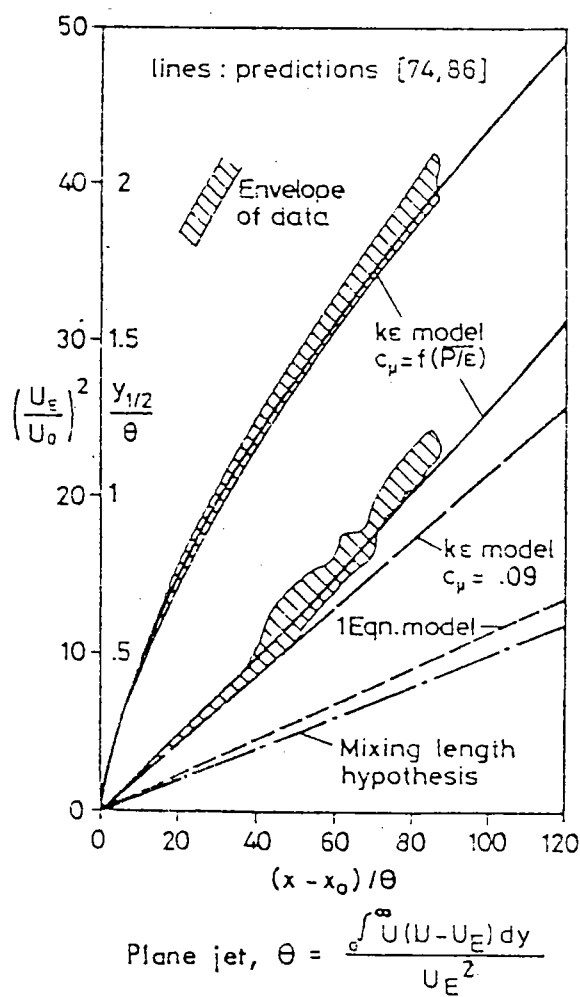


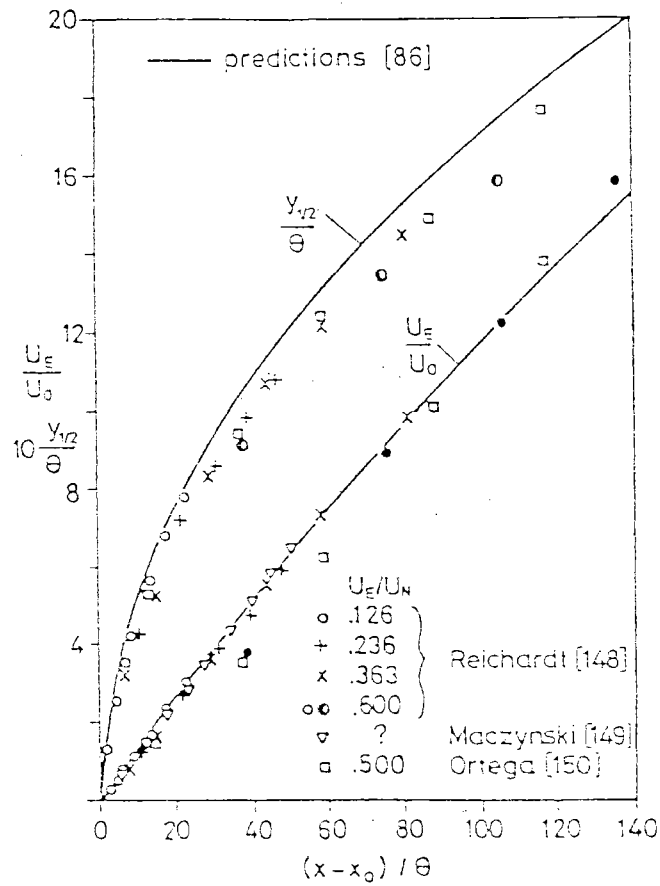
B. Ha Minh & Chassaing (1977) 利用 $k - \epsilon - E$ 模式，Bernald et al (1974) 使用單一方程式模式，計算突擴管流 (sudden pipe expansion flow)，速度分佈發展之計算結果與實驗值比較如下。單一方程式模式在回流區 (recirculation) 計算結果尚稱良好，但在再發展區 (re-development) 結果不好。 $k - \epsilon - E$ 模式則佳。



IV-2 射流與跡流 (Jets and Wakes)

A. Rodi et al 等使用 $k - \epsilon$ 模式計算平面射流 (plane jet)
) 圓形射流、跡流、計算結果與實驗值比較，成果滿意。

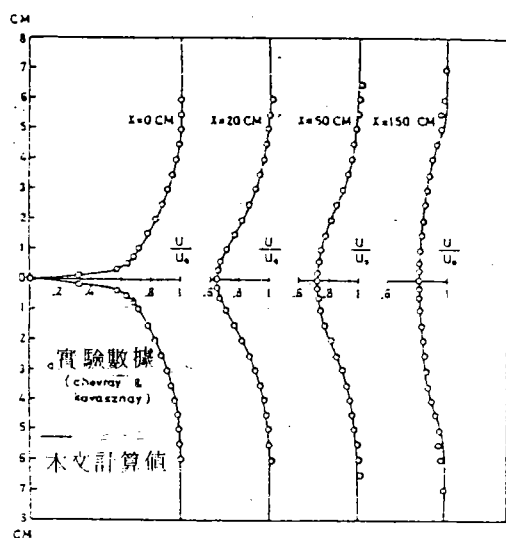




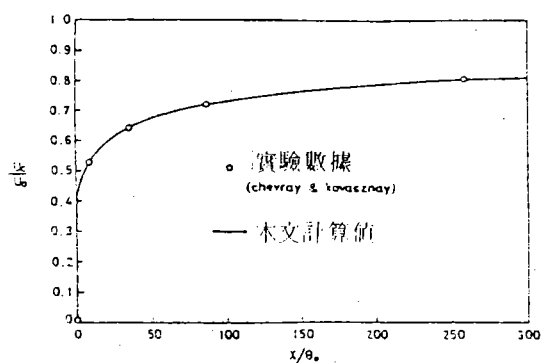
$$\text{Round jet, } \Theta = \left(\frac{2 \pi \int_0^\infty U(U-U_E) r dr}{U_E^2} \right)^{1/2}$$

B. 跡流 (Wake Flow)

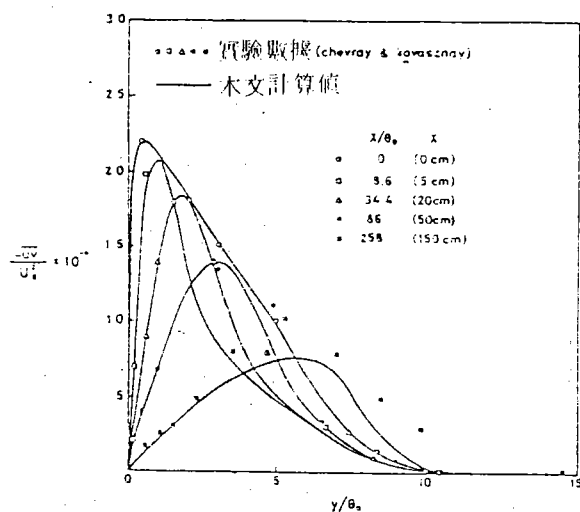
黃等 (1984) 使用雙尺度 $k - \epsilon$ 模式計算平面跡流，數值計算跡流流速分佈剖面，跡流中心軸向速度變化，紊流應力及跡流寬度等之變化。計算結果與 Chevray & Kovasznay (1969) 實驗結果比較，結果滿意。



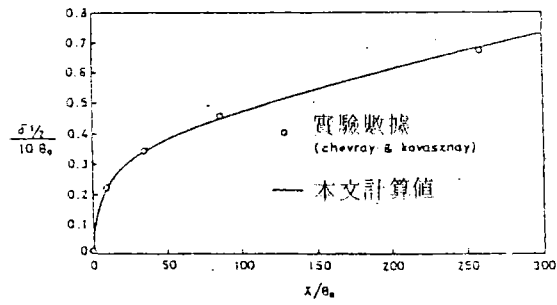
平板尾端跡流速度分佈剖面



平板尾端跡流中心軸向速度之變化



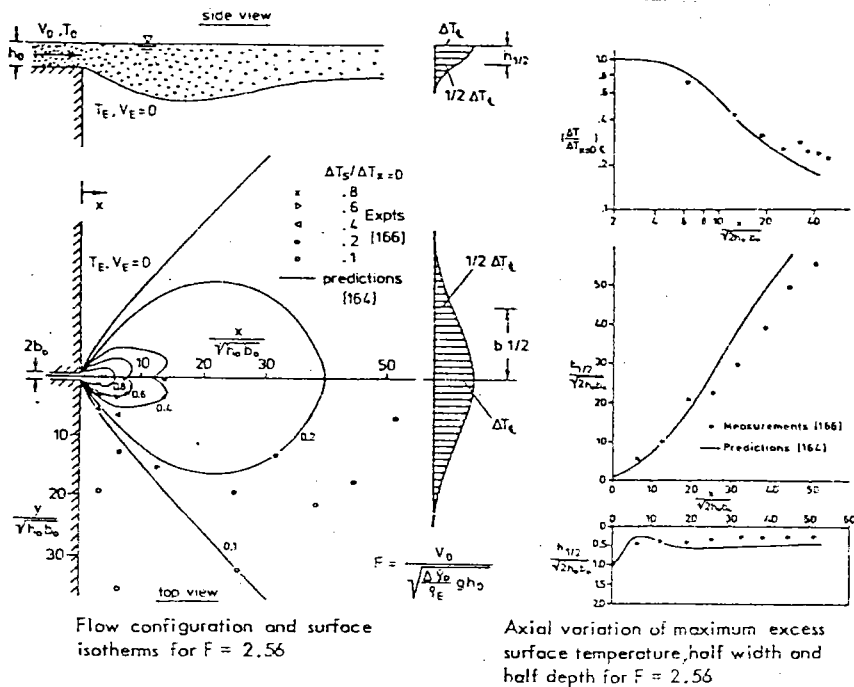
平板尾端跡流之剪應力分佈剖面



平板尾端跡流寬度之變化

IV-3 河川、湖泊或海洋放流 (Discharges into Steam , Lakes or Ocean) 。

Mc Guirk and Rodi 使用 $k - \epsilon$ 模式計算三維度廢熱水放流，計算結果與實驗值如下：



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3. 微分方程式分類

陳慶生

偏微分方程式的分類

一方程式包含某（未知）函數對於二個或更多個獨立變數的一次或更高次之偏導數時，稱為偏微分方程式，此方程式內所含偏導數之最高階數，即為偏微分方程式之階數，若偏微分方程式內之應變數（即函數）及其所有各種偏導數，均為一次方時則該方程式稱為綫性。

一偏微分方程式在其各自變數空間的區域 R 以內之解答為一函數，其在包含 R 之某領區內具有各偏導數，且在 R 內任何處之值，經代入後均能滿足原偏微分方程式之關係，一般而言，一偏微分方程式的全部解答為數可能甚多，例如下列函數：

$$u = x^2 - y^2, \quad u = e^x \cos y, \quad u = \ln(x^2 + y^2)$$

彼此雖全不相同，但均為下列偏微分方程式之解答，

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

對於某一已知物理問題之偏微分方程式，其唯一解答（Unique Solution）可由該問題所附加之物理情況求出，例如某些問題中所求解函數在其邊界上之值均屬已知，（此即所謂邊界條件），又如在以時間 t 為自變數之另一類問題中，其解答在 $t = 0$ 時之值，常早已設完（此即所謂初始條件）。對於一偏微分方程式決定其唯一解答

所需附加的獨立邊界條件與初始條件的個數則與方程式本身的特性有關，往後的討論將僅限於二階綫性偏微分方程式（只考慮兩個獨立變數），其一般型式為：

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + F u = G \quad \dots\dots\dots(1)$$

其中A、B、……，G，只是x、y的函數。

方程式(1)可依下面判別式之正負性質來分類

$$\triangle = B^2 - 4AC \quad (2)$$

在一座標面上某點如果 $\triangle > 0$ 則該方程式屬於雙曲綫型（hyperbolic type），如果 $\triangle = 0$ 則為拋物綫型（Parabolic type），如果(1)式中A、B、C亦為u以及u一次微分量的函數，則(1)式稱之為準綫性方程式（quasi-linear equation），在這種情況下，方程式在座標面上某點之類型不只與該點的位置同時也和解答u有關，舉例來說，下面方程式：

$$(1-y)u_{xx} + 2xu_{xy} + (1+y)u_{yy} = H$$

在圓 $x^2 + y^2 = 1$ 之內屬於橢圓型，在圓外為雙曲綫型，在圓上為拋物綫型，而方程式

$$u_{xx} + u u_{yy} = H$$

之類型則與u之正負號有關，上述方程式之分類是基於在不同的方程式類型下所需邊界條件的性質有所不同，如果方程式的類型與解答本身有關，則在處理所需邊界條件時亦與解答有關，這樣自然會引起一些困難，

方程式(1)可對應到一個特性方程（Characteristic equation）

$$A dy^2 - B dx dy + C dx^2 = 0 \quad (3)$$

如果 $A \neq 0$ 則(3)式之解答爲：

$$\frac{dy}{dx} = \frac{1}{2A} (B \pm \sqrt{B^2 - 4AC}) \quad (4.a.b)$$

如果(1)式屬於雙曲綫型則(4)式在 xy 一平面上每一點定義出兩個斜率，稱爲該點的特性方向，如果 $A = 0$ ，則一個特性方向爲平行於 Y 軸，另一方向則由方程式 $Bdy = Cdx$ 來決定，如果(4)式右邊是方程式(1)解答 u 的函數，則特性函數會隨邊界條件不同而改變。

對於一個雙曲綫型的微分方程式(1)，在 xy 一平面上取一曲綫段 Γ ，稱之爲 initial arc，若曲綫 Γ 上任一點切綫方向都不是該點所對應的特性方向，而且函數 u 及 u 的正向微分量 (normal derivative) $\frac{\partial u}{\partial n}$ 在曲綫 Γ 上爲已知，則綫性微分方程式(1)在空間中任一點 p 存在有唯一解，如果通過 p 之兩組特性曲綫與 Γ 相交於 Q_1, Q_2 如圖 1，則函數 u 在 p 點之值只與曲綫 Γ 上 Q_1, Q_2 之間的已知數據有關由 Γ, r_1, r_2 所決定的有限區域 R 稱之爲 P 點的決定區域

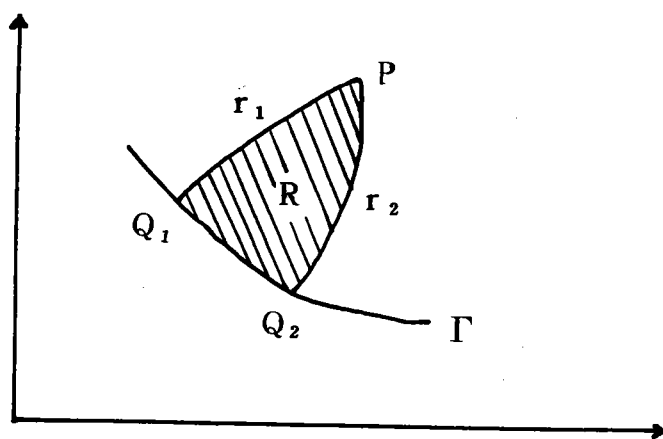


圖 1.

雙曲綫型微分方程式的典型例子爲波動方程式：

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (4)$$

(4)式所對應的特性座標爲：

$$v = x + c t, \quad z = x - c t \quad (5)$$

將 v 、 z 取爲新的自變數， u 化成 v 、 z 之函數，則(4)式可簡化成

$$\frac{\partial^2 u}{\partial v \partial z} = 0 \quad (6)$$

由(6)式積分兩次可得 $u = \int h(v) dv + \phi(z)$ ，其中 h ， ϕ 爲任意函數，如果積分項以 $\phi(v)$ 表之，則可得：

$$u = \phi(v) + \phi(z)$$

由(5)式關係，我們可寫成

$$u(x, t) = \phi(x + c t) + \phi(x - c t) \quad (7)$$

此即波動方程式(4)之 d'Alembert 解答

通常 ϕ ， ψ 二函數可由初始條件以決定之，爲簡便起見，我們可設想已知初使速度爲零 $u_t(x, 0) = 0$ ，而初使位移爲 $u(x, 0) = f(x)$ 微分(7)式得：

$$\frac{\partial u}{\partial t} = c \phi'(x + c t) - c \psi'(x - c t) \quad (8)$$

上式之撇號，係代表各函數分別對於整個變數 $x + c t$ 及 $x - c t$ 之導數，由(7)，(8)兩式及初始條件，我們可寫出：

$$u(x, 0) = \phi(x) + \psi(x) = f(x)$$

$$u_t(x, 0) = c \phi'(x) - c \psi'(x) = 0$$

由後一方程式， $\psi' = \phi'$ ，故 $\psi = \phi + k$ ，將此關係代入前一方程式得：
 $2\phi + k = f$ 或 $\phi = (f - k)/2$ ，利用這些函數 ϕ 及 ψ ，我們可寫出(4)式如下：

$$u(x, t) = \frac{1}{2} [f(x + c t) + f(x - c t)]$$

我們所獲致之結果顯示，兩個初始條件以及邊界條件即可唯一地決定雙曲綫型方程式之解答。

拋物綫型方程式之一般型式爲：

$$u_{xx} = H(x, t, u, u_x, u_t) \quad (9)$$

其中 t 表示時間，例如下式：

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left(k \frac{\partial u}{\partial x} \right) \quad (10)$$

可表成(9)之型式，而且當 k 與 u 無關則(10)式爲綫型，它代表一絕緣柱的溫度分佈，其中 k 表示熱傳係數，如果圓柱無限沿伸在 x 一軸方向，如果在 $t = 0$ 時圓柱之溫度爲已知， $u(x, 0) = f(x)$

$-\infty < x < \infty$ ，則對於 $t > 0$ 的 $u(x, t)$ 即可求得，爲解決此一問題，我們假設 $u(x, t) = F(x)G(t)$ 代入(10)式，得到：

$$F'' + P^2 F = 0$$

$$\text{及 } G + KP^2 G = 0$$

從而得到下列二函數

$$F(x) = A \cos px + B \sin px$$

$$\text{及 } G(t) = e^{-kP^2 t}$$

其中 A 、 B 爲任意常數，故(10)式之解答爲：

$$u(x, t, p) = FG = (A \cos px + B \sin px) e^{-kP^2 t} \quad (11)$$

依照通常之方式設 P 爲一固定數值之倍數，所得出以(11)式函數組成之任何級數，當 $t = 0$ 時，將導致一個週期性函數，但初始值 $f(x)$ 並非週期函數，由此可以看出此處必需利用 Fourier 積分以代替 Fourier 級數，因(11)式內之 A 和 B 均爲任意常數，我們可設想其爲 P 之函數而寫成 $A = A(p)$ 及 $B = B(p)$ 因爲(10)爲綫型，故可以

$$\begin{aligned}
 u(x, t) &= \int_0^{\infty} u(x, t, p) dp \\
 &= \int_0^{\infty} [A(p) \cos px + B(p) \sin px] e^{-kp^2 t} dp
 \end{aligned}
 \tag{12}$$

作為(10)之解答，由(12)及初始條件可得：

$$\begin{aligned}
 u(x, 0) &= \int_0^{\infty} [A(p) \cos px + B(p) \sin px] dp \\
 &= f(x)
 \end{aligned}
 \tag{13}$$

由(13)可得：

$$\begin{aligned}
 A(p) &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos pv dv \\
 B(p) &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin pv dv
 \end{aligned}$$

由上式得

$$\begin{aligned}
 u(x, 0) &= \frac{1}{\pi} \int_0^{\infty} \left[\int_{-\infty}^{\infty} f(v) \cos(px - pv) dv \right] dp \\
 &= \frac{1}{2\sqrt{\pi kt}} \int_{-\infty}^{\infty} f(v) \exp \left\{ -\frac{(x-v)^2}{4kt} \right\} dv
 \end{aligned}
 \tag{14}$$

若 $f(x)$ 對所有之 x 而言均為有界限，而且在每一有限區間內均能積分之；則可證明函數(14)滿足方程式(10)及初始條件，故函數(14)即為本問題之答案。

拋物綫型方程式與雙曲綫型方程式之區別在於只要一個初始條件即可決定函數 $u(x, t)$ ，對於拋物綫型方程式判別式 $\Delta = 0$ ，則特性曲綫只有一組由 $t = \text{常數}$ 決定，對於任一點 $p(x_0, t_0)$ 綫下所有的初始值及邊界值均為 p 點的決定區域，如圖 2 所示。

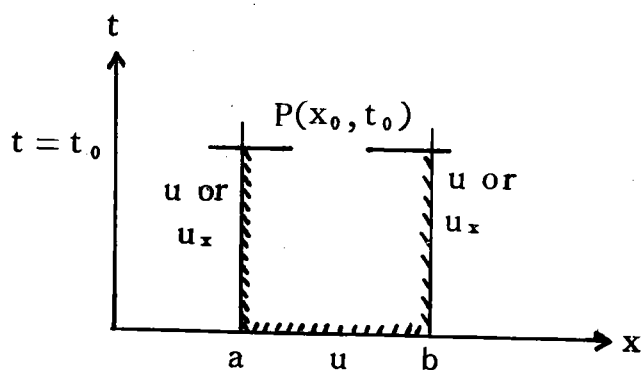


圖 2

對於橢圓型方程式，其判別式 $\Delta < 0$ ，由(4)式知不存在有實數特性值，因此對於空間任一點其決定區域是整個由討論邊界所決定的空間，因此對於任何一點的解答，它必須和所有的邊界值有關，最常見的橢圓方程式為 Poisson's equation

$$u_{xx} + u_{yy} + Q(x, y) = 0 \quad (15)$$

當 $Q(x, y) = 0$ 則(15)式退化成 Laplace 方程式， $u(x, y)$ 可解釋成二次度空間(x, y —平面) Steady State 的溫度分佈，在這種情況下可知對任一有限區間 R (由封閉曲線 c 所決定)，圖 3 如果沿著 c 上每一點其溫度值 u 或正向溫度導數 u_n 為已知則 R 內之溫度分佈便可決定，如果 R 為一multiple connected domain 則必須多加一些條件以便解答是唯一的。

如果一個問題中同時已知 u 及 u_n 沿著某一曲線，對橢圓方程而言，這不是一個“well posed”的問題，亦即解答 u 可能不存在除非已知的兩個條件滿足某些更嚴緊的條件 (亦即已知條件為相依條件)。

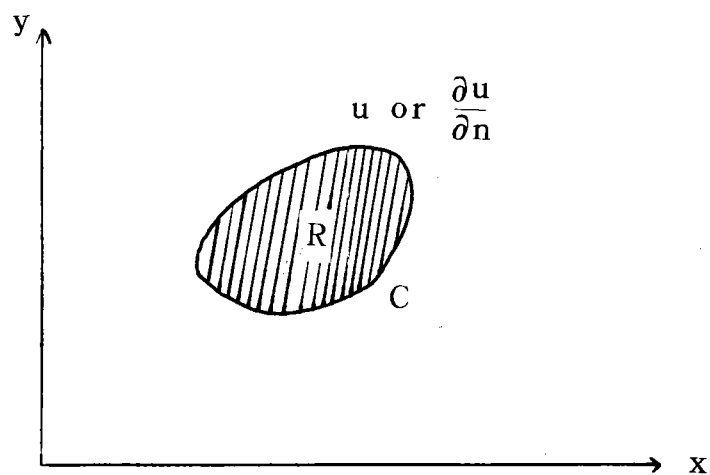


圖 3

由前面的討論，我們得到對於雙曲綫型方程式空間任一點存在有兩個特性方向，要唯一決定方程式之解答需兩個獨立的初始條件以及適當的邊界條件，對於拋物綫型方程式空間任一點只有一個特性方向，所需的初始條件只有一個，對於橢圓型方程式空間任一點不存有特性方向，僅由邊界條件即可決定方程式的唯一解答。

摘自: Anderson, D A. et al.: Computational Fluid
Mechanics and Heat Transfer, McGraw-Hill
Book Co., 1984.

4. 有限差分法

BASICS OF FINITE-DIFFERENCE METHODS

3-1 INTRODUCTION

In this chapter, basic concepts and techniques needed in the formulation of finite-difference representations are developed. In the finite-difference approach, the continuous problem domain is "discretized" so that the dependent variables are considered to exist only at discrete points. Derivatives are approximated by differences resulting in an algebraic representation of the partial differential equation (PDE). Thus, a problem involving calculus has been transformed into an algebraic problem.

The nature of the resulting algebraic system depends on the character of the problem posed by the original PDE (or system of PDE's). Equilibrium problems usually result in a system of algebraic equations which must be solved simultaneously throughout the problem domain in conjunction with specified boundary values. Marching problems result in algebraic equations which usually can be solved one at a time (although it is often convenient to solve them several at a time). Several considerations determine whether the solution so obtained will be a good approximation to the exact solution of the original PDE. Among these considerations are truncation error, consistency, and stability, all of which will be discussed in the present chapter.

3-2 FINITE DIFFERENCES

One of the first steps to be taken in establishing a finite-difference procedure for solving a PDE is to replace the continuous problem domain by a finite-difference mesh or grid. As an example, suppose that we wish to solve a PDE for which $u(x, y)$ is the

dependent variable in the square domain $0 \leq x \leq 1$, $0 \leq y \leq 1$. We establish a grid on the domain by replacing $u(x, y)$ by $u(i \Delta x, j \Delta y)$. Points can be located according to values of i and j so difference equations are usually written in terms of the general point (i, j) and its neighbors. This labeling is illustrated in Fig. 3-1. Thus, if we think of $u_{i,j}$ as $u(x_0, y_0)$ then

$$\begin{aligned} u_{i+1,j} &= u(x_0 + \Delta x, y_0) & u_{i-1,j} &= u(x_0 - \Delta x, y_0) \\ u_{i,j+1} &= u(x_0, y_0 + \Delta y) & u_{i,j-1} &= u(x_0, y_0 - \Delta y) \end{aligned}$$

Often in the treatment of marching problems, the variation of the marching coordinate is indicated by a superscript, such as u_j^{n+1} , rather than a subscript. Many different finite-difference representations are possible for any given PDE and it's usually impossible to establish a "best" form on an absolute basis. First of all, the accuracy of a difference scheme may depend on the exact form of the equation and problem being solved and secondly, our selection of a "best" scheme will be influenced by the aspect of the procedure which we are trying to optimize, i.e., accuracy, economy, programming simplicity, etc.

The idea of a finite-difference representation for a derivative can be introduced by recalling the definition of the derivative for the function $u(x, y)$ at $x = x_0, y = y_0$.

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} \quad (3-1)$$

Here, if u is continuous, it is expected that $[u(x_0 + \Delta x, y_0) - u(x_0, y_0)]/\Delta x$ will be a "reasonable" approximation to $\partial u/\partial x$ for a "sufficiently" small but finite Δx . In fact, the mean value theorem assures us that the difference representation is exact for some point within the Δx interval. The difference approximation can be put on a more formal basis through the use of either a Taylor-series expansion or Taylor's formula with a remainder. Developing a Taylor-series expansion for $u(x_0 + \Delta x, y_0)$ about (x_0, y_0) gives:

$$\begin{aligned} u(x_0 + \Delta x, y_0) &= u(x_0, y_0) + \left. \frac{\partial u}{\partial x} \right|_0 \Delta x + \left. \frac{\partial^2 u}{\partial x^2} \right|_0 \frac{(\Delta x)^2}{2!} + \dots \\ &+ \left. \frac{\partial^{n-1} u}{\partial x^{n-1}} \right|_0 \frac{(\Delta x)^{n-1}}{(n-1)!} + \left. \frac{\partial^n u}{\partial x^n} \right|_{\xi} \frac{(\Delta x)^n}{n!} \quad x_0 \leq \xi \leq (x_0 + \Delta x) \end{aligned} \quad (3-2)$$

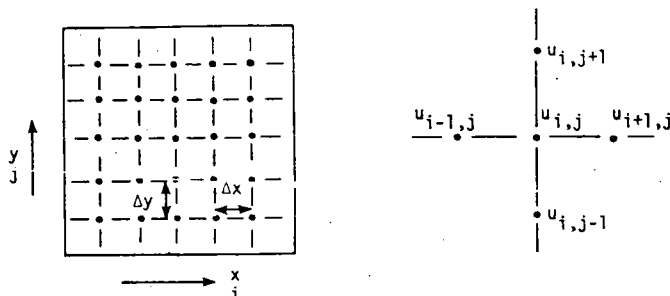


Figure 3-1 A typical finite-difference grid.

where the last term can be identified as the remainder. Thus, we can form the "forward" difference by rearranging Eq. (3-2)

$$\left(\frac{\partial u}{\partial x}\right)_{x_0, y_0} = \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} - \frac{\partial^2 u}{\partial x^2}\bigg|_0 \frac{\Delta x}{2!} - \dots \quad (3-3)$$

Switching now to the i, j notation for brevity, we consider

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j} - u_{i,j}}{\Delta x} + \text{Truncation error} \quad (3-4)$$

where $(u_{i+1,j} - u_{i,j})/\Delta x$ is obviously the finite-difference representation for $\partial u/\partial x|_{i,j}$. The truncation error is the difference between the partial derivative and its finite-difference representation. We can characterize the limiting behavior of the truncation error (T.E.) by using the order of (O) notation whereby we write,

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j} - u_{i,j}}{\Delta x} + O(\Delta x)$$

where $O(\Delta x)$ has a precise mathematical meaning. Here, when the truncation error is written as $O(\Delta x)$ we mean $|\text{T.E.}| \leq K|\Delta x|$ for $\Delta x \rightarrow 0$ (sufficiently small Δx). K is a positive real constant. As a practical matter, the order of the truncation error in this case is found to be Δx raised to the largest power which is common to all terms in the truncation error.

To give a more general definition of the O notation, when we say $f(x) = O[\phi(x)]$, we mean that there exists a constant K , independent of x such that $|f(x)| \leq K|\phi(x)|$ for all x in S where f and ϕ are real or complex functions defined in S . We often restrict S by $x \rightarrow \infty$ (sufficiently large x) or as is most common in finite-difference applications, $x \rightarrow 0$ (sufficiently small x). More details on the O notation can be found in Whittaker and Watson (1927).

Note that $O(\Delta x)$ tells us nothing about the exact size of the T.E. but rather how it behaves as Δx tends toward zero. If another difference expression had a T.E. = $O[(\Delta x)^2]$, we might expect or hope that the T.E. of the second representation would be smaller than the first for a convenient Δx but we could only be *sure* that this would be true if we refined the mesh "sufficiently," and "sufficiently" is a quantity which is hard to estimate.

An infinite number of difference representations can be found for $\partial u/\partial x|_{i,j}$. For example we could expand "backwards"

$$u(x_0 - \Delta x, y_0) = u(x_0, y_0) - \left(\frac{\partial u}{\partial x}\right)_0 \Delta x + \frac{\partial^2 u}{\partial x^2}\bigg|_0 \frac{(\Delta x)^2}{2} - \frac{\partial^3 u}{\partial x^3}\bigg|_0 \frac{(\Delta x)^3}{6} + \dots \quad (3-5)$$

and obtain the "backward" difference representation,

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i,j} - u_{i-1,j}}{\Delta x} + O(\Delta x) \quad (3-6)$$

We can subtract Eq. (3-5) from Eq. (3-2), rearrange, and obtain the "central" difference

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + O(\Delta x)^2 \quad (3-7)$$

42 FUNDAMENTALS OF FINITE-DIFFERENCE METHODS

We can also add Eq. (3-2) and Eq. (3-5) and rearrange to obtain an approximation to the second derivative

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} + O(\Delta x)^2 \quad (3-8)$$

It should be emphasized that these are only a few examples of the possible ways in which first and second derivatives can be approximated.

It is convenient to utilize difference operators to represent finite differences when particular forms are used repetitively. Here we define the first forward difference of $u_{i,j}$ with respect to x at the point i, j as

$$\Delta_x u_{i,j} = u_{i+1,j} - u_{i,j} \quad (3-9)$$

Thus, we can express the forward finite-difference approximation for the first partial derivative as

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j} - u_{i,j}}{\Delta x} + O(\Delta x) = \frac{\Delta_x u_{i,j}}{\Delta x} + O(\Delta x) \quad (3-10)$$

Similarly, derivatives with respect to other variables such as y can be represented by

$$\frac{\Delta_y u_{i,j}}{\Delta y} = \frac{u_{i,j+1} - u_{i,j}}{\Delta y}$$

The first backward difference of $u_{i,j}$ with respect to x at i, j is denoted by

$$\nabla_x u_{i,j} = u_{i,j} - u_{i-1,j} \quad (3-11)$$

It follows that the first backward-difference approximation to the first derivative can be written as

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i,j} - u_{i-1,j}}{\Delta x} + O(\Delta x) = \frac{\nabla_x u_{i,j}}{\Delta x} + O(\Delta x) \quad (3-12)$$

The central-difference operators $\bar{\delta}$, δ and δ^2 will be defined as

$$\bar{\delta}_x u_{i,j} = u_{i+1/2,j} - u_{i-1/2,j} \quad (3-13)$$

$$\delta_x u_{i,j} = u_{i+1/2,j} - u_{i-1/2,j} \quad (3-14)$$

$$\delta_x^2 u_{i,j} = \delta_x(\delta_x u_{i,j}) = u_{i+1,j} - 2u_{i,j} + u_{i-1,j} \quad (3-15)$$

and an averaging operator, μ as

$$\mu_x u_{i,j} = \frac{u_{i+1/2,j} + u_{i-1/2,j}}{2} \quad (3-16)$$

It is convenient to have specific operators for certain common central differences although two of them can be easily expressed in terms of first-difference operators

$$\bar{\delta}_x u_{i,j} = \Delta_x u_{i,j} + \nabla_x u_{i,j} \quad (3-17)$$

$$\delta_x^2 u_{i,j} = \Delta_x u_{i,j} - \nabla_x u_{i,j} = \Delta_x \nabla_x u_{i,j} \quad (3-18)$$

Using the newly defined operators, the central-difference representation for the first partial derivative can be written as

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + O(\Delta x)^2 = \frac{\delta_x u_{i,j}}{2\Delta x} + O(\Delta x)^2 \quad (3-19)$$

and the central-difference representation of the second derivative as

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} + O(\Delta x)^2 = \frac{\delta_x^2 u_{i,j}}{(\Delta x)^2} + O(\Delta x)^2 \quad (3-20)$$

Higher-order forward- and backward-difference operators are defined as

$$\Delta_x^n u_{i,j} = \Delta_x (\Delta_x^{n-1} u_{i,j}) \quad (3-21)$$

and

$$\nabla_x^n u_{i,j} = \nabla_x (\nabla_x^{n-1} u_{i,j}) \quad (3-22)$$

As an example, a forward second-derivative approximation is given by:

$$\begin{aligned} \frac{\Delta_x^2 u_{i,j}}{(\Delta x)^2} &= \frac{\Delta_x (u_{i+1,j} - u_{i,j})}{(\Delta x)^2} = \frac{u_{i+2,j} - u_{i+1,j} - u_{i+1,j} + u_{i,j}}{(\Delta x)^2} \\ &= \frac{u_{i+2,j} - 2u_{i+1,j} + u_{i,j}}{(\Delta x)^2} = \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} + O(\Delta x) \end{aligned} \quad (3-23)$$

We can show that forward- and backward-difference approximations to derivatives of any order can be obtained from

$$\left(\frac{\partial^n u}{\partial x^n}\right)_{i,j} = \frac{\Delta_x^n u_{i,j}}{(\Delta x)^n} + O(\Delta x) \quad (3-24)$$

and

$$\left(\frac{\partial^n u}{\partial x^n}\right)_{i,j} = \frac{\nabla_x^n u_{i,j}}{(\Delta x)^n} + O(\Delta x) \quad (3-25)$$

Central-difference representations of derivatives of orders greater than the second can be expressed in terms of Δ and ∇ or δ . A more complete development on the use of difference operators can be found in many textbooks on numerical analysis such as Hildebrand (1956).

Most of the PDE's arising in fluid mechanics and heat transfer involve only first- and second-partial derivatives, and generally we strive to represent these derivatives using values at only two or three grid points. Within these restrictions, the most frequently used first-derivative approximations on a grid for which $\Delta x = h = \text{constant}$ are

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j} - u_{i,j}}{h} + O(h) \quad (3-26)$$

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i,j} - u_{i-1,j}}{h} + O(h) \quad (3-27)$$

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j} - u_{i-1,j}}{2h} + O(h^2) \quad (3-28)$$

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{-3u_{i,j} + 4u_{i+1,j} - u_{i+2,j}}{2h} + O(h^2) \quad (3-29)$$

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{3u_{i,j} - 4u_{i-1,j} + u_{i-2,j}}{2h} + O(h^2) \quad (3-30)$$

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{1}{2h} \left(\frac{\bar{\delta}_x u_{i,j}}{1 + \delta_x^2/6} \right) + O(h^4) \quad (3-31)$$

The most common three-point second-derivative approximations for a uniform grid, $\Delta x = h = \text{constant}$, are

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = \frac{u_{i,j} - 2u_{i+1,j} + u_{i+2,j}}{h^2} + O(h) \quad (3-32)$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = \frac{u_{i,j} - 2u_{i-1,j} + u_{i-2,j}}{h^2} + O(h) \quad (3-33)$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + O(h^2) \quad (3-34)$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = \frac{\delta_x^2 u_{i,j}}{h^2(1 + \delta_x^2/12)} + O(h^4) \quad (3-35)$$

The compact, three-point schemes given by Eqs. (3-31) and (3-35) having fourth-order truncation errors deserve a further word of explanation. (See also Orszag and Israeli, 1974.) Letting $\partial u/\partial x|_{i,j} = v_{i,j}$, Eq. (3-31) is to be interpreted as

$$\left(1 + \frac{\delta_x^2}{6}\right) v_{i,j} = \frac{\bar{\delta}_x u_{i,j}}{2h}$$

or

$$\frac{1}{6} (v_{i+1,j} + 4v_{i,j} + v_{i-1,j}) = \frac{\bar{\delta}_x u_{i,j}}{2h} \quad (3-36)$$

which provides an *implicit* formula for the derivative of interest, $v_{i,j}$. The $v_{i,j}$ can be determined from the $u_{i,j}$ by solving a tridiagonal system of simultaneous algebraic equations, which can usually be accomplished quite efficiently. Tridiagonal systems commonly occur in connection with the use of implicit difference schemes for second-order PDE's arising from marching problems and will be defined and discussed in some detail in Chapter 4. For now it is sufficient to think of a tridiagonal system as the arrangement of unknowns which would occur if each difference equation in a system only involved a single unknown variable evaluated at 3 adjacent grid locations. The interpretation of Eq. (3-35) proceeds in a similar manner providing an *implicit* representation of $\partial^2 u/\partial x^2|_{i,j}$. Some difference approximations for derivatives which involve more than three grid points are given in Table 3-1. For completeness, a few common difference representations for mixed partial derivatives are presented in Table 3-2. These will prove useful for schemes to be discussed in later chapters. The mixed derivative approximations in Table 3-2 can be verified by using the Taylor-series expansion for two variables

Table 3-1 Difference approximations using more than three points

Derivative	Finite-difference representation	Equation
$\frac{\partial^3 u}{\partial x^3} \Big _{i,j} =$	$\frac{u_{i+2,j} - 2u_{i+1,j} + 2u_{i-1,j} - u_{i-2,j}}{2h^3} + O(h^2)$	(3-38)
$\frac{\partial^4 u}{\partial x^4} \Big _{i,j} =$	$\frac{u_{i+2,j} - 4u_{i+1,j} + 6u_{i,j} - 4u_{i-1,j} + u_{i-2,j}}{h^4} + O(h^2)$	(3-39)
$\frac{\partial^2 u}{\partial x^2} \Big _{i,j} =$	$\frac{-u_{i+3,j} + 4u_{i+2,j} - 5u_{i+1,j} + 2u_{i,j}}{h^2} + O(h^2)$	(3-40)
$\frac{\partial^3 u}{\partial x^3} \Big _{i,j} =$	$\frac{-3u_{i+4,j} + 14u_{i+3,j} - 24u_{i+2,j} + 18u_{i+1,j} - 5u_{i,j}}{2h^3} + O(h^2)$	(3-41)
$\frac{\partial^2 u}{\partial x^2} \Big _{i,j} =$	$\frac{2u_{i,j} - 5u_{i-1,j} + 4u_{i-2,j} - u_{i-3,j}}{h^2} + O(h^2)$	(3-42)
$\frac{\partial^3 u}{\partial x^3} \Big _{i,j} =$	$\frac{5u_{i,j} - 18u_{i-1,j} + 24u_{i-2,j} - 14u_{i-3,j} + 3u_{i-4,j}}{2h^3} + O(h^2)$	(3-43)
$\frac{\partial u}{\partial x} \Big _{i,j} =$	$\frac{-u_{i+2,j} + 8u_{i+1,j} - 8u_{i-1,j} + u_{i-2,j}}{12h} + O(h^4)$	(3-44)
$\frac{\partial^2 u}{\partial x^2} \Big _{i,j} =$	$\frac{-u_{i+2,j} + 16u_{i+1,j} - 30u_{i,j} + 16u_{i-1,j} - u_{i-2,j}}{12h^2} + O(h^4)$	(3-45)

Table 3-2 Difference approximations for mixed partial derivatives

Derivative	Finite-difference representation	Equation
$\frac{\partial^2 u}{\partial x \partial y} \Big _{i,j} =$	$\frac{1}{\Delta x} \left(\frac{u_{i+1,j} - u_{i+1,j-1}}{\Delta y} - \frac{u_{i,j} - u_{i,j-1}}{\Delta y} \right) + O(\Delta x, \Delta y)$	(3-46)
$\frac{\partial^2 u}{\partial x \partial y} \Big _{i,j} =$	$\frac{1}{\Delta x} \left(\frac{u_{i,j+1} - u_{i,j}}{\Delta y} - \frac{u_{i-1,j+1} - u_{i-1,j}}{\Delta y} \right) + O(\Delta x, \Delta y)$	(3-47)
$\frac{\partial^2 u}{\partial x \partial y} \Big _{i,j} =$	$\frac{1}{\Delta x} \left(\frac{u_{i,j} - u_{i,j-1}}{\Delta y} - \frac{u_{i-1,j} - u_{i-1,j-1}}{\Delta y} \right) + O(\Delta x, \Delta y)$	(3-48)
$\frac{\partial^2 u}{\partial x \partial y} \Big _{i,j} =$	$\frac{1}{\Delta x} \left(\frac{u_{i+1,j+1} - u_{i+1,j}}{\Delta y} - \frac{u_{i,j+1} - u_{i,j}}{\Delta y} \right) + O(\Delta x, \Delta y)$	(3-49)
$\frac{\partial^2 u}{\partial x \partial y} \Big _{i,j} =$	$\frac{1}{\Delta x} \left(\frac{u_{i+1,j+1} - u_{i+1,j-1}}{2\Delta y} - \frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y} \right) + O[\Delta x, (\Delta y)^2]$	(3-50)
$\frac{\partial^2 u}{\partial x \partial y} \Big _{i,j} =$	$\frac{1}{\Delta x} \left(\frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y} - \frac{u_{i-1,j+1} - u_{i-1,j-1}}{2\Delta y} \right) + O[\Delta x, (\Delta y)^2]$	(3-51)
$\frac{\partial^2 u}{\partial x \partial y} \Big _{i,j} =$	$\frac{1}{2\Delta x} \left(\frac{u_{i+1,j+1} - u_{i+1,j-1}}{2\Delta y} - \frac{u_{i-1,j+1} - u_{i-1,j-1}}{2\Delta y} \right) + O[(\Delta x)^2, (\Delta y)^2]$	(3-52)
$\frac{\partial^2 u}{\partial x \partial y} \Big _{i,j} =$	$\frac{1}{2\Delta x} \left(\frac{u_{i+1,j+1} - u_{i+1,j}}{\Delta y} - \frac{u_{i-1,j+1} - u_{i-1,j}}{\Delta y} \right) + O[(\Delta x)^2, \Delta y]$	(3-53)
$\frac{\partial^2 u}{\partial x \partial y} \Big _{i,j} =$	$\frac{1}{2\Delta x} \left(\frac{u_{i+1,j} - u_{i+1,j-1}}{\Delta y} - \frac{u_{i-1,j} - u_{i-1,j-1}}{\Delta y} \right) + O[(\Delta x)^2, \Delta y]$	(3-54)

$$\begin{aligned}
u(x_0 + \Delta x, y_0 + \Delta y) &= u(x_0, y_0) + \left(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y} \right) u(x_0, y_0) \\
&\quad + \frac{1}{2!} \left(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y} \right)^2 u(x_0, y_0) \\
&\quad + \cdots + \frac{1}{n!} \left(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y} \right)^n u(x_0 + \theta \Delta x, y_0 + \theta \Delta y) \\
&\quad 0 \leq \theta \leq 1
\end{aligned} \tag{3-37}$$

3-3 DIFFERENCE REPRESENTATION OF PARTIAL DIFFERENTIAL EQUATIONS

3-3.1 Truncation Error

As a starting point in our study of truncation error, let us consider the heat equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \tag{3-55}$$

Using a forward-difference representation for the time derivative and a central-difference representation for the second derivative, we can approximate the heat equation by

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{\alpha}{(\Delta x)^2} (u_{j+1}^n - 2u_j^n + u_{j-1}^n) \tag{3-56a}$$

However, we noted in Section 3-2 that truncation errors were associated with the forward- and central-difference representations used in Eq. (3-56a). If we rearrange Eq. (3-55) to put zero on the right-hand side and include the truncation errors associated with the difference representation of the derivatives we obtain

$$\begin{aligned}
\underbrace{\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2}}_{\text{PDE}} &= \underbrace{\frac{u_j^{n+1} - u_j^n}{\Delta t} - \frac{\alpha}{(\Delta x)^2} (u_{j+1}^n - 2u_j^n + u_{j-1}^n)}_{\text{FDE}} \\
&\quad + \underbrace{\left[-\frac{\partial^2 u}{\partial t^2} \right]_{n,j} \frac{\Delta t}{2} + \alpha \left[\frac{\partial^4 u}{\partial x^4} \right]_{n,j} \frac{(\Delta x)^2}{12} + \cdots}_{\text{T.E.}}
\end{aligned} \tag{3-56b}$$

The truncation errors associated with all derivatives in any one PDE should be obtained by expanding about the same point (n, j in the above discussion).

The difference representation given by Eq. (3-56a) will be referred to as the *simple explicit scheme* for the heat equation. An *explicit* scheme is one for which only one unknown appears in the difference equation in a manner which permits evaluation in terms of known quantities. Since the parabolic heat equation governs a marching problem for which an initial distribution of u must be specified, u 's at the

time level n can be considered as known. If the second derivative term in the heat equation was approximated by u 's at the $n + 1$ time level, three unknowns would appear in the difference equation and the procedure would be known as *implicit*, indicating that the algebraic formulation would require the simultaneous solution of several equations involving the unknowns. The differences between implicit and explicit schemes will be discussed further in Chapter 4.

The quantity in brackets (note that only the leading terms have been written out utilizing Taylor-series expansions) in Eq. (3-56b) is identified as the *truncation error* for this finite-difference representation of the heat equation and is defined as the difference between the partial differential equation and the difference approximation to it. That is, T.E. = PDE - FDE. The *order* of the truncation error in this case is $O(\Delta t) + O[(\Delta x)^2]$ which is frequently expressed in the form $O[\Delta t, (\Delta x)^2]$. Naturally, we solve only the finite-difference equations and hope that the truncation error is small. If we don't feel a little uneasy at this point, perhaps we should. How do we know that our difference representation is acceptable and that a marching solution technique will work in the sense of giving us an approximate solution to the PDE? In order to be acceptable, our difference representation for this marching problem needs to meet the conditions of *consistency* and *stability*.

3-3.2 Consistency

Consistency deals with the extent to which the finite-difference equations approximate the partial differential equations. The difference between the PDE and the finite-difference approximation has already been defined as the truncation error of the difference representation. A finite-difference representation of a PDE is said to be consistent if we can show that the difference between the PDE and its difference representation vanishes as the mesh is refined, i.e., $\lim_{\text{mesh} \rightarrow 0} (\text{PDE} - \text{FDE}) = \lim_{\text{mesh} \rightarrow 0} (\text{T.E.}) = 0$. This should always be the case if the order of the truncation error vanishes under grid refinement [i.e., $O(\Delta t)$, $O(\Delta x)$, etc.]. An example of a questionable scheme would be one for which the truncation error was $O(\Delta t/\Delta x)$ where the scheme would not formally be consistent unless the mesh were refined in a manner such that $\Delta t/\Delta x \rightarrow 0$. The DuFort-Frankel (DuFort and Frankel, 1953) differencing of the heat equation,

$$\frac{u_j^{n+1} - u_j^{n-1}}{2 \Delta t} = \frac{\alpha}{(\Delta x)^2} (u_{j+1}^n - u_j^{n+1} - u_j^{n-1} + u_{j-1}^n) \quad (3-57)$$

for which the leading terms in the truncation error are

$$+ \frac{\alpha}{12} \frac{\partial^4 u}{\partial x^4} \Big|_{n,j} (\Delta x)^2 - \alpha \frac{\partial^2 u}{\partial t^2} \Big|_{n,j} \left(\frac{\Delta t}{\Delta x} \right)^2 - \frac{1}{6} \frac{\partial^3 u}{\partial t^3} \Big|_{n,j} (\Delta t)^2$$

serves as an example. All is well if

$$\lim_{\Delta t, \Delta x \rightarrow 0} \left(\frac{\Delta t}{\Delta x} \right) = 0$$

but if Δt and Δx were to approach zero at the same rate such that $\Delta t/\Delta x = \beta$, then the DuFort-Frankel scheme is consistent with the hyperbolic equation,

$$\frac{\partial u}{\partial t} + \alpha \beta^2 \frac{\partial^2 u}{\partial t^2} = \alpha \frac{\partial^2 u}{\partial x^2}$$

3-3.3 Stability

Numerical stability is a concept applicable in the strict sense only to marching problems. A stable numerical scheme is one for which errors from any source (round-off, truncation, mistakes) are not permitted to grow in the sequence of numerical procedures as the calculation proceeds from one marching step to the next. Generally, concern over stability occupies much more of our time and energy than does concern over consistency. Consistency is relatively easy to check and most schemes which are conceived will be consistent just due to the methodology employed in their development. Stability is much more subtle and usually a bit of hard work is required in order to establish analytically that a scheme is stable. Much more will be presented in Section 3-6 on stability and some very workable methods will be developed for establishing the stability limits for linear partial differential equations. It will be possible to extend these guidelines to nonlinear equations in an approximate sense.

Using these guidelines, the DuFort-Frankel scheme, Eq. (3-57), for the heat equation would be found to be unconditionally stable whereas the simple explicit scheme would be stable only if $r = [\alpha \Delta t / (\Delta x)^2] \leq \frac{1}{2}$. This restriction would limit the size of the marching step permitted for any specified spatial mesh.

A scheme using a central time difference and having a more favorable truncation error $O[(\Delta t)^2, (\Delta x)^2]$,

$$\frac{u_j^{n+1} - u_j^{n-1}}{2 \Delta t} = \frac{\alpha}{(\Delta x)^2} (u_{j+1}^n - 2u_j^n + u_{j-1}^n) \quad (3-58)$$

is unconditionally unstable and therefore cannot be used for real calculations despite the fact that it looks to be more accurate, in terms of truncation error, than the ones given previously which will work.

Sometimes instability can be identified with a physical implausibility. That is, conditions which would result in an unstable numerical procedure would also imply unacceptable modeling of physical processes. To illustrate this, we rearrange the simple explicit representation of the heat equation, Eq. (3-56a), so that the unknown appears on the left. Letting $r = \alpha \Delta t / (\Delta x)^2$, our difference equation becomes

$$u_j^{n+1} = r(u_{j+1}^n + u_{j-1}^n) + (1 - 2r)u_j^n \quad (3-59)$$

Suppose that at time t , $u_{j+1}^n = u_{j-1}^n = 100^\circ\text{C}$ and $u_j^n = 0^\circ\text{C}$. This arrangement is shown in Fig. 3-2. If $r > \frac{1}{2}$ we see that the temperature at point j at time level $n+1$ will exceed the temperature at the two surrounding points at time level n . This seems unreasonable since we expect heat to flow from the warmer region to a colder region but not vice versa. The maximum temperature which we would expect to find at point j at time level $n+1$ is 100°C . If $r = 1$, for example, u_j^{n+1} would equal 200°C by Eq. (3-59).

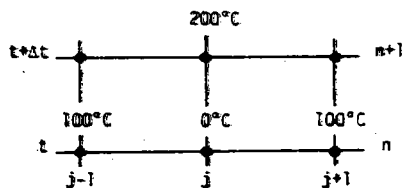


Figure 3-2 Physical implausibility resulting from $r = 1$.

3-3.4 Convergence for Marching Problems

Generally, we find that a consistent, stable scheme is convergent. Convergence here means that the solution to the finite-difference equation approaches the true solution to the PDE having the same initial and boundary conditions as the mesh is refined. A proof of this is available for initial value (marching) problems governed by linear PDE's. The theorem, due to Lax (see Richtmyer and Morton, 1967) is stated here without proof.

Lax's Equivalence Theorem. Given a properly posed initial value problem and a finite-difference approximation to it that satisfies the consistency condition, stability is the necessary and sufficient condition for convergence.

We might add that most computational work proceeds as though this theorem applies also to nonlinear PDE's although the theorem has never been proven for this more general category of equations.

3-3.5 Round-off and Discretization Errors

Any computed solution, including sometimes an "exact" analytic solution to a PDE, may be affected by rounding to a finite number of digits in the arithmetic operations. These errors are called *round-off errors* and we are especially aware of their existence in obtaining machine solutions to finite-difference equations because of the large number of dependent, repetitive operations which are usually involved. In some types of calculations, the magnitude of the round-off error is proportional to the number of grid points in the problem domain. In these cases, refining the grid may decrease the truncation error but increase the round-off error.

Discretization error is the error in the solution to the PDE caused by replacing the continuous problem by a discrete one and is defined as the difference between the exact solution of the PDE (round-off free) and the exact solution of the finite-difference equations (round-off free). In terms of the definitions developed thus far, the difference between the exact solution of the PDE and the computer solution to the finite-difference equations would be equal to the sum of the discretization error and the round-off error associated with the finite-difference calculation. We can also observe that the discretization error is the error in the solution which is caused by the truncation error in the difference representation of the PDE plus any errors introduced by the treatment of boundary conditions.

3-3.6 A Comment on Equilibrium Problems

Throughout our discussion of stability and convergence, the focus was on marching problems (parabolic and hyperbolic PDE's). Despite this emphasis on initial value problems, most of the material presented in this chapter also applies to equilibrium problems. The exception is the concept of stability. We should observe, however, that the important concept of consistency applies to difference representations of PDE's of all classes.

The "convergence" of the solution of the difference equation to the exact solution of the PDE might be aptly termed truncation or discretization convergence. The solution to equilibrium problems (elliptic equations) leads us to a system of simultaneous algebraic equations which needs to be solved only once, rather than in a marching manner. Thus, the concept of stability developed previously is not directly applicable as stated. To achieve "truncation convergence" for equilibrium problems it would seem that it is only necessary to devise a solution scheme in which the error in solving the simultaneous algebraic equations can be controlled as the mesh size is refined without limit. Many common schemes are iterative (Gauss-Seidel iteration is one example) in nature and for these we want to ensure that the iterative process converges. Here convergence means that the iterative process is repeated until the magnitude of the difference between the function at the $k + 1$ and the k iteration levels is as small as we wish for each grid point, i.e., $|u_{i,j}^{k+1} - u_{i,j}^k| < \epsilon$. This is known as *iteration convergence*. It would appear that (no proof can be cited) truncation convergence will be assumed for a consistent representation to an equilibrium problem if it can be shown that the iterative method of solution converges even for arbitrarily small choices of mesh sizes.

It is possible to use direct (noniterative) methods to solve the algebraic equations associated with equilibrium problems. For these methods we would want to be sure that the errors inherent in the method, especially round-off errors, do not get out of control as the mesh is refined and the number of points tends toward infinity.

In closing this section we should mention that there are aspects to the iterative solution of equilibrium problems which resemble the marching process in initial value problems and a sense in which stability concerns in the marching problems correspond to iterative convergence concerns in the solution to equilibrium problems.

3-3.7 Conservation Form and Conservative Property

Two different ideas will be discussed in this section. The first has to do with the PDE's themselves. The terms "conservation form," "conservation-law form," "conservative form," and "divergence form" are all equivalent and partial differential equations having this form have the property that the coefficients of the derivative terms are either constant or, if variable, their derivatives appear nowhere in the equation. Normally, for the PDE's which represent a physical conservation statement, this means that the divergence of a physical quantity can be identified in the equation. As an example, the conservative form of the equation for mass conservation (continuity equation) is

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0 \quad (3-60)$$

which can be written in vector notation as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{V} = 0$$

A nonconservative or nondivergence form would be

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + \rho \frac{\partial u}{\partial x} + v \frac{\partial \rho}{\partial y} + \rho \frac{\partial v}{\partial y} + w \frac{\partial \rho}{\partial z} + \rho \frac{\partial w}{\partial z} = 0 \quad (3-61)$$

As a second example, we consider the one-dimensional heat conduction equation for a substance whose density, ρ , specific heat, c , and thermal conductivity, k , all vary with position. The conservative form of this equation is

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) \quad (3-62)$$

whereas a nonconservative form would be

$$\rho c \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} + \frac{\partial k}{\partial x} \frac{\partial T}{\partial x} \quad (3-63)$$

In Eq. (3-62) the right-hand side can be identified as the negative of the divergence of the heat flux vector specialized for one-dimensional conduction. A difference formulation based on a PDE in nondivergence form may lead to numerical difficulties in situations where the coefficients may be discontinuous as in flows containing shock waves.

The second idea to be developed in this section is that of the *conservative property of a finite-difference representation*. The PDE's of interest in this book all have their basis in physical laws such as the conservation of mass, momentum, and energy. Such a PDE represents a conservation statement at a point. We strive to construct finite-difference representations which provide a good approximation to the PDE in a small, local neighborhood involving a few grid points. The same conservation principles which gave rise to the PDE's also apply to arbitrarily large regions (control volumes). In fact, in deriving the PDE's we usually start with the control-volume form of the conservation statement. If our finite-difference representation approximates the PDE closely in the neighborhood of each grid point, then we have reason to expect that the related conservation statement will be approximately enforced over a larger control volume containing a large number of grid points in the interior. Those finite-difference schemes which maintain the discretized version of the conservation statement exactly (except for round-off errors) for any mesh size over an arbitrary finite region containing any number of grid points is said to have the *conservative property*. For some problems this property is crucial.

The key word in the definition above is "exactly." All consistent schemes should approximately enforce the appropriate conservation statement over large regions, but schemes having the conservative property do so exactly (except for round-off

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errors), because of exact cancellation of terms. To illustrate this concept, we will consider a problem requiring the solution of the continuity equation for steady flow. The PDE can be written as

$$\nabla \cdot \rho \mathbf{V} = 0$$

We will assume that the PDE is approximated by a suitable finite-difference representation and solved throughout the flow. For an arbitrary control volume which could include the entire problem domain or any fraction of it, conservation of mass for steady flow requires that the net mass efflux be zero (mass flow rate in equals mass flow rate out). This is observed formally by applying the divergence theorem to the governing PDE,

$$\iiint_R \nabla \cdot \rho \mathbf{V} dR = \iint_S \rho \mathbf{V} \cdot \mathbf{n} dS = 0$$

To see if the finite-difference representation for the PDE has the conservative property, we must establish that the discretized version of the divergence theorem is satisfied. We normally check this for a control volume consisting of the entire problem domain. To do this the integral on the left is evaluated by summing the difference representation of the PDE at all grid points. If the difference scheme has the conservative property, all terms will cancel except those which represent fluxes at the boundaries. It should be possible to rearrange the remaining terms to obtain identically a finite-difference representation of the integral on the right. For this example the result will be a verification that the mass flux into the control volume equals the mass flux out. If the difference scheme used for the PDE is not conservative, the numerical solution may permit the existence of small mass sources or sinks.

Schemes having the conservative property occur in a natural way when differencing starts with the divergence form of the PDE. For some equations and problems, the divergence form is not an appropriate starting point. For these situations, use of a control volume method (Section 3-4.4) for obtaining the difference scheme is helpful. This difference representation will usually have the conservative property if care is taken to ensure that the expressions used to represent fluxes across the interface of two adjacent control volumes are the same in the difference form of the conservation statement for each of the two control volumes.

The conservative property issue has been actively discussed and debated over the short history of computational fluid mechanics and heat transfer. However, the conservative property is not the only important figure of merit for a difference representation. PDE's represent more than a conservation statement at a point. As shown by solution forms in Chapter 2, PDE's also contain information on characteristic directions and domains of dependence. Proper representation of this information is also important. Many useful finite-difference equations do not have the conservative property and in a few instances, prove to be more accurate in some sense than those that do. The importance of maintaining the conservation statement with high accuracy over a finite region is highly problem dependent. All consistent formulations, whether

or not they have the conservative property, can provide an adequate representation for most problems if the grid is refined sufficiently.

3-4 FURTHER EXAMPLES OF METHODS FOR OBTAINING FINITE-DIFFERENCE EQUATIONS

As we start with a given PDE and a finite-difference mesh, several procedures are available to us for developing finite-difference equations. Among these are:

- a. Taylor-series expansions
- b. polynomial fitting
- c. integral method (called the micro-integral method by some)
- d. control-volume approach

It's sometimes possible to obtain exactly the same finite-difference representation by using all four methods. In our introduction to the subject, we'll lean most heavily on the use of Taylor-series expansions utilizing polynomial fitting on occasion in treating boundary conditions.

3-4.1 Use of Taylor Series

We now demonstrate how one might proceed on a slightly more formal basis with Taylor-series expansions to develop difference expressions satisfying specified constraints. Suppose we want to develop a difference approximation for $\partial u / \partial x$ having a truncation error of $O(\Delta x)^2$ using at most values $u_{i-2,j}$, $u_{i-1,j}$, and $u_{i,j}$. With these constraints and objectives, it would appear logical to write Taylor-series expressions for $u_{i-2,j}$ and $u_{i-1,j}$ expanding about the point (i, j) and attempt to solve for $\partial u / \partial x$ from the resulting equations in such a way as to obtain a truncation error of $O(\Delta x)^2$:

$$u_{i-2,j} = u_{i,j} + \frac{\partial u}{\partial x}(-2\Delta x) + \frac{\partial^2 u}{\partial x^2} \frac{(2\Delta x)^2}{2!} + \frac{\partial^3 u}{\partial x^3} \frac{(-2\Delta x)^3}{3!} + \dots \quad (3-64)$$

$$u_{i-1,j} = u_{i,j} + \frac{\partial u}{\partial x}(-\Delta x) + \frac{\partial^2 u}{\partial x^2} \frac{(\Delta x)^2}{2!} + \frac{\partial^3 u}{\partial x^3} \frac{(-\Delta x)^3}{3!} + \dots \quad (3-65)$$

It's often possible to determine the required form of the difference representation by inspection or simple substitution. To proceed by substitution, we will rearrange Eq. (3-64) to put $\partial u / \partial x$ on the left-hand side such that

$$\frac{\partial u}{\partial x} = \frac{u_{i,j} - u_{i-2,j}}{2\Delta x} + \frac{\partial^2 u}{\partial x^2} \Delta x + O(\Delta x)^2$$

As is, the representation is $O(\Delta x)$ because of the term $(\partial^2 u / \partial x^2) \Delta x$. We can substitute for $\partial^2 u / \partial x^2$ in the above equation using Eq. (3-65) to obtain the desired result. A more formal procedure to obtain the desired expression is sometimes useful. To proceed more formally we first multiply Eq. (3-64) by a and Eq. (3-65) by b and add the two equations. If $-2a - b = 1$, then the coefficient of $\partial u / \partial x$ will be 1 after the

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addition, and if $2a + b/2 = 0$, then the terms involving $\partial^2 u / \partial x^2)_{i,j}$ which would contribute a T.E. of $O(\Delta x)$ to the final result will be eliminated. A solution to the equations

$$-2a - b = 1 \quad 2a + \frac{b}{2} = 0$$

is given by $a = \frac{1}{2}$, $b = -2$. Thus, if we multiply Eq. (3-64) by $\frac{1}{2}$, Eq. (3-65) by -2 , add the results and solve for $\partial u / \partial x)_{i,j}$ we obtain

$$\left(\frac{\partial u}{\partial x} \right)_{i,j} = \frac{u_{i-2,j} - 4u_{i-1,j} + 3u_{i,j}}{2\Delta x} + O[(\Delta x)^2]$$

which can be recognized as Eq. (3-30). A careful check on the details of this example will reveal that it was really necessary to include terms involving $\partial^3 u / \partial x^3)_{i,j}$ in the Taylor-series expansions in order to determine whether or not these terms would cancel in the algebraic operations and reduce the truncation error even further to $O(\Delta x)^3$. Fortuitous cancellation of terms occurs frequently enough to warrant close attention to this point.

We should observe that it is sometimes necessary to carry out the inverse of the above process. That is, suppose we had obtained the approximation represented by Eq. (3-30) by some other means and we wanted to investigate the consistency and truncation error of such an expression. For this, the use of Taylor-series expansions would be invaluable and the recommended procedure would be to substitute the Taylor-series expressions from Eq. (3-64) and Eq. (3-65) above for $u_{i-2,j}$ and $u_{i-1,j}$ into the difference representation to obtain an expression of the form $\partial u / \partial x)_{i,j} + \text{T.E.}$ on the right-hand side. At this point the T.E. has been identified and if $\lim_{\Delta x \rightarrow 0} (\text{T.E.}) = 0$ the difference representation is consistent.

As a slightly more complex example, we will develop a finite-difference approximation with truncation error $O(\Delta y)^2$ for $\partial u / \partial y$ at point (i, j) using at most $u_{i,j}$, $u_{i,j+1}$, $u_{i,j-1}$ when the grid spacing is not uniform. We will adopt the notation that $\Delta y_+ = y_{i,j+1} - y_{i,j}$ and $\Delta y_- = y_{i,j} - y_{i,j-1}$ as indicated in Fig. 3-3.

We recall that for equal spacing, the central-difference representation for a first derivative was equivalent to the arithmetic average of a forward and backward representation. That is, for $\Delta y_+ = \Delta y_- = \Delta y$

$$\left(\frac{\partial u}{\partial y} \right)_{i,j} = \frac{\bar{\delta}_y u_{i,j}}{2\Delta y} = \frac{\Delta y_+ u_{i,j} + \Delta y_- u_{i,j}}{2\Delta y} + O[(\Delta y)^2]$$

We might wonder if, for unequal spacing, use of a geometrically weighted average will preserve the second-order accuracy:

$$\left(\frac{\partial u}{\partial y} \right)_{i,j} \approx \frac{\Delta y_+ u_{i,j}}{\Delta y_+} \left(\frac{\Delta y_-}{\Delta y_+ + \Delta y_-} \right) + \frac{\Delta y_- u_{i,j}}{\Delta y_-} \left(\frac{\Delta y_+}{\Delta y_+ + \Delta y_-} \right) + O[(\Delta y)^2] \quad (3-66)$$

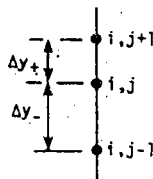


Figure 3-3 Notation for unequal y spacing.

The truth of the above statement may be evident to some, but it can be verified from basics by use of Taylor-series expansions about point (i, j) . Letting $\Delta y_+/\Delta y_- = \alpha$, and adopting the more compact subscript notation to denote differentiation, $u_y = \partial u/\partial y)_{i,j}$, $u_{yy} = \partial^2 u/\partial y^2)_{i,j}$, etc., we obtain

$$u_{i,j+1} = u_{i,j} + u_y \alpha \Delta y_- + u_{yy} \frac{(\alpha \Delta y_-)^2}{2!} + u_{yyy} \frac{(\alpha \Delta y_-)^3}{3!} + u_{yyyy} \frac{(\alpha \Delta y_-)^4}{4!} + \dots \quad (3-67)$$

$$u_{i,j-1} = u_{i,j} + u_y (-\Delta y_-) + u_{yy} \frac{(-\Delta y_-)^2}{2!} + u_{yyy} \frac{(-\Delta y_-)^3}{3!} + u_{yyyy} \frac{(-\Delta y_-)^4}{4!} + \dots \quad (3-68)$$

As before, we will multiply Eq. (3-67) by a and Eq. (3-68) by b , add the results and solve for $\partial u/\partial y)_{i,j}$. Requiring that the coefficient of $\partial u/\partial y)_{i,j} \Delta y_-$ be equal to one after the addition, gives $a\alpha - b = 1$. For the final result to have a truncation error $O(\Delta y)^2$ or better, the coefficient of u_{yy} must be zero after the addition which requires that $\alpha^2 a + b = 0$. A solution to these two algebraic equations can be obtained readily as $a = 1/\alpha(\alpha + 1)$, $b = -\alpha/(\alpha + 1)$. Thus

$$\frac{\partial u}{\partial y} \Big|_{i,j} = \frac{a \times \text{Eq. (3-67)} + b \times \text{Eq. (3-68)}}{\Delta y_-} + O(\Delta y)^2$$

The final result can be written as

$$\frac{\partial u}{\partial y} \Big|_{i,j} = \frac{u_{i,j+1} + (\alpha^2 - 1)u_{i,j} - \alpha^2 u_{i,j-1}}{\alpha(\alpha + 1) \Delta y_-} \quad (3-69)$$

which can be rearranged further into the form given by Eq. (3-66).

Our Taylor series examples thus far have illustrated procedures for obtaining a finite-difference approximation to a single derivative. However, our main interest is in correctly approximating an entire PDE at an arbitrary point in the problem domain. For this reason, we must be careful to use the same expansion point in approximating all derivatives in the PDE by the Taylor-series method. If this is done, then the truncation error for the entire equation can be obtained by adding the truncation error for each derivative.

There is no requirement that the expansion point be (i, j) as indicated by the following examples where the order of the truncation error and the most convenient expansion points are indicated. The geometric arrangement of points used in the difference equation is indicated by the sketch of the difference "molecule."

Fully implicit form for the heat equation, Eq. (3-55):

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{\alpha}{(\Delta x)^2} (u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}) \quad \text{T.E.} = O[\Delta t, (\Delta x)^2] \quad (3-70)$$

The difference molecule for this scheme is shown in Fig. 3-4, and point $(n + 1, j)$ is indicated as the most convenient expansion point.

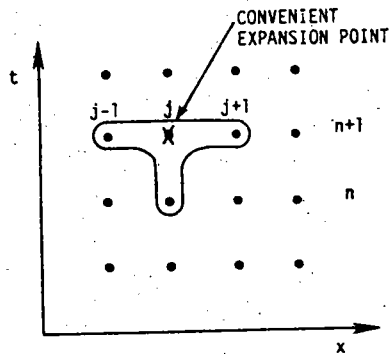


Figure 3-4 Difference molecule, fully implicit form for heat equation.

Crank-Nicolson form for the heat equation:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{\alpha}{2(\Delta x)^2} [u_{j+1}^{n+1} + u_{j+1}^n - 2(u_j^{n+1} + u_j^n) + u_{j-1}^{n+1} + u_{j-1}^n] \quad (3-71a)$$

$$\text{T.E.} = O[(\Delta t)^2, (\Delta x)^2]$$

The difference molecule for the Crank-Nicolson scheme is shown in Fig. 3-5, and point $(n + \frac{1}{2}, j)$ is designated as the most convenient expansion point.

It is interesting to note that the *order* of the truncation error for difference representations of a complete PDE (not a single derivative term, however) is not dependent upon the choice of expansion point in the evaluation of this error by the Taylor-series method. We will demonstrate this point by considering the Crank-Nicolson scheme. The truncation error for the Crank-Nicolson scheme was most conveniently determined by expanding about the point $(n + \frac{1}{2}, j)$ to obtain the results stated above. Using this point resulted in the elimination of the maximum number of terms from the Taylor series by cancellation. Had we used point (n, j) or even $(n - 1, j)$ as the expansion point, the conclusion on the order of the truncation error would have been the same. To reach this conclusion, however, we often must examine the truncation error very carefully. To illustrate, evaluating the truncation error of

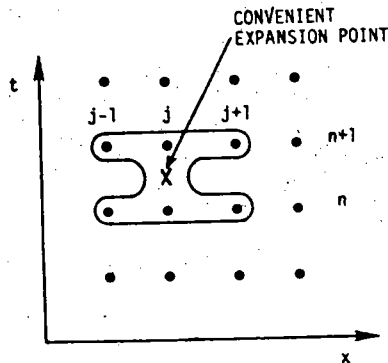


Figure 3-5 Difference molecule, Crank-Nicolson form for heat equation.

the Crank-Nicolson scheme by using expansions for $u_{j-1}^n, u_{j+1}^n, u_{j-1}^{n+1}, u_{j+1}^{n+1}, u_j^{n+1}$ about point (n, j) in Eq. (3-71a) gives, after rearrangement,

$$u_t - \alpha u_{xx} = -u_{tt} \frac{\Delta t}{2} + \alpha u_{txx} \frac{\Delta t}{2} + O(\Delta x)^2 + O(\Delta t)^2 \quad (3-71b)$$

At first glance, we are tempted to conclude that the truncation error for the Crank-Nicolson scheme becomes $O(\Delta t) + O(\Delta x)^2$, when evaluated by expanding about point (n, j) , because of the appearance of the terms $-u_{tt} \Delta t/2$ and $\alpha u_{txx} \Delta t/2$. However, we can recognize these two terms as $-\Delta t/2(\partial/\partial t)(u_t - \alpha u_{xx})$ where the quantity in the second set of parentheses is the left-hand side of Eq. (3-71b). Thus, we can differentiate Eq. (3-71b) with respect to t and multiply both sides by $-\Delta t/2$ to learn that $-\Delta t/2(\partial/\partial t)(u_t - \alpha u_{xx}) = O(\Delta t)^2 + O(\Delta x)^2$. From this we conclude that the truncation error for the Crank-Nicolson scheme is $O(\Delta t)^2 + O(\Delta x)^2$ when evaluated about either point (n, j) or point $(n + \frac{1}{2}, j)$. Use of other points will give the same results for the order of the truncation error. This example illustrates that the leading terms in the truncation error should be examined very carefully to see if they can be identified as a multiple of a derivative of the original PDE. If they can, they should be replaced by expressions of higher order.

3-4.2 Use of Polynomial Fitting

Many applications of polynomial fitting are observed in computational fluid mechanics and heat transfer. The technique can be used to develop the entire finite-difference representation for a PDE. However, the technique is perhaps most commonly employed in the treatment of boundary conditions or in gleanings information from the solution in the neighborhood of the boundary.

Let's consider some specific examples.

Example 3-1 In this example, the derivative approximations needed to represent a PDE will be obtained by assuming that the solution to the PDE can be approximated locally by a polynomial. The polynomial is then "fitted" to the points surrounding the general point (i, j) utilizing values of the function at the grid points. A sufficient number of points can be used to determine the coefficients in the polynomial exactly. The polynomial can then be differentiated to obtain the desired approximation to the derivatives. Let's consider Laplace's equation which governs the two-dimensional temperature distribution in a solid under steady-state conditions:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \quad (3-72)$$

We suppose that both the x and y dependency of temperature can be expressed by a second-degree polynomial. For example, holding y fixed, we assume that temperatures at various x locations in the neighborhood of point (i, j) can be determined from

$$T(x, y_0) = a + bx + cx^2$$

For convenience we let $x = 0$ at point (i, j) , and $\Delta x = \text{constant}$. Clearly,

$$\begin{aligned}\frac{\partial T}{\partial x}\bigg|_{i,j} &= b \\ \frac{\partial^2 T}{\partial x^2}\bigg|_{i,j} &= 2c\end{aligned}$$

The coefficients a , b , and c can be evaluated in terms of temperatures at specific grid points and Δx . To do so, we must make some choices as to which neighboring grid points to use and this choice determines the geometrical arrangement of the difference molecule; that is, whether the resulting derivative approximations are central, forward, or backward differences. Here we'll choose points $(i-1, j)$, (i, j) , and $(i+1, j)$ and obtain:

$$\begin{aligned}T(i, j) &= a \\ T(i+1, j) &= a + b \Delta x + c(\Delta x)^2 \\ T(i-1, j) &= a - b \Delta x + c(\Delta x)^2\end{aligned}$$

from which we determine that

$$\begin{aligned}b &= \frac{\partial T}{\partial x}\bigg|_{i,j} = \frac{T_{i+1,j} - T_{i-1,j}}{2 \Delta x} \\ c &= \frac{1}{2} \frac{\partial^2 T}{\partial x^2}\bigg|_{i,j} = \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{2(\Delta x)^2}\end{aligned}$$

Thus,

$$\frac{\partial^2 T}{\partial x^2}\bigg|_{i,j} = \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2} \quad (3-73)$$

This represents an exact result if indeed a second-degree polynomial expresses the correct variation of temperature with x . In the general case, we only suppose that the second-degree polynomial is a good approximation to the solution. The truncation error of the expression, Eq. (3-73), can be determined by substituting Taylor-series expansions about point (i, j) for $T_{i+1,j}$ and $T_{i-1,j}$ into Eq. (3-73). The truncation error is found to be $O(\Delta x)^2$ and will involve only fourth-order and higher derivatives, which are equal to zero when the temperature variation is given by a second-degree polynomial.

A finite-difference approximation for $\partial^2 T / \partial y^2$ can be found in a like manner. We notice that arbitrary decisions need to be made in the process of polynomial fitting which will influence the form and truncation error of the result; particularly, these decisions influence which of the neighboring points will appear in the difference expression. We also observe that there is nothing unique about the procedure of polynomial fitting which guarantees that the difference approximation for the PDE is the best in any sense or that the numerical scheme is stable (when used for a marching problem)

Example 3-2 Suppose we have solved the finite-difference form of the energy equation for the temperature distribution near a solid boundary and we need to estimate the heat flux at the location. Our finite-difference solution gives us only the temperature at discrete grid points. From Fourier's law, the boundary heat flux is given by $q_w = -k \partial T / \partial y|_{y=0}$. Thus, we need to approximate $\partial T / \partial y|_{y=0}$ by a difference representation which uses the temperature obtained from the finite-difference solution to the energy equation. One way to proceed is to assume that the temperature distribution near the boundary is a polynomial and to "fit" such a polynomial, i.e., straight line, parabola, or third-degree polynomial, etc. to the finite-difference solution which has been determined at discrete points. By requiring that the polynomial match the finite-difference solution for T at certain discrete points, the unknown coefficients in the polynomial can be determined.

For example, if we assume that the temperature distribution near the boundary is again a second-degree polynomial of the form $T = a + by + cy^2$, then, referring to Fig. 3-6, we note that $\partial T / \partial y|_{y=0} = b$. Further, for equally spaced mesh points we can write

$$T_1 = a$$

$$T_2 = a + b \Delta y + c(\Delta y)^2$$

$$T_3 = a + b(2 \Delta y) + c(2 \Delta y)^2$$

from which we can determine that

$$a = T_1$$

$$b = \frac{-3T_1 + 4T_2 - T_3}{2 \Delta y}$$

and

$$c = \frac{T_1 - 2T_2 + T_3}{2(\Delta y)^2}$$

Thus, we can evaluate the wall heat flux by the approximation

$$q_w = -k \left. \frac{\partial T}{\partial y} \right|_{y=0} \simeq -kb = \frac{k}{2 \Delta y} (3T_1 - 4T_2 + T_3)$$

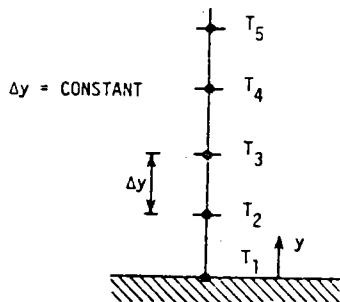


Figure 3-6 Finite-difference grid near wall.

It is natural to inquire about the truncation error of this approximation for $\partial T/\partial y|_{y=0}$. This may be established by expressing T_2 and T_3 in terms of Taylor-series expansions about the boundary point and substituting these evaluations into the difference expression for $\partial T/\partial y|_{y=0}$. Alternatively, we can identify the second-degree polynomial as a truncated Taylor-series expansion about $y = 0$:

Second-degree polynomial:

$$T = a + by + cy^2$$

Taylor series:

$$T = T(0) + \left. \frac{\partial T}{\partial y} \right|_0 y + \left. \frac{\partial^2 T}{\partial y^2} \right|_0 \frac{y^2}{2!} + \underbrace{\left. \frac{\partial^3 T}{\partial y^3} \right|_0 \frac{y^3}{3!} + \dots}_{\text{T.E.}}$$

Thus, the approximation $T \simeq a + by + cy^2$ is equivalent to utilizing the first three terms of a Taylor-series expansion with the resulting T.E. in the expression for T being $O(\Delta y)^3$. Solving the Taylor series for an expression for $\partial T/\partial y|_{y=0}$ involves division by Δy which reduces the truncation error in the expression for $\partial T/\partial y|_{y=0}$ to $O(\Delta y)^2$.

Example 3-3 Suppose that the energy equation is being solved for the temperature distribution near the wall as in Example 3-2, but now the wall heat flux is specified as a boundary condition. We may then want to use polynomial fitting to obtain an expression for the boundary temperature which is called for in the difference equations for internal points. In other words, if $q_w = -k \partial T/\partial y|_{y=0}$ is given, how can we evaluate T at $y = 0$, i.e., (T_1) in terms of q_w/k and T_2, T_3 , etc.? Here we might assume that $T = a + by + cy^2 + dy^3$ near the wall and that $\partial T/\partial y|_{y=0} = b = -q_w/k$ (given). Our objective is to evaluate T_1 which in this case equals a . Referring to Fig. 3-6, we can write

$$T_2 = a - \frac{q_w}{k} \Delta y + c(\Delta y)^2 + d(\Delta y)^3$$

$$T_3 = a - \frac{q_w}{k} (2\Delta y) + c(2\Delta y)^2 + d(2\Delta y)^3$$

$$T_4 = a - \frac{q_w}{k} (3\Delta y) + c(3\Delta y)^2 + d(3\Delta y)^3$$

These three equations can be solved for a, c , and d in terms of $T_2, T_3, T_4, q_w/k$, and Δy . The desired result, T_1 as a function of $T_2, T_3, q_w/k$, and Δy , follows directly from $T_1 = a$ and is given by

$$T_1 = \frac{1}{11} \left(16T_2 - 9T_3 + 2T_4 + \frac{6\Delta y q_w}{k} \right) + O[(\Delta y)^4] \quad (3-74)$$

The truncation error in the above expression can be established by substituting Taylor-series expansions about (i, j) for the temperatures on the right-hand side or by identifying the polynomial as a truncated series by inspection. We will close this discussion on polynomial fitting by listing some expressions for wall values

Table 3-3 Some useful results from polynomial fitting

Polynomial degree	Wall value of function or derivative	Equation
1	$\left. \frac{\partial T}{\partial y} \right _{i,j} = \frac{T_{i,j+1} - T_{i,j}}{h} + O(h)$	(3-75)
1	$T_{i,j} = T_{i,j+1} - h \left. \frac{\partial T}{\partial y} \right _{i,j} + O(h^2)$	(3-76)
2	$\left. \frac{\partial T}{\partial y} \right _{i,j} = \frac{1}{2h} (-3T_{i,j} + 4T_{i,j+1} - T_{i,j+2}) + O(h^2)$	(3-77)
2	$T_{i,j} = \frac{1}{3} \left[4T_{i,j+1} - T_{i,j+2} - 2h \left. \frac{\partial T}{\partial y} \right _{i,j} \right] + O(h^3)$	(3-78)
3	$\left. \frac{\partial T}{\partial y} \right _{i,j} = \frac{1}{6h} (-11T_{i,j} + 18T_{i,j+1} - 9T_{i,j+2} + 2T_{i,j+3}) + O(h^3)$	(3-79)
3	$T_{i,j} = \frac{1}{11} \left[18T_{i,j+1} - 9T_{i,j+2} + 2T_{i,j+3} - 6h \left. \frac{\partial T}{\partial y} \right _{i,j} \right] + O(h^4)$	(3-80)
4	$\left. \frac{\partial T}{\partial y} \right _{i,j} = \frac{1}{12h} (-25T_{i,j} + 48T_{i,j+1} - 36T_{i,j+2} + T_{i,j+3} - 3T_{i,j+4}) + O(h^4)$	(3-81)
4	$T_{i,j} = \frac{1}{25} \left[48T_{i,j+1} - 36T_{i,j+2} + 16T_{i,j+3} - 3T_{i,j+4} - 12h \left. \frac{\partial T}{\partial y} \right _{i,j} \right] + O(h^5)$	(3-82)

of a function and its first derivative in terms of values of the function. These expressions are useful, for example, in extracting a value of the function at the wall, if the wall value of the first derivative is specified. The results in Table 3-3 were obtained from polynomial fitting, assuming that $T(y)$ can be expressed as a polynomial of degree up to the fourth, and that $\Delta y = h = \text{constant}$.

3-4.3 The Integral Method

The integral method provides yet another means for developing difference approximations to PDE's. We consider again the heat equation as the specimen equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \quad (3-83)$$

The strategy is to develop an algebraic relationship among the u 's at neighboring grid points by integrating the heat equation with respect to the independent variables t and x over the local neighborhood of point (n, j) . The point (n, j) will also be identified as point (t_0, x_0) . Grid points are spaced at intervals of Δx and Δt . We arbitrarily decide to integrate both sides of the equation over the interval t_0 to $t_0 + \Delta t$ and $x_0 - \Delta x/2$ to $x_0 + \Delta x/2$. Choosing $t_0 - \Delta t/2$ to $t_0 + \Delta t/2$ would lead to an inherently unstable difference equation. Unfortunately, at this point we have no way of knowing which choice for the integration interval would be the right or wrong one relative to stability of the solution method. This can only be determined by a trial calculation or application of the methods for stability analysis to be presented in

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Section 3-6. The order of integration is chosen for each side in a manner to take advantage of exact differentials:

$$\int_{x_0 - \Delta x/2}^{x_0 + \Delta x/2} \left(\int_{t_0}^{t_0 + \Delta t} \frac{\partial u}{\partial t} dt \right) dx = \alpha \int_{t_0}^{t_0 + \Delta t} \left(\int_{x_0 - \Delta x/2}^{x_0 + \Delta x/2} \frac{\partial^2 u}{\partial x^2} dx \right) dt \quad (3-84)$$

The inner level of integration can be done exactly giving

$$\begin{aligned} & \int_{x_0 - \Delta x/2}^{x_0 + \Delta x/2} [u(t_0 + \Delta t, x) - u(t_0, x)] dx \\ &= \alpha \int_{t_0}^{t_0 + \Delta t} \left[\frac{\partial u}{\partial x} \left(t, x_0 + \frac{\Delta x}{2} \right) - \frac{\partial u}{\partial x} \left(t, x_0 - \frac{\Delta x}{2} \right) \right] dt \end{aligned} \quad (3-85)$$

For the next level of integration, we take advantage of the mean-value theorem for integrals which assures us that for a continuous function $f(y)$

$$\int_{y_1}^{y_1 + \Delta y} f(y) dy = f(\bar{y}) \Delta y \quad (3-86)$$

where $\bar{y}$ is some value of y in the interval $y_1 \leq \bar{y} \leq y_1 + \Delta y$. Thus, any value of y on the interval will provide an approximation to the integral and we can write

$$\int_{y_1}^{y_1 + \Delta y} f(y) dy \simeq f(\tilde{y}) \Delta y \quad \text{for } y_1 \leq \tilde{y} \leq y_1 + \Delta y$$

As we invoke the mean-value theorem to further simplify Eq. (3-85), we arbitrarily select x_0 on the left-hand side and $t_0 + \Delta t$ on the right-hand side as the locations within the intervals of integration at which to evaluate the integrands:

$$\begin{aligned} [u(t_0 + \Delta t, x_0) - u(t_0, x_0)] \Delta x &= \alpha \left[\frac{\partial u}{\partial x} \left(t_0 + \Delta t, x_0 + \frac{\Delta x}{2} \right) \right. \\ &\quad \left. - \frac{\partial u}{\partial x} \left(t_0 + \Delta t, x_0 - \frac{\Delta x}{2} \right) \right] \Delta t \end{aligned} \quad (3-87)$$

To express the result in purely algebraic terms requires that the first derivatives, $\partial u / \partial x$, on the right-hand side be approximated by finite differences. We could achieve this by falling back on our experience to date and simply utilize central differences. Alternatively, we can continue to pursue a purely integral approach and invoke the mean-value theorem for integrals again observing that

$$\begin{aligned} u(t_0 + \Delta t, x_0 + \Delta x) &= u(t_0 + \Delta t, x_0) + \int_{x_0}^{x_0 + \Delta x} \frac{\partial u}{\partial x} (t_0 + \Delta t, x) dx \\ &\simeq u(t_0 + \Delta t, x_0) + \frac{\partial u}{\partial x} \left(t_0 + \Delta t, x_0 + \frac{\Delta x}{2} \right) \Delta x \end{aligned} \quad (3-88)$$

from which we can write

$$\frac{\partial u}{\partial x} \left(t_0 + \Delta t, x_0 + \frac{\Delta x}{2} \right) \approx \frac{u(t_0 + \Delta t, x_0 + \Delta x) - u(t_0 + \Delta t, x_0)}{\Delta x} \quad (3-89)$$

In evaluating the integral in Eq. (3-88) through the mean-value theorem, we have arbitrarily evaluated the integrand at the midpoint of the interval. Hence, the final result is only an approximation. Treating the other first derivative in a similar manner permits the approximation to the heat equation to be written as

$$\begin{aligned} [u(t_0 + \Delta t, x_0) - u(t_0, x_0)] \Delta x = & \frac{\alpha}{\Delta x} [u(t_0 + \Delta t, x_0 + \Delta x) - 2u(t_0 + \Delta t, x_0) \\ & + u(t_0 + \Delta t, x_0 - \Delta x)] \Delta t \end{aligned} \quad (3-90)$$

Reverting back to the n, j notation whereby n denotes time (t) and j denotes space (x), we can rearrange the above in the form

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{\alpha}{(\Delta x)^2} (u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}) \quad (3-91)$$

which can be recognized as the fully implicit representation of the heat equation given in Section 3-4.1. The choice of $t_0 + \Delta t$ as the location to use in utilizing the mean-value theorem for the second integration on the right-hand side is responsible for the implicit form. If t_0 had been chosen instead, an explicit formulation would have resulted. We note that a statement of the truncation error does not evolve naturally as part of this method for developing difference equations but must be determined as a separate step.

3-4.4 Control-Volume Approach

A distinctly different point of view is taken in utilizing the control-volume approach than is adopted in working with any of the other methods considered thus far. In the Taylor-series and integral methods we accepted the PDE as the correct and appropriate form of the conservation principle (physical law) governing our problem and merely turned to mathematical tools to develop algebraic approximations to derivatives. We never considered again the physical law represented by the PDE. Physical reasoning had been used previously in deriving the PDE but then put aside. The Taylor-series and integral methods then proceed in a rather formal, mechanical way operating on the PDE.

In the control-volume method, we examine the PDE governing the problem at hand and recall the physical law or conservation statement which the PDE represents. In general, we translate the PDE to a statement in words as it might apply to a control volume in the neighborhood of a grid point in a finite-difference mesh. Here we are recognizing the discrete nature of the finite-difference model at the outset. We now proceed to work out a mathematical statement of the physical conservation principle following steps somewhat reminiscent of the procedures used by some to derive PDE's from physical laws except that we do not take the limit of shrinking the control

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volume to a point. When the governing PDE can be written in divergence form, we can be guided in this process by employing the Gauss divergence theorem to obtain the correct mathematical formulation for the physical law for a control volume. In practice, the control-volume method has a history of leading quickly to expressions that prove to be more accurate than other possibilities near boundaries, probably because the method keeps the discrete nature of the solution method in view at all times.

As an example, consider two-dimensional steady-state conduction in a solid where for constant thermal conductivity we find that the temperature distribution must satisfy Laplace's equation, Eq. (3-72).

A grid is first established on the problem domain. Points should be placed on the boundaries since temperatures at boundary points arise in the specification of boundary conditions. The problem domain is then equitably divided up into control volumes about each grid point. It will be most convenient to establish the boundaries of the control volumes halfway between adjacent grid points. Following this procedure, internal grid points will only be in the geometric center of the control volumes if the mesh spacing is constant, i.e., $\Delta x = c_1$, $\Delta y = c_2$.

We'll first consider the control volume labeled A in Fig. 3-7 which is representative of all internal (nonboundary) points. In steady state, the net rate at which heat flows into control volume A must be zero. This is the conservation statement which leads to the establishment of Laplace's equation as governing the temperature distribution at a point in the problem domain. We can also work backwards from the PDE to obtain the proper statement for a control volume by using the divergence theorem. First, we must recognize that the heat-flux vector is related to the temperature distribution through Fourier's law

$$\mathbf{q} = -k \nabla T$$

Thus, when k is constant, Eq. (3-72) can be interpreted as

$$-\nabla \cdot \mathbf{q} = \nabla \cdot (k \nabla T) = 0$$

Applying the divergence theorem gives

$$\iiint_R \nabla \cdot (k \nabla T) dR = \iint_S (k \nabla T) \cdot \mathbf{n} dS = 0$$

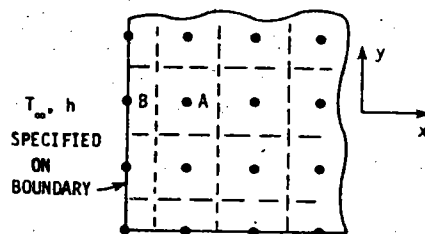


Figure 3-7 Finite-difference grid for control-volume method.

The integral on the right represents the net flow of heat into any arbitrary fixed region. Applying this to evaluate the inflow of heat through each of the four boundaries of the control volume about point (i, j) gives

$$-k \Delta y \left. \frac{\partial T}{\partial x} \right|_{i-1/2, j} + k \Delta y \left. \frac{\partial T}{\partial x} \right|_{i+1/2, j} - k \Delta x \left. \frac{\partial T}{\partial y} \right|_{i, j-1/2} + k \Delta x \left. \frac{\partial T}{\partial y} \right|_{i, j+1/2} = 0$$

The $1/2$ in the subscripts refers to evaluation at the boundaries of the control volume which are halfway between mesh points. The energy balance is exact if the derivatives represent suitable average values for the boundaries concerned. Approximating the derivatives by central differences we find that

$$k \Delta y \frac{(T_{i-1, j} - T_{i, j})}{\Delta x} + k \Delta y \frac{(T_{i+1, j} - T_{i, j})}{\Delta x} + k \Delta x \frac{(T_{i, j-1} - T_{i, j})}{\Delta y} + k \Delta x \frac{(T_{i, j+1} - T_{i, j})}{\Delta y} = 0$$

Dividing through by $k \Delta x \Delta y$ we can write the result as

$$\frac{T_{i-1, j} - 2T_{i, j} + T_{i+1, j}}{(\Delta x)^2} + \frac{T_{i, j-1} - 2T_{i, j} + T_{i, j+1}}{(\Delta y)^2} = 0 \quad (3-92)$$

Each of these two terms can be recognized as approximations to second derivatives, $(\partial^2 T / \partial x^2)$ for the first and $\partial^2 T / \partial y^2$ for the second) which were obtained earlier by the Taylor-series expansion method.

Let's now consider the control volume on the boundary, labeled B in Fig. 3-7. In this example we will assume that the boundary conditions are convective. For the continuous (nondiscrete) problem this is formulated mathematically by $h(T_\infty - T_{i, j}) = -k \partial T / \partial x|_{i, j}$ where the point (i, j) is the point on the physical boundary associated with control volume B. If we were to proceed with the Taylor-series approach to this boundary condition, we would likely next seek a difference representation for $\partial T / \partial x|_{i, j}$. If a simple forward difference is used, the difference equation would be

$$h(T_\infty - T_{i, j}) = \frac{k}{\Delta x} (T_{i, j} - T_{i+1, j}) \quad (3-93)$$

In the control-volume approach, however, we are forced to observe that there is some material associated with the boundary point and conduction may occur along the boundary. The energy balance on the control volume would account for possible transfer across all four boundaries

$$h \Delta y (T_\infty - T_{i, j}) + k \Delta y \left. \frac{\partial T}{\partial x} \right|_{i+1/2, j} + \frac{k \Delta x}{2} \left. \frac{\partial T}{\partial y} \right|_{i, j+1/2} - \frac{k \Delta x}{2} \left. \frac{\partial T}{\partial y} \right|_{i, j-1/2} = 0 \quad (3-94)$$

Using central differences to approximate the derivatives, we can write

$$h \Delta y (T_{\infty} - T_{i,j}) + k \Delta y \frac{(T_{i+1,j} - T_{i,j})}{\Delta x} + \frac{k \Delta x}{2} \frac{(T_{i,j+1} - T_{i,j})}{\Delta y} + \frac{k \Delta x}{2} \frac{(T_{i,j-1} - T_{i,j})}{\Delta y} = 0 \quad (3-95)$$

Dividing through by k , we can write the result as

$$\frac{h \Delta y}{k} T_{\infty} + \frac{\Delta y}{\Delta x} T_{i+1,j} + \frac{\Delta x}{2 \Delta y} (T_{i,j+1} + T_{i,j-1}) - \left(\frac{h \Delta y}{k} + \frac{\Delta y}{\Delta x} + \frac{\Delta x}{\Delta y} \right) T_{i,j} = 0 \quad (3-96)$$

which is somewhat different than Eq. (3-93) which followed from the most obvious application of the Taylor-series method to approximate the derivative $\partial T / \partial x$, which occurred in the formal mathematical statement of the boundary condition.

Looking back over the methodology of the control-volume and Taylor-series methods, we can note that the Taylor-series method readily provided difference approximations to derivatives and the representation for the complete PDE was made up from the addition of several such representations. In contrast, the control-volume method employs the conservation statement or physical law represented by the entire PDE and appears incapable of providing a finite-difference representation just to a derivative alone. That is, we would not know how to respond to a request to develop a difference representation to $\partial u / \partial x$ alone by the control-volume method. The distinctive characteristic of the control-volume approach is that a "balance" of some physical quantity is made on the region in the neighborhood of a grid point. The discrete nature of the problem domain is always taken into account in the control-volume approach which ensures that the physical law is satisfied over a finite region rather than only at a point as the mesh is shrunk to zero. It would appear that difference equations developed by the control-volume approach would almost certainly have the conservative property.

It is difficult to appreciate the subtle differences which may occur in the difference representations obtained for the same PDE by using the four different methods discussed in this section without working a large number of examples. In many cases, and especially for simple, linear equations, the resulting difference equations can be identical. That is, four different approaches can give the same result. There is no guarantee that difference equations developed by any of the methods will be numerically stable so that the same difference scheme developed by four methods could turn out to be worthless. The differences in the results obtained from using the different methods are more likely to become evident in coordinate systems other than rectangular and when the PDE's being approximated are in nonconservative form.

3-5 INTRODUCTION TO THE USE OF IRREGULAR MESHES

Naturally, it's most convenient to let the mesh increments such as Δx and Δy be constant throughout the computational domain. However, in many instances this is

not possible due to boundaries which do not coincide with the regular mesh or the need to reduce the mesh spacing in certain regions in order to maintain the desired level of accuracy. These irregularities occur frequently enough in physical problems as to command a significant amount of attention from workers in computational fluid mechanics and heat transfer. Prior to resorting to the use of irregular meshes, it is well to look for possible analytical procedures such as a coordinate transformation to bring the boundaries and the finite-difference mesh into alignment. Coordinate transformations will be discussed in detail in Section 5-6.

3-5.1 Irregular Mesh Due to Shape of a Boundary

To illustrate this problem we will assume that the regular mesh used for a problem is square, i.e., $\Delta x = \Delta y = \text{constant}$ but that one of the boundaries is curved causing unequal spacing between the boundary and some internal grid points as illustrated in Fig. 3-8. We will assume that the governing PDE is Laplace's equation and that u is specified on the boundaries. Some of the procedures which can be used in dealing with irregular mesh points near boundaries are listed below:

1. Use an especially fine but regular mesh near the boundary and define the grid point closest to the actual boundary as the boundary point for computational purposes. Unless a coarser mesh is used away from the boundary resulting in irregular mesh problems where the transition in spacing is made, this method could require a very large number of grid points to achieve reasonable accuracy.
2. Use linear interpolation to assign values of u to any internal point which is less than a regular mesh increment from the boundary. The interpolation is between the specified boundary values of u and values of u determined at neighboring points by the finite-difference equations applicable to internal points in the regular mesh. Referring to Fig. 3-8 we might interpolate to obtain u_p by either

$$u_{p_1} = u_A + \frac{\Delta x}{\Delta x + \alpha \Delta x} (u_C - u_A)$$

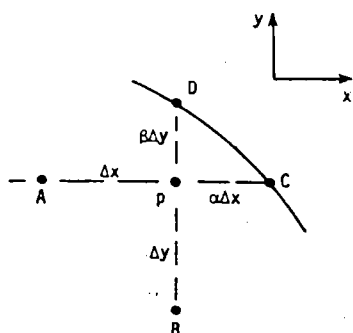


Figure 3-8 Irregular mesh caused by the shape of a boundary.

or

$$u_{p_1} = u_B + \frac{\Delta y}{\Delta y + \beta \Delta y} (u_D - u_B)$$

We could also use the average of the two interpolated values, $u_p = (u_{p_1} + u_{p_2})/2$.

3. Develop a finite-difference approximation to the governing PDE which is valid at internal points even when the mesh is irregular.

A difference expression valid on an arbitrary irregular mesh can be developed quite readily through the integral method by integrating about point x_0, y_0 and letting each integration interval extend halfway to a neighboring point. The mesh notation used is defined in Fig. 3-9. The starting point for the integral development of the difference expression is

$$\int_{y_0 - \Delta y_-/2}^{y_0 + \Delta y_+/2} \left(\int_{x_0 - \Delta x_-/2}^{x_0 + \Delta x_+/2} \frac{\partial^2 u}{\partial x^2} dx \right) dy + \int_{x_0 - \Delta x_-/2}^{x_0 + \Delta x_+/2} \left(\int_{y_0 - \Delta y_-/2}^{y_0 + \Delta y_+/2} \frac{\partial^2 u}{\partial y^2} dy \right) dx = 0 \quad (3-97)$$

Using the definition of an exact differential, this can be written as

$$\begin{aligned} & \int_{y_0 - \Delta y_-/2}^{y_0 + \Delta y_+/2} \left[\frac{\partial u}{\partial x} \left(x_0 + \frac{\Delta x_+}{2}, y \right) - \frac{\partial u}{\partial x} \left(x_0 - \frac{\Delta x_-}{2}, y \right) \right] dy \\ & + \int_{x_0 - \Delta x_-/2}^{x_0 + \Delta x_+/2} \left[\frac{\partial u}{\partial y} \left(x, y_0 + \frac{\Delta y_+}{2} \right) - \frac{\partial u}{\partial y} \left(x, y_0 - \frac{\Delta y_-}{2} \right) \right] dx = 0 \end{aligned}$$

Employing the mean-value theorem for integrals and using the central point of the interval to evaluate the integrands gives

$$\begin{aligned} & \left[\frac{\partial u}{\partial x} \left(x_0 + \frac{\Delta x_+}{2}, y_0 \right) - \frac{\partial u}{\partial x} \left(x_0 - \frac{\Delta x_-}{2}, y_0 \right) \right] \frac{\Delta y_+ + \Delta y_-}{2} \\ & + \left[\frac{\partial u}{\partial y} \left(x_0, y_0 + \frac{\Delta y_+}{2} \right) - \frac{\partial u}{\partial y} \left(x_0, y_0 - \frac{\Delta y_-}{2} \right) \right] \frac{\Delta x_+ + \Delta x_-}{2} = 0 \quad (3-98) \end{aligned}$$

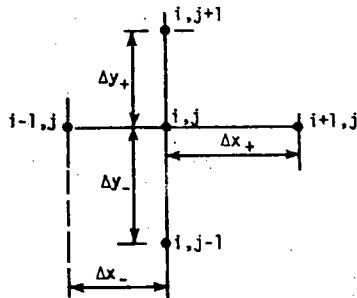


Figure 3-9 Notation for arbitrary irregular mesh.

Approximating these derivatives centrally as was done in Section 3-4.3 gives, after rearrangement, the following approximation for Laplace's equation in subscript notation:

$$\frac{2}{\Delta x_+ + \Delta x_-} \left(\frac{u_{i+1,j} - u_{i,j}}{\Delta x_+} - \frac{u_{i,j} - u_{i-1,j}}{\Delta x_-} \right) + \frac{2}{\Delta y_+ + \Delta y_-} \left(\frac{u_{i,j+1} - u_{i,j}}{\Delta y_+} - \frac{u_{i,j} - u_{i,j-1}}{\Delta y_-} \right) = 0 \quad (3-99)$$

When the above is specialized to the points near the irregular boundary depicted in Fig. 3-8, the derivative approximations appear as

$$\begin{aligned} \left(\frac{\partial^2 u}{\partial x^2} \right)_p &\simeq \frac{2}{\Delta x(1+\alpha)} \left(\frac{u_C - u_P}{\alpha \Delta x} - \frac{u_P - u_A}{\Delta x} \right) \\ \left(\frac{\partial^2 u}{\partial y^2} \right)_p &\simeq \frac{2}{\Delta y(1+\beta)} \left(\frac{u_D - u_P}{\beta \Delta y} - \frac{u_P - u_B}{\Delta y} \right) \end{aligned} \quad (3-100)$$

Equation (3-99) can also be developed by utilizing Taylor-series expansions. The unequal spacing makes the Taylor-series method noticeably more laborious whereas the integral approach proceeds for unequal spacing with no increase in effort. However, Taylor-series expansions about (i, j) should be substituted into Eq. (3-99) to establish the consistency and truncation error of these approximations. This will be left as an exercise for the reader. As a note of warning, we might recall that our second-derivative approximations on a regular mesh acquired second-order truncation errors only through fortuitous cancellation of terms from the forward and backward Taylor-series expansions. This cancellation will not occur if the mesh increments are unequal.

When approximately the same number of grid points are being used, we might expect this third method of treating irregular points near boundaries to be the most accurate because the governing PDE is being approximated at each internal point [not the case for method (2)] and the location of the boundary is not being altered as was done in method (1).

Irregular boundaries with derivative (Neumann) boundary conditions can also be treated, but the algebraic relationships become more complex. Some elementary examples illustrating the application of derivative boundary conditions to irregular boundaries can be found in Forsythe and Wasow (1960), James et al. (1967), and Allen (1954).

3-5.2 Irregular Mesh Not Caused by Shape of a Boundary

In some flow problems it is necessary to use especially small grid spacing near solid boundaries or shock waves where gradients in the dependent variables are especially large in order to obtain the desired accuracy or "resolution." In the interest of computational economy we strive to use a coarser grid away from these critical regions

so that we are forced to consider ways of treating an irregular or variable mesh in these problems. We can cite at least two ways to proceed:

1. We can employ a coordinate transformation so that unequal spacing in the original coordinate system becomes equal spacing in the new system but the PDE becomes altered somewhat in form. This will be described in detail later in our course of study.
2. The difference equation can be formulated in such a way that it remains valid when the spacing is irregular. Actually this is the same procedure as method (3) used above in connection with the irregular mesh caused by curved boundaries. Such a formulation for Laplace's equation is given as Eq. (3-99).

3-5.3 Concluding Remarks

The purpose of this section has been to introduce some of the problems and applicable solution procedures associated with irregular boundaries and unequal mesh spacing in general. Our coverage of the topic has been by no means complete. More advanced considerations on this topic tend to quickly become quite specialized and detailed. Good pedagogy suggests that we move on and see more of the forest before we spend any more time studying this tree. Some ideas on this topic will be developed further in connection with specific problems in fluid mechanics and heat transfer.

3-6 STABILITY CONSIDERATIONS

A finite-difference approximation to a partial differential equation may be consistent but the solution will not necessarily converge to the solution of the PDE. The Lax Equivalence theorem (see Section 3-3.4) states that a stable numerical method must also be used. We will address the question of stability in this section.

The problem of stability in numerical analysis is similar to the problem of stability encountered in a modern control system. The transfer function in a control system plays the role of the difference operator. Consider a marching problem in which initial values at time level n are known and values of the unknown at time level $n+1$ are required. The difference operator may be viewed as a "black box" which has a certain transfer function. A schematic representation would appear as shown in Fig. 3-10. The stability of such a system depends upon the operations performed by the black box on the input data. A control systems engineer would require that the transfer function have no poles in the right-half plane. Without this requirement, input signals would be falsely amplified and the output would be useless; in fact, it would grow without bound. Similarly, the way in which the difference operator alters the input information to produce the solution at the next time level is the central concern of stability analysis.

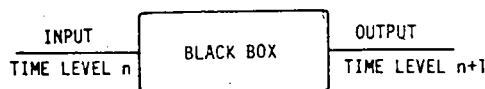


Figure 3-10 Schematic diagram of stability.

As a starting point for stability analysis, consider the simple explicit approximation to the heat equation

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{\alpha}{(\Delta x)^2} (u_{j+1}^n - 2u_j^n + u_{j-1}^n)$$

This may be solved for u_j^{n+1} to yield:

$$u_j^{n+1} = u_j^n + \alpha \frac{\Delta t}{(\Delta x)^2} (u_{j+1}^n - 2u_j^n + u_{j-1}^n) \quad (3-101)$$

Let the exact solution of this equation be denoted by D . This is the solution that would be obtained using a computer with infinite accuracy. Similarly, denote the numerical solution of Eq. (3-101) computed using a real machine with finite accuracy by N . If the analytical solution of the partial differential equation is A , then we may write

$$\text{Discretization error} = A - D$$

$$\text{Round-off error} = N - D$$

The question of stability of a numerical method examines the error growth while computations are being performed. O'Brien et al. (1951) pose the question of stability in the following manner:

1. Does the overall error due to round-off

$$\begin{bmatrix} \text{Grow} \\ \text{Not grow} \end{bmatrix} \Rightarrow \text{strong} \begin{bmatrix} \text{instability} \\ \text{stability} \end{bmatrix}$$

2. Does a single general round-off error

$$\begin{bmatrix} \text{Grow} \\ \text{Not grow} \end{bmatrix} \Rightarrow \text{weak} \begin{bmatrix} \text{instability} \\ \text{stability} \end{bmatrix}$$

The second question is the one most frequently answered because it can be treated much more easily from a practical point of view. The question of weak stability is usually answered by using a Fourier analysis. This method is also referred to as a von Neumann analysis. It is assumed that proof of weak stability using this method implies strong stability.

3-6.1 Fourier or von Neumann Analysis

Consider the finite-difference Eq. (3-101). Let ϵ represent the error in the numerical solution due to round-off errors. The numerical solution actually computed may be written

$$N = D + \epsilon \quad (3-102)$$

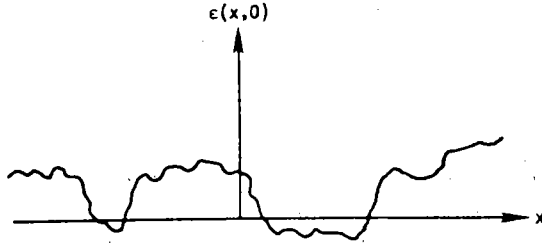


Figure 3-11 Initial error distribution.

This computed numerical solution must satisfy the difference equation. Substituting Eq. (3-102) into the difference Eq. (3-101) yields:

$$\frac{D_j^{n+1} + \epsilon_j^{n+1} - D_j^n - \epsilon_j^n}{\Delta t} = \alpha \left(\frac{D_{j+1}^n + \epsilon_{j+1}^n - 2D_j^n - 2\epsilon_j^n + D_{j-1}^n + \epsilon_{j-1}^n}{\Delta x^2} \right)$$

Since the exact solution D must satisfy the difference equation, the same is true of the error, i.e.,

$$\frac{\epsilon_j^{n+1} - \epsilon_j^n}{\Delta t} = \alpha \left(\frac{\epsilon_{j+1}^n - 2\epsilon_j^n + \epsilon_{j-1}^n}{\Delta x^2} \right) \quad (3-103)$$

In this case, the exact solution D and the error ϵ must both satisfy the same difference equation. This means that the numerical error and the exact numerical solution both possess the same growth property in time. Any perturbation of the input values at the n th time level will either be prevented from growing without bound for a stable system or will grow larger for an unstable system.

Consider a distribution of errors at any time in a mesh. We choose to view this distribution at time $t = 0$ for convenience. This error distribution is shown schematically in Fig. 3-11. We assume the error $\epsilon(x, t)$ can be written as a series of the form

$$\epsilon(x, t) = \sum_m b_m(t) e^{ik_m x} \quad (3-104)$$

where the period of the fundamental frequency ($m = 1$) is assumed to be $2L$. The interval of interest is taken to be L units in length and the wave number may be written

$$k_m = \frac{m\pi}{L} \quad m = 0, 1, 2, \dots, M$$

where M is the number of intervals Δx units long contained in length L . For instance, if an interval of length $2L$ is subdivided using five points, the value of M is 2 and the corresponding frequencies are

$$f_m = \frac{k_m}{2\pi} = \frac{m}{2L}$$

$$f_0 = 0 \quad m = 0$$

$$f_1 = \frac{1}{2L} \quad m = 1$$

$$f_2 = \frac{1}{L} \quad m = 2$$

The frequency measures the number of wave lengths in each $2L$ units of length. When $m = 0, f_0 = 0$ and this corresponds to a steady term in the assumed expansion.

Since the difference equation is linear, superposition may be used and we may examine the behavior of a single term of the series given in Eq. (3-104). Consider the term

$$\epsilon_m(x, t) = b_m(t) e^{ik_m x}$$

We seek solutions of the form

$$z^n e^{ik_m x}$$

which reduces to $e^{ik_m x}$ when $t = 0$ ($n = 0$). Toward this end, let

$$z = e^{a \Delta t}$$

so that

$$z^n = e^{an \Delta t} = e^{at}$$

and

$$\epsilon_m(x, t) = e^{at} e^{ik_m x} \quad (3-105)$$

where k_m is real but a may be complex.

If Eq. (3-105) is substituted into Eq. (3-103) we obtain

$$e^{a(r+\Delta t)} e^{ik_m x} - e^{at} e^{ik_m x} = r(e^{at} e^{ik_m(x+\Delta x)} - 2e^{at} e^{ik_m x} + e^{at} e^{ik_m(x-\Delta x)})$$

where $r = \alpha \Delta t / (\Delta x)^2$. If we divide by $e^{at} e^{ik_m x}$ and utilize the relation

$$\cos \beta = \frac{e^{i\beta} + e^{-i\beta}}{2}$$

the above expression becomes

$$e^{a \Delta t} = 1 + 2r(\cos \beta - 1)$$

where $\beta = k_m \Delta x$. Employing the trigonometric identity

$$\sin^2 \frac{\beta}{2} = \frac{1 - \cos \beta}{2}$$

the final expression is

$$e^{a \Delta t} = 1 - 4r \sin^2 \frac{\beta}{2} \quad (3-106)$$

Furthermore, since $\epsilon_j^{n+1} = e^{a \Delta t} \epsilon_j^n$ for each frequency present in the solution for the error, it is clear that if $|e^{a \Delta t}|$ is less than or equal to one, a general component of the error will not grow from one time step to the next. This requires that

$$\left| 1 - 4r \sin^2 \frac{\beta}{2} \right| \leq 1 \quad (3-107)$$

The factor $1 - 4r \sin^2 \beta/2$ (representing $\epsilon_j^{n+1}/\epsilon_j^n$) is called the *amplification factor* and will be denoted by G . Clearly, the influence of boundary conditions is not included in this analysis. In general, the Fourier stability analysis assumes that we have imposed periodic boundary conditions.

In evaluating the inequality Eq. (3-107), two possible cases must be considered:

1. Suppose $\left(1 - 4r \sin^2 \frac{\beta}{2}\right) > 0$ then $4r \sin^2 \frac{\beta}{2} > 0$
2. Suppose $\left(1 - 4r \sin^2 \frac{\beta}{2}\right) < 0$ then $4r \sin^2 \frac{\beta}{2} - 1 \leq 1$

The first condition is always satisfied since r is positive. The second inequality is satisfied only if $r \leq \frac{1}{2}$ which is the stability requirement for this method. This numerically places a constraint on the size of the time step relative to the size of the mesh spacing. The reason for the physically implausible temperatures calculated in the example at the end of Section 3-3.3 is now very clear. The step size Δt selected was too large by a factor of 2 and the solution began to diverge immediately. The stability of the calculation with $\alpha(\Delta t/\Delta x^2) = \frac{1}{2}$ can easily be verified. It should be noted that the amplification factor given by Eq. (3-106) could have been deduced by substituting a general form given by Eq. (3-104) into the difference equation. The proof is left as an exercise for the reader.

The application of the von Neumann or Fourier stability method is equally straightforward for hyperbolic equations. As an example, the first-order wave equation in one dimension is:

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0 \quad (3-108)$$

where c is the wave speed. This equation has one characteristic given by a solution of $x_t = c$. The solution of Eq. (3-108) is given by

$$u(x - ct) = \text{constant}$$

This solution requires the initial data prescribed at $t = 0$ to be propagated along the characteristics.

Lax (1954) proposed the following first-order method for solving equations of this form

$$u_j^{n+1} = \frac{u_{j+1}^n + u_{j-1}^n}{2} - c \frac{\Delta t}{\Delta x} \left(\frac{u_{j+1}^n - u_{j-1}^n}{2} \right) \quad (3-109)$$

The first term on the right-hand side represents an average value of the unknown at the previous time level while the second term is the difference form of the first derivative. If a term of the form

$$u_j^n = e^{at} e^{ik_m x}$$

is substituted into the difference equation, the amplification factor becomes

$$e^{\alpha \Delta t} = \cos \beta - i\nu \sin \beta$$

The stability requirement is

$$|\cos \beta - i\nu \sin \beta| \leq 1$$

where $\nu = c \Delta t / \Delta x$ is called the *Courant number*. Since the square of the absolute value of a complex number is the sum of the squares of the real and imaginary parts, the method is stable if

$$|\nu| \leq 1 \quad (3-110)$$

Again a conditional stability requirement must be placed on the time step and the spatial mesh spacing. This is called the Courant-Friedrichs-Lewy (CFL) condition and was discussed at length relative to the concepts of convergence and stability in an historically important paper by Courant et al. (1928). Some authorities consider this paper to be the starting point for the development of modern numerical methods for partial differential equations.

The amplification factor or growth factor for a particular numerical method depends upon mesh size and wave number or frequency. The amplification factor for the Lax finite-difference method may be written

$$G = \cos \beta - i\nu \sin \beta = |G| e^{i\phi} = \sqrt{\cos^2 \beta + \nu^2 \sin^2 \beta} e^{i \tan^{-1}(-\nu \tan \beta)} \quad (3-111)$$

where ϕ is the phase angle. Clearly the magnitude of G changes with Courant number ν and frequency parameter β . A good understanding of the amplification factor can be obtained from a polar plot. Figure 3-12 is a plot of Eq. (3-111) for several different

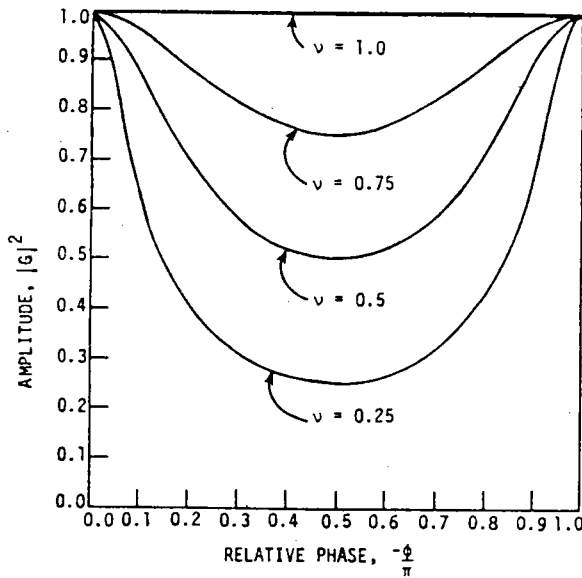


Figure 3-12 Amplitude-phase plot for the amplification factor of the Lax scheme.

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Courant numbers. Several interesting results can be deduced by a careful examination of this plot. The phase angle for the Lax method varies from 0 for the low frequencies to $-\pi$ for the high frequencies. This may be seen by computing the phase for both cases. For a Courant number of one, all frequency components are propagated without attenuation in the mesh. For Courant numbers less than one, the low and high frequency components are only mildly altered while the mid-range frequency signal content is severely attenuated. The phase is also shown and we can determine the phase error for any frequency from these curves.

A physical interpretation of the results provided by Eq. (3-110) for hyperbolic equations is important. Consider the second-order wave equation

$$u_{tt} - c^2 u_{xx} = 0 \quad (3-112)$$

This equation has characteristics

$$x + ct = \text{constant} = c_1$$

$$x - ct = \text{constant} = c_2$$

A solution at a point (x, t) depends upon data contained between the characteristics which intersect that point as sketched in Fig. 3-13. The analytic solution at (x, t) is influenced only by information contained between c_1 and c_2 .

The numerical stability requirement for many explicit numerical methods for solving hyperbolic PDE's is the CFL condition which is

$$\left| c \frac{\Delta t}{\Delta x} \right| \leq 1$$

This is the same as given in Eq. (3-110) and may be written as

$$\left(\frac{\Delta t}{\Delta x} \right)^2 \leq \frac{1}{c^2}$$

The characteristic slopes are given by $dt/dx = \pm 1/c$. The CFL condition requires that the analytic domain of influence lie within the numerical domain of influence. The numerical domain may include more than, but not less than, the analytical zone. Another interpretation is that the slope of the lines connecting $(j \pm 1, n)$ and $(j, n + 1)$ must be smaller in absolute value (flatter) than the characteristics. The CFL require-

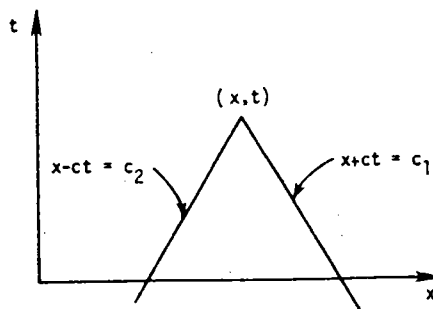


Figure 3-13 Characteristics of the second-order wave equation.

ment makes sense from a physical point of view. One would also expect the numerical solution to be degraded if too much unnecessary information is included by allowing $c(\Delta t/\Delta x)$ to become greatly different from unity. This is in fact what occurs numerically. The best results for hyperbolic systems using the most common explicit methods are obtained with Courant numbers near unity. This is consistent with our observations about attenuation associated with the Lax method as shown in Fig. 3-12.

Before we begin our study of stability for systems of equations, an example demonstrating the application of the von Neumann method to higher dimensional problems is in order.

Example 3-4 A solution of the two-dimensional heat equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} + \alpha \frac{\partial^2 u}{\partial y^2}$$

is desired using the simple explicit scheme. What is the stability requirement for the method?

The finite-difference equation for this problem is:

$$u_{j,k}^{n+1} = u_{j,k}^n + r_x(u_{j+1,k}^n - 2u_{j,k}^n + u_{j-1,k}^n) + r_y(u_{j,k+1}^n - 2u_{j,k}^n + u_{j,k-1}^n)$$

where $r_x = \alpha(\Delta t/\Delta x^2)$ and $r_y = \alpha(\Delta t/\Delta y^2)$. In this case a Fourier component of the form

$$u_{j,k}^n = e^{at} e^{ik_x x} e^{ik_y y}$$

is assumed. If $\beta_1 = k_x \Delta x$ and $\beta_2 = k_y \Delta y$ we obtain

$$e^{a\Delta t} = 1 + 2r_x(\cos \beta_1 - 1) + 2r_y(\cos \beta_2 - 1)$$

If the identity $\sin^2(\beta/2) = (1 - \cos \beta)/2$ is used, the amplification factor is

$$G = 1 - 4r_x \sin^2 \frac{\beta_1}{2} - 4r_y \sin^2 \frac{\beta_2}{2}$$

Thus for stability $|1 - 4r_x \sin^2(\beta_1/2) - 4r_y \sin^2(\beta_2/2)| \leq 1$, which is true only if $4r_x \sin^2 \beta/2 + 4r_y \sin^2 \beta/2 \leq 2$. The stability requirement is then $r_x + r_y \leq \frac{1}{2}$ or $\alpha \Delta t [1/(\Delta x)^2 + 1/(\Delta y)^2] \leq \frac{1}{2}$. This is similar to the analysis of the same method for the one-dimensional case but shows that the effective time step in two dimensions is reduced. This example was easily completed, but in general a stability analysis in more than a single space dimension and time is difficult. Frequently, the stability must be determined by computing the magnitude of the amplification factor for different values of r_x and r_y .

3-6.2 Stability Analysis for Systems of Equations

The previous discussion illustrates how the von Neumann analysis can be used to evaluate stability for a single equation. The basic idea used in this technique also provides a useful method of viewing stability for systems of equations. Systems of equa-

tions encountered in fluid mechanics and heat transfer can usually be written in the form

$$\frac{\partial \mathbf{E}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = 0 \quad (3-113)$$

where $\mathbf{E}$ and $\mathbf{F}$ are vectors and $\mathbf{F} = \mathbf{F}(\mathbf{E})$. In general this system of equations is nonlinear. In order to perform a linear stability analysis, we rewrite the system as

$$\frac{\partial \mathbf{E}}{\partial t} + \left[\frac{\partial \mathbf{F}}{\partial \mathbf{E}} \right] \frac{\partial \mathbf{E}}{\partial x} = 0 \quad (3-114)$$

or

$$\frac{\partial \mathbf{E}}{\partial t} + [\mathbf{A}] \frac{\partial \mathbf{E}}{\partial x} = 0$$

where $[\mathbf{A}]$ is the Jacobian matrix $[\partial \mathbf{F} / \partial \mathbf{E}]$. We locally linearize the system by holding $[\mathbf{A}]$ constant while the $\mathbf{E}$ vector is advanced through a single time step. A similar linearization is used for a single nonlinear equation permitting the application of the von Neumann method of the previous section.

For the sake of discussion, let us apply the Lax method to this system. The result is

$$\mathbf{E}_j^{n+1} = \frac{1}{2} \left([\mathbf{I}] + \frac{\Delta t}{\Delta x} [\mathbf{A}]^n \right) \mathbf{E}_{j-1}^n + \frac{1}{2} \left([\mathbf{I}] - \frac{\Delta t}{\Delta x} [\mathbf{A}]^n \right) \mathbf{E}_{j+1}^n \quad (3-115)$$

where the notation is as previously defined and $[\mathbf{I}]$ is the identity matrix. The stability of the difference equation can again be evaluated by applying the Fourier or von Neumann method. If a typical term of a Fourier series is substituted into Eq. (3-115), an expression is obtained

$$\mathbf{e}^{n+1}(k) = [\mathbf{G}(\Delta t, k)] \mathbf{e}^n(k) \quad (3-116)$$

where

$$[\mathbf{G}] = [\mathbf{I}] \cos \beta - i \frac{\Delta t}{\Delta x} [\mathbf{A}] \sin \beta \quad (3-117)$$

and $\mathbf{e}^n$ represents the Fourier coefficients of the typical term. The $[\mathbf{G}]$ matrix is called the amplification matrix. This matrix is now dependent upon step size and frequency or wave number, i.e., $[\mathbf{G}] = [\mathbf{G}(\Delta t, k)]$. For a stable finite-difference calculation, the largest eigenvalue of $[\mathbf{G}]$, $\sigma_{\max}$, must obey

$$|\sigma_{\max}| \leq 1 \quad (3-118)$$

This leads to the requirement that

$$\left| \lambda_{\max} \frac{\Delta t}{\Delta x} \right| \leq 1 \quad (3-119)$$

where $\lambda_{\max}$ is the largest eigenvalue of the $[\mathbf{A}]$ matrix, i.e., the Jacobian matrix of the system. A simple example to demonstrate this is of value.

Example 3-5 Determine the stability requirement necessary in solving the system of first-order equations

$$\frac{\partial u}{\partial t} + c \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} + c \frac{\partial u}{\partial x} = 0$$

using the Lax method. In this problem

$$\mathbf{E} = \begin{bmatrix} u \\ v \end{bmatrix}$$

and

$$\frac{\partial \mathbf{E}}{\partial t} + [\mathbf{A}] \frac{\partial \mathbf{E}}{\partial x} = 0$$

where

$$[\mathbf{A}] = \begin{bmatrix} 0 & c \\ c & 0 \end{bmatrix}$$

Thus, the maximum eigenvalue is c and the stability requirement is the usual CFL condition

$$\left| c \frac{\Delta t}{\Delta x} \right| \leq 1$$

It should be noted that the stability analysis presented above does not include the effect of boundary conditions even though a matrix notation for the system is used. The influence of boundary conditions is easily included for systems of difference equations.

Equation (3-116) shows that the stability of a finite-difference operator is related to the amplification matrix. We may also write Eq. (3-116) as

$$\mathbf{e}^{n+1}(k) = [G(\Delta t, k)]^n [\mathbf{e}^1(k)] \quad (3-120)$$

The stability condition (Richtmyer and Morton, 1967) requires that for some positive τ , the matrices $[G(\Delta t, k)]^n$ be uniformly bounded for

$$0 < \Delta t < \tau$$

$$0 \leq n \Delta t \leq T$$

for all k where T is the maximum time. This leads to the *von Neumann necessary condition* for stability which is

$$|\sigma_l(\Delta t, k)| \leq 1 + O(\Delta t) \quad (3-121)$$

for

$$0 < \Delta t < \tau$$

for each eigenvalue and wave number where σ_l represents the eigenvalues of $[G(\Delta t, k)]$.

The stability requirement used in previous examples required that the maximum eigenvalue have a modulus less than or equal to one. Clearly that requirement is more stringent than Eq. (3-118). The von Neumann necessary condition provides that local growth $c \Delta t$ can be acceptable and, in fact, must be possible in many physical problems. The classical example illustrating this point is the heat equation with a source term.

Example 3-6 Suppose we wish to solve the heat equation with a source term

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} + cu$$

using the simple explicit finite-difference method. If a Fourier stability analysis is performed, the amplification factor is

$$G = 1 - 4r \sin^2 \frac{\beta}{2} + c \Delta t$$

This shows that the solution of the difference equation may grow with time and still satisfy the von Neumann necessary condition. Physical insight must be used when the stability of a finite-difference method is investigated. One must recognize that for hyperbolic systems the strict condition less than or equal to one should be used. Hyperbolic equations are wave-like and do not possess solutions which increase exponentially with time.

We have investigated stability of various finite-difference methods by using the von Neumann method. If the influence of boundary conditions on stability is desired, we must use the *matrix method*. This is most easily demonstrated by applying the Lax method to solve the one-dimensional linear wave equation

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0$$

Assume that an array of m points is used to solve this problem and that the boundary conditions are periodic, i.e.,

$$u_{m+1}^n = u_1^n \quad (3-122)$$

If the Lax method is applied to this problem, a system of algebraic equations is generated which has the form

$$\mathbf{u}^{n+1} = [X] \mathbf{u}^n \quad (3-123)$$

where

$$\mathbf{u}^n = [u_1, u_2, \dots, u_m]^T \quad (3-124)$$

and

$$[X] = \begin{bmatrix} 0 & \frac{1-\nu}{2} & 0 & \cdot & \cdot & \cdot & \frac{1+\nu}{2} \\ \frac{1+\nu}{2} & 0 & \frac{1-\nu}{2} & \cdot & \cdot & \cdot & 0 \\ 0 & \frac{1+\nu}{2} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1-\nu}{2} & \cdot \\ \frac{1-\nu}{2} & 0 & \cdot & \cdot & \cdot & \frac{1+\nu}{2} & 0 \end{bmatrix} \quad (3-125)$$

The stability of the finite-difference calculation in Eq. (3-123) is governed by the eigenvalue structure of $[X]$. Since $[X]$ was formed assuming periodic boundary conditions, only the three diagonals noted in Eq. (3-125) and the two corner elements contribute to the calculation. This matrix is called an aperiodic matrix (Lomax et al., 1970). For matrices of the form

$$\begin{bmatrix} a_1 & a_2 & 0 & \cdot & \cdot & \cdot & a_0 \\ a_0 & a_1 & a_2 & 0 & \cdot & \cdot & 0 \\ 0 & a_0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_2 & 0 & \cdot & \cdot & \cdot & a_0 & a_1 \end{bmatrix} \quad (3-126)$$

the eigenvalues are given by

$$\lambda_j = a_1 + (a_0 + a_2) \cos \frac{2\pi}{m} (j-1) + i(a_0 - a_2) \sin \frac{2\pi}{m} (j-1)$$

$$j = 1, m$$

In this case a_0 , a_1 , and a_2 have the values

$$a_0 = \frac{1+\nu}{2} \quad a_1 = 0 \quad a_2 = \frac{1-\nu}{2}$$

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and the eigenvalues are

$$\lambda_j = \cos \frac{2\pi}{m} (j-1) + i\nu \sin \frac{2\pi}{m} (j-1) \quad (3-127)$$

The numerical method is thus stable if $|\nu| \leq 1$, i.e., if the CFL condition is satisfied. This shows that an analysis based upon the matrix operator associated with the Lax method yields the same stability requirement as previously derived for the simple wave equation. For periodic boundary conditions, the Fourier and matrix method yield virtually identical results. Another example is needed in order to demonstrate the effect of boundary conditions and the discreteness of the mesh.

As in the previous example, assume that the Lax method is used to solve the first-order linear wave equation. If a four-point mesh is used, special treatment is needed to enforce the boundary conditions at the first and fourth points. For simplicity we set u at the first point equal to a constant value for all time so the equation for the first point reads

$$u_1^{n+1} = u_1^n$$

Since we are computing a solution to the wave equation, the value of u_4 cannot be arbitrarily chosen. It must be consistent with the way the solution is propagated. We elect to set

$$u_4^{n+1} = u_3^n$$

which determines the boundary value from the interior solution. With this boundary condition treatment the $[X]$ matrix becomes

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1+\nu}{2} & 0 & \frac{1-\nu}{2} & 0 \\ 0 & \frac{1+\nu}{2} & 0 & \frac{1-\nu}{2} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The eigenvalues are easily computed and are

$$\lambda_1 = 1$$

$$\lambda_2 = 0$$

$$\lambda_{3,4} = \pm \frac{1}{2} \sqrt{(1-\nu)(3+\nu)}$$

Using the requirement that $|\lambda| \leq 1$ for stability, the restriction on ν is the usual CFL condition

$$|\nu| \leq 1$$

While the CFL condition was unaltered by the boundary conditions in this example, the stability bound is usually changed and this should be expected.

It is clear that the boundary conditions on the mesh are included in the matrix method. This means that the influence of boundary conditions on stability is automatically included if the matrix analysis is used. Unfortunately, a closed-form solution for the eigenvalues is usually not available for arbitrary end boundary conditions.

The treatment of stability presented in this section has included the Fourier (von Neumann) method and the matrix method of analysis. These two techniques are probably the most widely used to determine the stability of finite-difference schemes. Other methods of analyzing stability have been devised and are frequently very convenient to use. The works of Hirt (1968) and Warming and Hyett (1974) are typical of these techniques. A more comprehensive mathematical analysis of stability including many theorems and proofs is contained in the book by Richtmyer and Morton (1967).

5. 矩陣演算

周 宗 仁

5-1 矩陣之基本演算

1. 加算

2. 減算

3. 乘算

在一般情況下 $AB \neq BA$ 。例如若 A 為 $m \times n$ 次， B 為 $n \times \ell$ 次之矩陣， AB 相乘所得矩陣為 $m \times \ell$ 次，但 BA 之演算當 $m \neq \ell$ 時無法執行，即使可執行時，所得結果不一定會同於 AB 。

4. 轉置：將矩陣內相對應行列元素互換，通常以 A^T 表示。

5. 單位矩陣：對角元素為 1，其餘元素為 0 之正方形矩陣，一般以 I 表示。

6. 矩陣之基本變形

①兩行（列）之交換。

②對某一行（列）乘以不為 0 之常數。

③對某一行（列）以另一行（列）之常數倍相加。

7. 矩陣分割

若

$$A = \left(\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array} \right)$$

可將其分割成

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

5-2 聯立一次方程式之各種解法

- (1) Gauss - Jordan 法。
- (2) 三角分解法。
- (3) 平方根法。
- (4) Gauss - Seidel 法。
- (5) 最急下降法。
- (6) 逐次加速緩和法 (SOR 法)。
- (7) 直列型分離法。

$$\begin{bmatrix} A_1 & B_1 & 0 & 0 \\ C_2 & A_2 & B_2 & 0 \\ 0 & C_3 & A_3 & B_3 \\ & & \ddots & \ddots \\ & & & C_m & A_m \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ \vdots \\ D_{m-1} \\ D_m \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ \vdots \\ F_{m-1} \\ F_m \end{Bmatrix} \quad (1)$$

由第(1)式得

$$A_1 D_1 + B_1 D_2 = F_1 \quad (2)$$

$$C_2 D_2 + A_2 D_2 + B_2 D_3 = F_2 \quad (3)$$

在(2)及(3)式中，消去 D_1 ，並令

$$A_2 - C_2 A_1^{-1} B_1 = \bar{A}_2$$

$$F_2 - C_2 A_1^{-1} F_1 = \overline{F}_2$$

則可得

$$\overline{A}_2 D_2 + B_2 D_3 = \overline{F}_2$$

同理，上式再與原聯立一次方程式中之第3式消去 D_2 可得

$$\overline{A}_3 D_3 + B_3 D_4 = \overline{F}_3$$

⋮
⋮
⋮

最後得

$$\overline{A}_{m-1} D_{m-1} + B_{m-1} D_m = \overline{F}_{m-1}$$

$$\overline{A}_m D_m = \overline{F}_m$$

因此，首先可由上式求得 D_m ，再逐次分別求得 D_{m-1} ， D_{m-2} ，
.....， D_1 。

(8) 條件惡劣之矩陣

若矩陣內有部份元素之絕對值遠小於其他元素之絕對值，或趨近於0時，在數值上呈接近特異狀態（near singular），此時之解極易產生很大的誤差。例如

$$\begin{bmatrix} 1.0 & 10.0 \\ 10.0 & 100.1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 1.0 \\ 10.1 \end{bmatrix}$$

上式之解為

$$\begin{cases} x_1 = -9 \\ x_2 = 1 \end{cases}$$

現若將 (1, 1) 元素之值只改變 10^{-3} ，即

$$\begin{bmatrix} 0.999 & 10.0 \\ 10.0 & 100.1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 1.0 \\ 10.1 \end{bmatrix}$$

上式之解為

$$\begin{cases} x_1 = 9000 \\ x_2 = -899 \end{cases}$$

此現象稱為 ill-conditioned。對此類矩陣在處理上應特別加以注意。

5-3 分割法之應用

若正方形矩陣以 A 表示，而其逆矩陣為 B 時，欲求 B 時，可將 A、B 作下列分割而求之。

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

即

$$\begin{bmatrix} A_{11} B_{11} + A_{12} B_{21} & A_{11} B_{12} + A_{12} B_{22} \\ A_{21} B_{11} + A_{22} B_{21} & A_{21} B_{12} + A_{22} B_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

而得

$$A_{11} B_{11} + A_{12} B_{21} = I \quad (1)$$

$$A_{21} B_{11} + A_{22} B_{21} = 0 \quad (2)$$

$$A_{11} B_{12} + A_{12} B_{22} = 0 \quad (3)$$

$$A_{21} B_{12} + A_{22} B_{22} = I \quad (4)$$

由(2)，(3)式得

$$B_{21} = -A_{22}^{-1} A_{21} B_{11} \quad (5)$$

$$B_{12} = -A_{11}^{-1} A_{12} B_{22} \quad (6)$$

將(5), (6)式分別代入(1), (4)式得

$$B_{11} = (A_{11} - A_{12} A_{22}^{-1} A_{21})^{-1}$$

$$B_{22} = (A_{22} - A_{21} A_{11}^{-1} A_{12})^{-1}$$

將上式分別代入(5)及(6)式, 即可得 B_{21} 及 B_{12} 值。

5-4 複數聯立一次方程式之解法

若 $AX = B$ 中, A , B 及 X 之元素均為複數, 若將其以下列形式表示。

$$A = C + iD$$

$$B = F + iG$$

$$X = Y + iZ$$

但 C , D , F , G , Y 及 Z 為實矩陣, 則將其代入原式而得

$$\begin{bmatrix} C & -D \\ D & C \end{bmatrix} \begin{Bmatrix} Y \\ Z \end{Bmatrix} = \begin{Bmatrix} F \\ G \end{Bmatrix}$$

利用前節所述分割法可求得 A 之逆矩陣之實數部 R 及虛數部 Q 為

$$R = (C + DC^{-1}D)^{-1}$$

$$Q = -C^{-1}DR$$

因此得

$$X = (RF - QG) + i(QF + RG)$$

六、邊界條件及處理方法

劉 肖 孔

海域模式的一般原理頗為簡單，即是「基於流體力學的基本動力方程式，利用偏微分的數值法及計算機來解析潮、流等物理現象。」

然而，在微分學中，這是一組「最初」與「邊界」條件問題。如果沒有適當的邊界條件，則無法求解。非但如此，在一般性的高階數值模式中，百分之八十以上的電腦程式及邏輯均花費在處理邊界的「特殊情況」上。在「近似解法」方面愈精確的數值模式，其邊界條件的處理越複雜。模式的數值解法雖各有不同，但在邊界條件的處理原則上却大同小異。

模式所依據的「方程式系列」，每個方程式，均需有適當的邊界處理方法。以最簡單的連續方程式為例，如果我們假定海底的滲透量為零，則模式最底層的垂直流速向量（ W ）必設為零。而且在時空的變化上均需為零。否則即失去了質量上的「維原性」。

這個很顯然的條件，常稱為「底床」的連續性邊界條件（Kinematic boundary condition at the lower boundary），它是一個「必要條件」而非「充分條件」。因為其他的原因，亦可能失去模式的「質量維原性（Conservation of mass）」。例如在「差分」與「平均」的方法上，如果不能維持平衡，亦可能產生「不維原」的副作用。

如果一個海域數值模式在質量上不能「維原」，則在長時期的積分過程中，水會愈來愈少，在未考慮蒸發與滲透的情況下，這個模式自然不能被接受。除此之外，一個模式在「動量」、「能量」、「旋

量」(Vorticity)上亦應保持其「維原性」。除了模式本身的積分法則以外，最重要也最困難的工作之一即是邊界點的計算法則。本章即對此類問題提出討論。

6.1 固體邊界的處理

6.1.1 動量轉移之各種假定及其效應

在水平方向，如要滿足「連續性邊界條件」， x ， y 方向的流速分向量 u ， v 與固體邊界的垂直點，必設為零（見圖 6.1）。

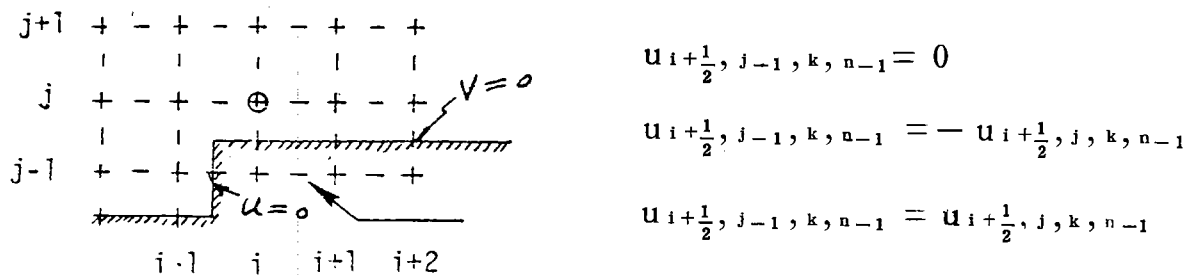


Fig 6.1 Location of closed boundary on the grid.

在固體邊界內的流速，如 $u_{i+\frac{1}{2}, j-\frac{1}{2}, k, n-1}$ （圖 6.1）。照常理，自然應設定為零。因為位於固體之內，自然無流速可言。這種假定，當模式格點間距離愈小的時候，與實際動量轉移的情況越接近。因為在理論上，該點與其「相對內點」的平均值，雖然位於固體的邊界點，却永遠無法等於零。

在這種情況下，唯一能滿足此種條件的方法即是將固體邊界內，流速項設定一「假想值」，其大小與其相對內點相同，方向則相反。如斯，則在固體邊界點的平均值即永遠等於零。在理論上永遠滿足其極限值。然而，此種方法亦有其缺點。即是由流場至邊界間的動量轉移量偏高。換言之，即是固體邊界對流場的「阻力影響」較實際情況為大。此種異常的效果亦可由邊界附近的「假想流速分佈」來解釋。就是：根據這樣的假設和計算法，由邊界點至內點間的流速分佈為

線性增加（呈三角形流速分佈）。然而實際情況却更接近於「對數分佈」。

所以，當計算格點愈大，雷諾係數愈高時，誤差愈大。那種情況下，應將外點的方向設定與內點方向相同，大小亦相同，邊界效應，即變為零。

在實際應用上，第一種方法不失為一種簡單而精確的折衷辦法。

6.1.2 溫度與鹽度的邊界值處理方法

在計算溫度與鹽度時，如有固體邊界點涉及，則需注意溫度與鹽度平衡方程式中之各項計算，是否在「平均」及「差分」過程中有「鹽份及熱度」外溢的現象。否則即會失去模式在溫度與鹽度的維原性。常需檢查的項目為「對流項（如 $u \frac{\partial S}{\partial x}$ ， $u \frac{\partial T}{\partial x}$ ， $u \frac{\partial P}{\partial x}$ 等等）」與「擴散項（如 $D \times \frac{\partial^2 S}{\partial x^2}$ 等）」。

「對流項」的處理方法較為簡單，因為在 Kinematic boundary condition 中，已將各固體邊界點的流速，設定為零，因此其「副作用」即將固體邊界點的對流項亦乘之以零。

然而擴散項却需特別處理。一般的方法是在寫電腦程式時，使用一種對稱的 Algorithm，使一般性的「擴散計算法則」中包含一個變數項，在任何固體邊界附近時，讓此變數有相互抵消為零的效果。因此對當地固體邊界，即無擴散的作用。

以上這種方法，需用實際電腦程式來解釋，較易明瞭（筆者以透明片討論）。因為擴散項為二次導來式，所以涉及計算點很多，故而非常不容易檢驗，尤其是「高階近似值解法」，則涉及的計算點更多。在四週環海的島嶼附近常常發生問題。

筆者過去常採用「實驗法」來檢驗模式在此方面的正確性。即是在計算進行過程中，將全域中的「溫度」、「鹽度」、「水位」、「

內格點動能」及「污染物」的總合列出，視其變化。如果僅有 truncation error，則程式即有「正確」的可能。

6.2 開口邊界之處理方法

6.2.1 邊界水位之估算及輸入法

天文潮是一般海域模式所必需的開口邊界條件，如果模式所包含的區域，其邊界位於深海，則理論天文潮在該地區的變化不大，即可採用。如果模式的邊界位於淺海地區，一般採用兩種方法，即為「實測法」與「大小模式內插法」。筆者依實例，加以說明。

6.2.1.1 淺海模式邊界實測法

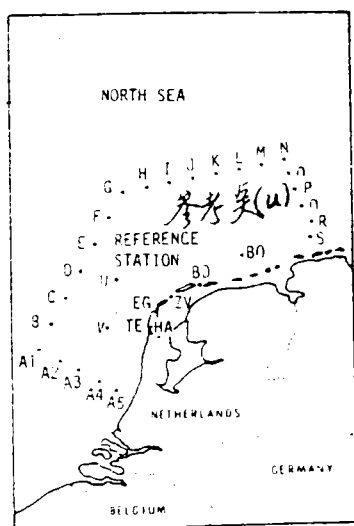
荷蘭位於北海之南端，其外海均為淺水區域（見圖 6.2），所以「淺水化」以後的天文潮在該地區的變化頗為複雜，因此採用實測法較為適當。如圖 6.2 所示，所有邊界點均為臨時性「壓力式」水位站。其中 u 站為「一級固定」測站，有長期的氣象及其他重要資料，所以被選定為參考點。筆者以參考點為中心，使用「交錯法」，計算該點與所有邊界點之間的「轉移函數」（圖 6.2 之右側插圖）。

使用實測資料，筆者共求得三十一組轉移函數群（Liu et al, 1974）。因此今後在該地區，如建立其他格點不同大小的模式，僅以內插法，（每一個週期）即可求得所需的開口邊界條件。

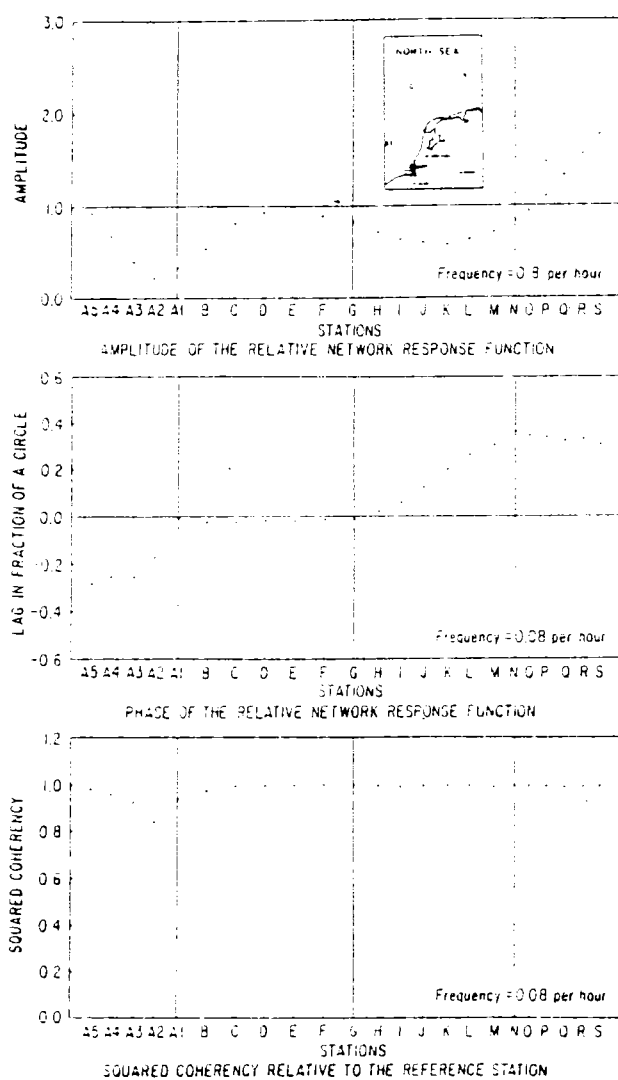
利用一級測站（參考點）向模式邊界預測水位，經過實地驗證結果，其誤差值大約在百分之一點四左右。（見圖 6.3）。由誤差圖中可以看出，其中多半均為任意誤差，系統誤差量並非很大。

6.2.1.2 大小模式內插法（Nested model）

實測法雖然很理想，但是花費很大，荷蘭對水利工程至為重視，因此在儀器的購置、船舶的使用以及人員的調度方面均係大手筆



Relative location of tide gauges (from Liu et al., 1974).



潮位

潮的延角

相關函數

FIG. 32. Amplitude, phase, and squared coherency of the computed frequency response function for the semidiurnal tidal component (frequency = 0.80 per hr) (from Liu et al., 1974)

圖 6.2 荷蘭水政部爲了建立該國的外海數值模式，曾在模式的外邊界放置廿八個測站。上圖爲筆者利用反應函數法來推算該模式的邊界潮位。中國海域模式筆者選訂蘭嶼爲參考點，用類似方法推算中國海域及台灣海域內各點的天文潮。

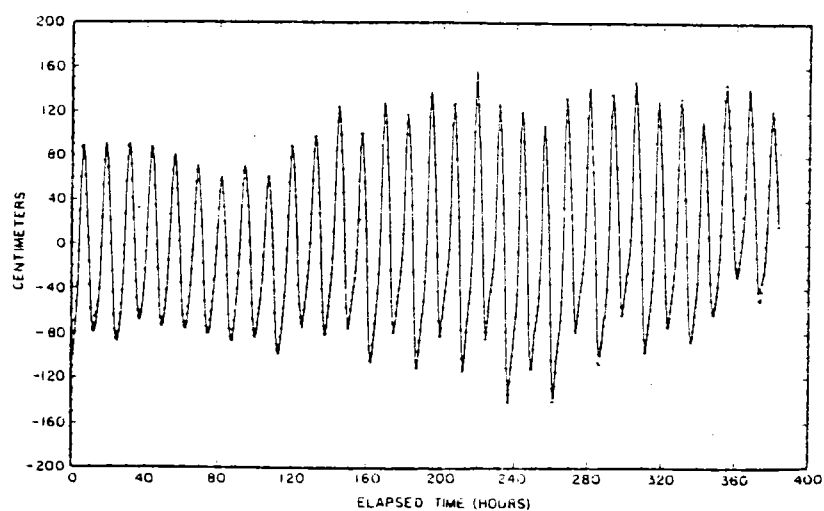


FIG. 6.3. Predicted water level at a boundary station from the reference station for a model of the southern Dutch coast. (from Liu *et al.*, 1974).

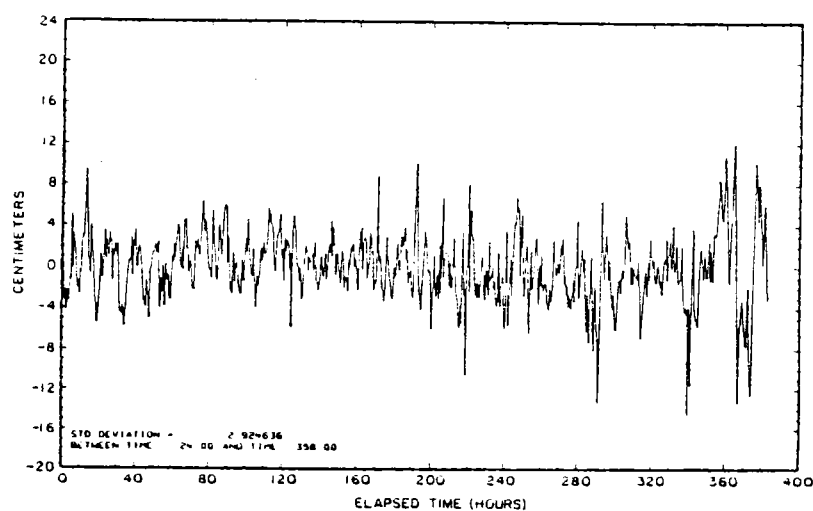


FIG. 6.4. Difference between the predicted and observed water levels at a boundary station for a model of the southern Dutch coast. (from Liu *et al.*, 1974).

圖 6.3 利用參考點（一級測站）向模式邊界預測水位，經過實地驗證結果，其誤差值大約在百分之一點四左右。

作法。其中尚有另外一個重要因素，即是在十餘年前，模式的技術未能有目前的進展，所以在同樣的情況下，中國海域模式所採用的方法較前者更為經濟有效。

爲了要計算台灣附近淺水天文潮的推算系數，則必須先建立中國海域數值模式，以中國海大陸棚邊界的深水大洋天文潮爲開口邊界條件。此種方法的細節，請參閱行政院科技顧問組的「中國海域三度空間數值模式」。(圖 6.4)。

6.2.2 邊界流速之估算及輸入法

依照類似的原理，模式所包含的近岸區域如有長期性的海流存在，則必須參考實測資料。否則使用大小模式內插法，將岸區的模式包納在更大的模式之內。目前最大的太平洋海流模式，其解析度約爲一度見方，使用平均地衡風及剪力資料輸入，「起動」(Spin-up)兩年之後，所計算的黑潮總流量約在每秒四千萬立方公尺左右，與實測量雖很接近，然而在其流速分佈的解析度方面，尚有改進的餘地。

6.2.3 邊界鄰點的流速特別計算法

模式邊界點以外的格點，因係外點，所以不包括在計算項目之內，該處所有向量(如流速)及非向量(如塩度)均爲未知數。模式邊界點計算時，即不能使用該種變數。以計算圖 6.5 中之 u 爲例，僅能包括「壓力梯度」及垂直剪力兩項，其他非線性項如「對流」及「擴散」等均需包括模式外點的未知數。因此數值模式在邊界附近的精確度比內部地區要差。

上述爲該點 u 向量的「線性」計算方法，同一地點， v 向量即可包括對流項，地球自轉效應及 y 方向的「擴散項」。

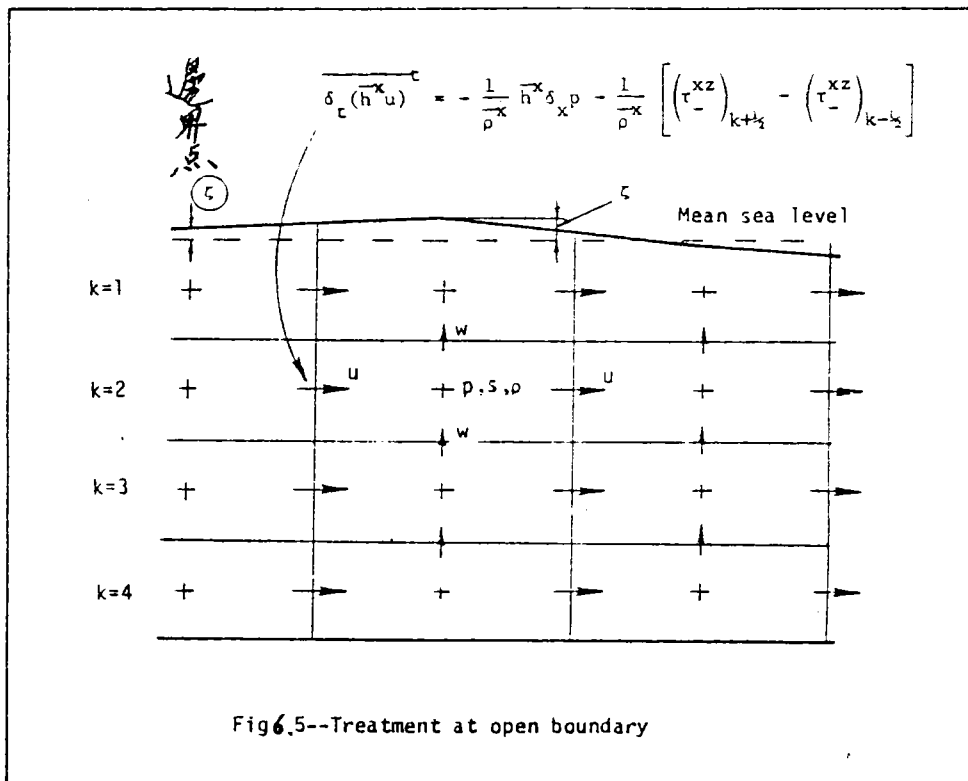
$$\begin{aligned} \overline{\delta_t (\bar{h}^y \bar{v})}^t &= -\delta_y (\bar{h}^y \bar{v}^y \bar{v}^y) - \bar{h}^y \delta_z (\bar{v}^z \bar{w}^y) \\ &\quad - f \bar{h}^y \bar{u}^{xy} - \frac{1}{\rho^y} \bar{h}^y \delta_y p - \frac{1}{\rho^y} [(\tau_{yz}^y)]_{k+\frac{1}{2}} \end{aligned}$$

$$- \left(\tau_{yz}^- \right)_{k-\frac{1}{2}} \Big] + \delta_y \left\{ \overline{h^y} A_y^y \delta_y v \right\}_-$$

$$\text{at } i, j + \frac{1}{2}, k, n$$

爲便於解釋，在此我們僅以簡單的有限差分法的 Staggered-grid 來舉例說明。在原則上，愈是高階的近似解法，雖然在內部精確度稍高，在邊界點附近，由於涉及點數多，所以處理愈加複雜。

Arakawa 氏之近似解法卽爲一例。



在建立程式之初，如果在開口邊界處即產生迴旋流，應注意檢驗程式。因該地的非線性迴流項不包括在計算項目之內。

6.2.4 邊界鄰點「溫度」、「塩度」及「反覆流」之計算方法

根據前節的同樣原理，在邊界鄰點無法計算「熱度」、「塩度」、「污染物」及「紊流動能」的擴散效應。在該地，僅能計算由內點

至外點的對流效應。以溫度與鹽度為例：（亦請參閱圖 6.5）

$$S_{i,j,k,n+1}$$

$$= S_{i,j,k,n} - (S_{i,j,k,n} - S_{i+1,j,k,n}) \frac{\Delta t}{\Delta x} \cdot u_{i+\frac{1}{2},j,k,n}$$

$$T_{i,j,k,n+1}$$

$$= T_{i,j,k,n} - (T_{i,j,k,n} - T_{i+1,j,k,n}) \frac{\Delta t}{\Delta x} \cdot u_{i+\frac{1}{2},j,k,n}$$

當海流流向為由內至外時，所需的各點數值均為已知數，故無問題。然而當「流向」反轉，由外向內迴流時，則必需要估算迴流時間，以便將該地之溫度、鹽度之「回復值」重新設定。在本文後之附錄中有電腦程式實例，可用以舉例說明，提出討論。

6.2.5 注入流所產生之「非連續性」處理方法

當模式的邊界點為電廠出水口或是衛生下水道出水口時，由於當地溫度及大腸菌濃度超出附近水域很多時，常易產生 discontinuity 的非連續性作用（見 Liu & Leendertse, 1978 中之討論）。

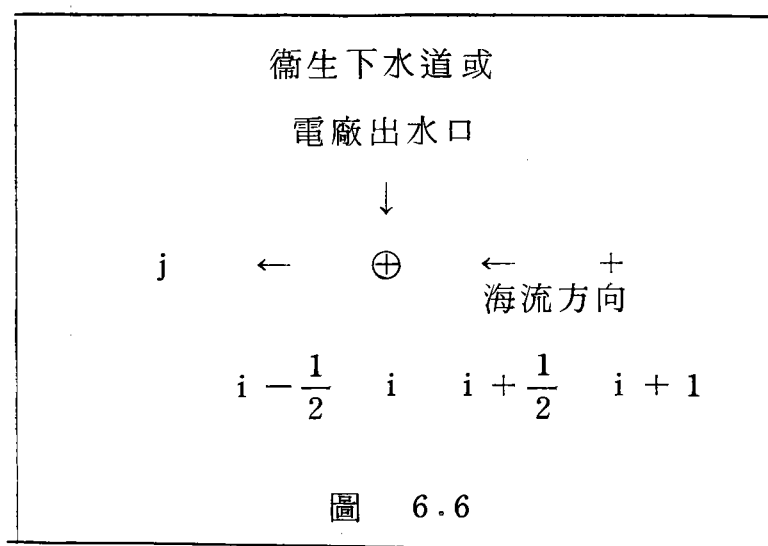


圖 6.6

以大腸菌的濃度計算為例，一般都市的污水大腸菌含量約為 $10^6 - 10^7$ MPN/ml，高出附近海水一萬倍之多。所以當下游海流或河流與注入流成垂直方向時，由於對流項的「差分與平均」過程（

例如 $\bar{h} \cdot u \bar{T}$)，使海域與河流上游點的大腸菌濃度（或溫度）計算結果呈「負值」的不合理現象。注入流的溫度愈高，海流流速越大，呈負值的趨勢亦愈。

其實，這只是個「顯然」現象而已，如將附近鄰點之濃度總和加以檢驗，並無誤差。

有幾種「上游差分」方法（upwind differencing）可以使當地的濃度平滑。但以筆者管見，此類方法僅係局部的 Smoothing，反使檢驗過程不易進行。不如聽其自然，如為美化起見，最常用的方法為 exponential weighting，在原則上即是 Smoothing 與注入點的距離成反比。

6.3 由實測資料，使用「交絡法」訂定模式邊界條件的實例

訂立海域模式邊界條件最理想的方法是實測法。然而，有了實測資料以後，尚需從事頗多的分析工作，其實測結果方能有效使用。

分析過程上經常涉及兩個問題，一個是部份資料不全的問題，第二個是實測站位與計算格點不能完全吻合的問題。兩者均涉及「時」、「地」的內插計算。

十二年以前，荷蘭水政部為了訂立外海數值模式的邊界條件，曾從事大規模的外海實測工作。為了解決上述的兩項問題，筆者曾使用「交絡法」（Cross-spectral analysis），計算模式所需的邊界條件，其中細節，請見附錄拙著。其他兩位著者，一為模式界先進權威—林德西博士，另一位係荷蘭水政部，主管數值模式及實測工作的協調人。

附錄中另一實例為美國紐約水質模式的邊界條件訂立方法。

附錄 6A

ESTIMATION OF BOUNDARY CONDITIONS FOR COASTAL MODELS

by

S. K. Liu,¹ J. J. Leendertse,¹ and J. Voogt²

ABSTRACT

In this study, frequency response and transfer function techniques are used together with cross-spectral and fast Fourier transform methods to determine the proper boundary values for computing the flow field of a coastal sea. Tide data containing considerable perturbations from swell and meteorological disturbances are analyzed.

In computing the frequency response estimates, the effect of noise in the input is treated by a cancelling technique and by the choice of a reference station to evaluate the interdependencies among the other stations at the boundary. The usefulness of the network frequency response function is threefold: (1) future conditions can be simulated using observed water levels at any single location, (2) boundary information for models of different grid size can be obtained by interpolation, and (3) missing data at a given location can be estimated optimally using data at neighboring stations and the network response function. The paper discusses an example of such an application, the determination of a boundary of a two-dimensional model of Jamaica Bay, New York City, U.S.A.

INTRODUCTION

One of the major difficulties in coastal and estuarine hydrodynamic computation is obtaining good boundary information. For example, the computation requires the time histories of water levels at open boundaries as one of the major input forcing functions from which is derived the internal flow field. Field measurements at boundaries, as well as within the prototype, are also needed during various phases of model development and adjustment. However, such field data often contain noise generated by instruments, meteorological disturbances, or short-period waves. Often, part of the records of the critical period may even be missing. This paper deals mainly with problems such as these encountered frequently in hydrodynamic and water quality modeling. The analyses used are the estimation of network frequency response function, cross-spectral computation, noise cancellation, and numerical convolution.

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ESTIMATION OF NETWORK FREQUENCY RESPONSE FUNCTIONS

The frequency response function $H(f)$, as its name indicates, describes the amplitude and phase relationship between one fluctuation with a certain frequency f as input and the resulting fluctuation as output.

In determining network frequency response relationships, the statistical approach using cross-spectral estimates is used instead of the classic method employing the deterministic Fourier or Laplace transforms, because the sampling variability, confidence limits, and the phase of the frequency response function can only be estimated using cross-spectral procedures.

A description of the computational method used in this paper is presented by Liu, (1) and by Leendertse and Liu. (2,3) Reference is also made to the handbook of Jenkins and Watts (4) and the thesis of Goodman (5) on this subject. A brief outline of the computational method is given below.

With respect to the interdependency (or the lack of it) between two random time series, x_t and y_t , we see that if the random processes are jointly stationary such that the joint distribution depends only on time differences, then the degree of interdependency can be measured by the cross-covariance function. In discrete time this is defined as

$$\hat{\gamma}_{xy}(k) = \frac{1}{n-k} \sum_{t=1}^{n-k} (x_t - \bar{x}) (y_{t+k} - \bar{y}) \quad (1)$$

$$\hat{\gamma}_{yx}(k) = \frac{1}{n-k} \sum_{t=1}^{n-k} (y_t - \bar{y}) (x_{t-k} - \bar{x})$$

for $k = 0, 1, 2, \dots, m$, where m is the largest time lag chosen, n the total number of data points, and $\bar{x}, \bar{y}$ the mean values of the series $\{x\}, \{y\}$.

The relationship between these two stochastic processes can also be expressed by the integral equation

$$y(t) - \mu = \int_0^{\infty} h(\tau) [X(t - \tau) - \mu_x] d\tau + N(t) \quad (2)$$

where $h(\tau)$ is the impulse response function, μ_x, μ_y are the mean values of the two processes, and $N(t)$ is the uncorrelated error term.

Wiener (6) showed that the optimal estimates of $h(\tau)$ should satisfy the following integral equation:

$$\gamma_{xy}(t) = \int_0^{\infty} \hat{h}(\tau) \gamma_{xx}(t - \tau) d\tau \quad (3)$$

The solution to Eq. (3) may be obtained by Fourier transformation. Because covariance function and spectral density function form a Fourier transform pair, thus

$$P_{xy}(f) = H(f) \cdot P_{xx}(f) \quad \text{or} \quad H(f) = P_{xy}(f)/P_{xx}(f) \quad (4)$$

where $P_{xy}(f)$ is the cross-spectrum between $x(t)$ and $y(t)$, $P_{xx}(f)$ is the auto-spectrum of $x(t)$, and the complex valued function $H(f)$ is the frequency response function. The possibility of using a statistical approach using spectral densities to give the frequency response was suggested by Lee.⁽⁷⁾ However, it was later⁽⁵⁾ that a quantitative basis for applying the method with finite sample records corrupted by measurement noise became available.

For computing the cross-spectral density function estimate by a numerical Fourier transform, the even and odd parts of the cross-covariance function are determined by

$$\hat{A}(k) = \frac{1}{2}[\gamma_{xy}(k) + \gamma_{yx}(k)] \quad (5)$$

$$\hat{B}(k) = \frac{1}{2}[\gamma_{xy}(k) - \gamma_{yx}(k)] \quad (6)$$

from which the co-spectral density function is estimated:

$$\hat{C}_{xy}(f) = 2\tau \left[\hat{A}(0) + 2 \sum_{k=1}^{k=m-1} \hat{A}(k) \cos(2\pi f k \tau) + \hat{A}(m) \cos(2\pi f m \tau) \right] \quad (7)$$

The quadrature spectral density function is estimated by

$$\hat{Q}_{xy}(f) = 2\tau \left[2 \sum_{k=1}^{k=m-1} \hat{B}(k) \sin(2\pi f k \tau) + \hat{B}(m) \sin(2\pi f m \tau) \right] \quad (8)$$

The spectral density functions of input and output are determined in a similar manner. If, in Eq. (1), the output series is replaced by the input series, we obtain the auto-covariance function of the input,

$$\gamma_{xx}(k) = \frac{1}{n-k} \sum_{t=1}^{t=n-k} (x_t - \bar{x})(x_{t+k} - \bar{x}) \quad (9)$$

from which the input spectral density function is determined:

$$\hat{P}_{xx}(k) = 2\tau \left[\gamma_{xx}(0) + 2 \sum_{k=1}^{k=m-1} \gamma_{xx}(k) \cos(2\pi f k \tau) + \gamma_{xx}(m) \cos(2\pi f m \tau) \right] \quad (10)$$

The output spectral density function is determined similarly.

The time interval τ used in the analysis influences the highest frequency that can be determined by the analysis method. At least two samples per cycle are required to define a frequency component in a data set; thus the highest frequency determined is

$$f_c = \frac{1}{2\tau} \quad (11)$$

This frequency is the so-called Nyquist frequency. If higher frequencies are present in the data, these are aliased as lower frequencies.

The spectral density functions are determined for particular frequencies f . These frequencies are calculated only at the special discrete frequencies of harmonic number k , where

$$f = \frac{kf_c}{m} \quad k = 0, 1, 2, 3 \dots m \quad (12)$$

The total number of discrete frequencies determined depends on the maximum lag $m\tau$.

For each spectral function, the discrete value found is a kind of average value in a certain range or band. The bandwidth for the computations is

$$B = \frac{1}{m\tau} \quad (13)$$

One would tend to determine the spectral functions with small bandwidth -- thus in much detail -- by choosing a large value for m . Unfortunately, this considerably affects the accuracy of the result.

It should be understood that the analysis method gives estimates of the function only. Since we are dealing with data that is not deterministic, each sample record used for analysis differs from another and the results obtained from these records will also differ somewhat.

The estimates of spectral density functions described above are so-called "raw" estimates, which have certain undesirable properties. If a strong periodic component is present, the analysis may show small positive and negative values in the frequency bands adjacent to that in which the periodic component is present. This phenomenon is called "leakage," and the negative values it produces are particularly bothersome. To counter it, frequency smoothing is used, by which the estimate at a particular frequency is computed as a weighted average of the particular frequency and the adjacent frequency.

For example, the smoothed co-spectral density function can be taken as

$$\bar{C}_{xy}(f) = \bar{C}_{xy}\left(\frac{kf_c}{m}\right) = .25\hat{C}_{xy}\left(\frac{(k-1)f_c}{m}\right) + .5\hat{C}_{xy}\left(\frac{kf_c}{m}\right) + .25\hat{C}_{xy}\left(\frac{(k+1)f_c}{m}\right) \quad (14)$$

This frequency smoothing is called "hanning," which is equivalent to the Tukey lag window.⁽⁴⁾ Other methods of minimizing the effect of leakage are available,^(1,4) but these are not applied in this investigation.

For the absolute value of the smoothed cross-spectral density we obtain the following estimate:

$$|\hat{P}_{xy}(f)| = \left(\hat{C}_{xy}(f)^2 + \hat{Q}_{xy}(f)^2 \right)^{1/2} \quad (15)$$

Subsequently, the estimated amplitude of the frequency response function is

$$|\hat{H}(f)| = \hat{P}_{xy}(f) / \hat{P}_{xx}(f) = \hat{A}_{xy}(f) \quad (16)$$

and its estimated phase spectrum is

$$\hat{\phi}(f) = \tan^{-1} \left[\hat{Q}_{xy}(f) / \hat{C}_{xy}(f) \right] \quad (17)$$

The frequency response function thus obtained is an optimal estimate (in a least square sense), assuming that the system is linear. Even if we assume that we have a linear system whose input $\{x\}$ may be measured exactly, the output may still contain measurement errors. In the case analyzed here, the output may be influenced by wind and system nonlinearities. The measured output then contains the transformed input signal plus measurement noise, etc. (in our case noise caused by wind and nonlinearities).

It is now possible to introduce a measure of the linear relation between the two series, called the coherency function $\Omega_{xy}(f)$. The squared coherency is estimated to be

$$\Omega_{xy}^2(f) = \frac{|\hat{P}_{xy}(f)|^2}{\hat{P}_{xx}(f) \hat{P}_{yy}(f)} \quad (18)$$

If the system is completely linear, the squared coherency is unity; if the two time series are completely uncorrelated, then the coherency would be zero. If the coherency is less than unity but greater than zero, then there is noise in the measurements, the system is not linear, or the output $\{y\}$ of the system is due to an input $\{x\}$ as well as other inputs.

In determining the behavior of a system it is often useful to see how the noise is distributed over the frequency range. The estimated spectral density function of the noise is expressed by

$$\hat{P}_{x\Delta y}(f) = [1 - \Omega_{xy}^2(f)] \hat{P}_{yy}(f) \quad (19)$$

In a strict sense, when the cross-spectral estimates $\hat{P}_{xy}(f)$ are used to determine the frequency response function, the measurement noise at the input will cause the frequency response function to be underestimated, as can be seen from the equation

$$\hat{H}(f) = \hat{P}_{xy}(f) / \hat{P}_{xx}(f) = \hat{P}_{xy}(f) / [\hat{P}_{xx}(f) + \hat{P}_{nnx}(f)] < H(f) \quad (20)$$

in which $P_{xx}(f)$ is the true value of input spectra and $P_{nnx}(f)$ is the spectrum of extraneous noise components in the input. Notice that the uncorrelated noise at the output does not cause bias.

The amount of measurement noise and the error due to it can be estimated by computing the amplitude function forward and backward between two sets of records (i.e., switching x series and y series in the computational sequence). By going from y to x, the noise in y would cause the transfer from y to x to be underestimated, which is equivalent to the overestimation of x in the relative amplification factor. However, by going from x to y, the noise in y no longer influences the value of backward transfer, but the noise in x is now the important factor. In each direction the transfer function is underestimated. Therefore, the best estimate of the transfer function from y to x is

$$\frac{1}{2}[1/A_{yx}(f) + A_{xy}(f)] \quad (21)$$

The error would be cancelled out if the random measurement noise level in both records were about the same.

The computed coherency $\hat{\Omega}_{xy}^2(f)$ also contains some bias if the phase difference between two stations is appreciable. This bias can be reduced by a process called alignment (see Ref. 4 for details), using the peak of the cross-covariance as a guide to make the required shift.

DATA ANALYSIS

One of the applications of the network frequency response analysis is estimating open boundary conditions for a two-dimensional mathematical model of the Netherlands coast in the North Sea (Fig. 1). In order to determine the proper boundary conditions for the model, 29 bottom pressure recorders were installed by the Netherlands Rijkswaterstaat during the months of May and June 1971. Hourly water level data were first analyzed without astronomical prejudices, thus allowing for all possible frequencies and their higher harmonics that were present.

The frequency domain mapping was carried out using arbitrary-radix algorithms of fast Fourier transforms. During the transformation, Tukey's⁽⁸⁾ interim data taper window was applied to eliminate leakage from the peaks. The Fourier line spectra for stations U (reference station) and A₁ are shown in Fig. 2. The contribution from the meteorological disturbances, located in the frequency range below 0.04 per hour, and the higher harmonic components induced by the diurnal-semidiurnal components can be noted in the graphs.

Frequency response analyses and cross-spectral computations were then carried out between the reference station U and the 21 stations located at the model open boundary. The graphic results from a typical analysis are shown in Fig. 3. In the top row the adjusted bottom pressures at station A are shown. The computed spectra at station A are also shown. It will be noted that the higher harmonics of the lunar component, which are the quarter-diurnal tide (M_4 at $f = .16 \text{ hr}^{-1}$) and the sixth-diurnal

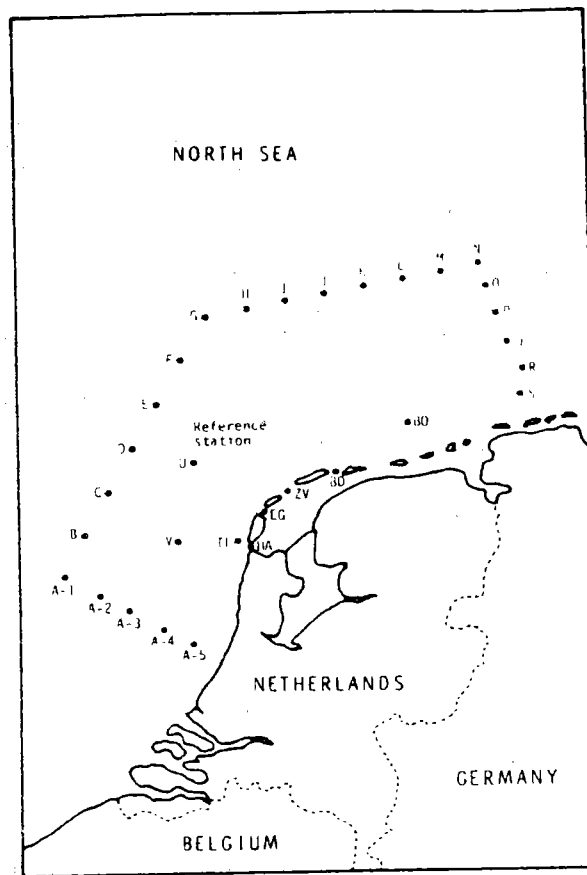


Figure 1 Relative location of tide gauges

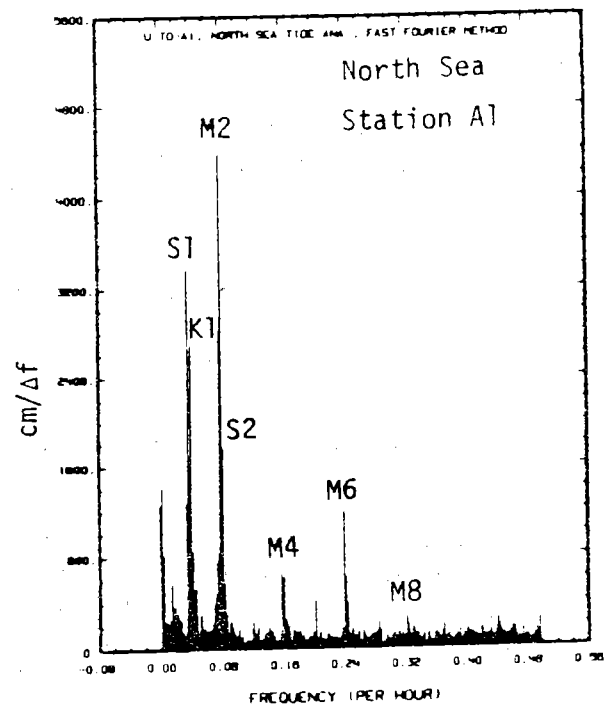
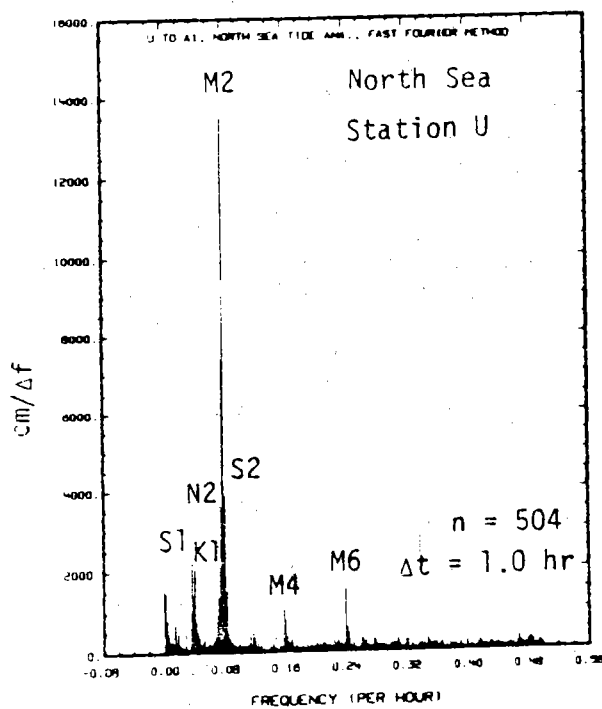


Figure 2 Analysis of tidal components and other oscillations in the observed data using fast Fourier transform and cosine data tapering filter

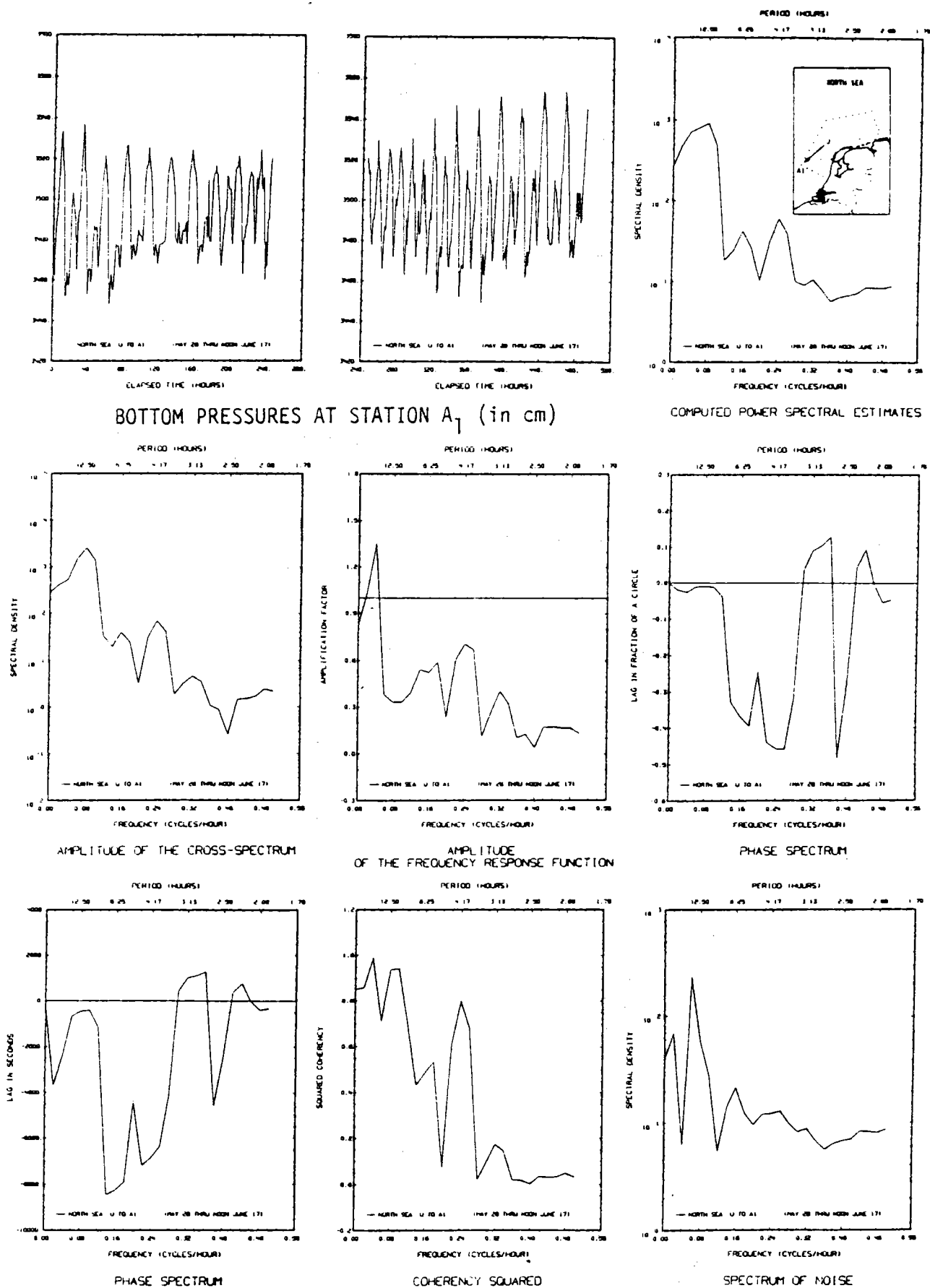


Figure 3 Frequency response analysis between Station U (reference station) and Station A1

tide (M_6 at $f = .24 \text{ hr}^{-1}$), contain much less energy than the semi-diurnal tide at $f = .08 \text{ hr}^{-1}$. This can also be found in the cross-spectrum between the reference station U and A_1 in the first graph in the second row of Fig. 3. The amplitude of the frequency response estimate between U and A_1 shown in the middle graph, indicates that the amplification factor is 1.36 for the diurnal tide ($f = 0.04$), but only 0.3 for the semi-diurnal tide ($f = 0.08$). The phase of the response function is shown in the third graph of the second row in fractions of a circle. The phase is also shown in the bottom row as the lag in seconds. The computed square coherency and the spectrum of the uncorrelated components are shown in the bottom row.

The spatial distribution of the amplitude and phase of the frequency response function and the spatial distribution of the coherency of the records for the semidiurnal component from the reference station U and the boundary stations are shown in Fig. 4. The decrease in amplification near station A_2 indicates the passing of the amphidromic point located approximately halfway between the English and Dutch coasts (see Proudman and Doodson, Fig. 5, Ref. 9). The amplitude, phase, and squared coherency of the computed frequency response function for the quarter-diurnal harmonic ($f = 0.16/\text{hr}$) along the boundary network are shown in Fig. 6.

Frequency response function for points between gauges can be interpolated for models of different grid size. The impulse response function $h(k)$ between the reference station U and the boundary stations is obtained by inverse Fourier transform from the co-, quad-, and auto-spectra of the reference station.

$$\hat{h}(k) = \frac{\Delta f}{2} \left\{ \sum_{h=0}^m \left[\frac{\bar{C}_{sy}(h)}{\bar{P}_{xx}(h)} \right] \cos \frac{hk\pi}{m} + \sum_{h=0}^m \left[\frac{\bar{Q}_{xy}(h)}{\bar{P}_{xx}(h)} \right] \sin \frac{hk\pi}{m} \right\} \quad (22)$$

for $k = 0, 1, 2, 3 \dots m$
 $h = 0, \dots m$

The water levels at these boundary points for any future condition can then be generated optimally from the measured information at station U (or from any other single station) by the convolution formula:

$$y(n\Delta t) = \Delta t \sum_{k=0}^n x(k\Delta t) \hat{h}(n\Delta t - k\Delta t) \quad (23)$$

RECONSTRUCTION OF BOUNDARY INFORMATION

The aforementioned approach was used for reconstructing the open boundary information of a two-dimensional mathematical model of Jamaica Bay, New York City, U.S.A., as shown in Fig. 7.(2) During a large-scale field observation of water quality for comparing simulated with observed

NORTH SEA TIDE ANALYSIS

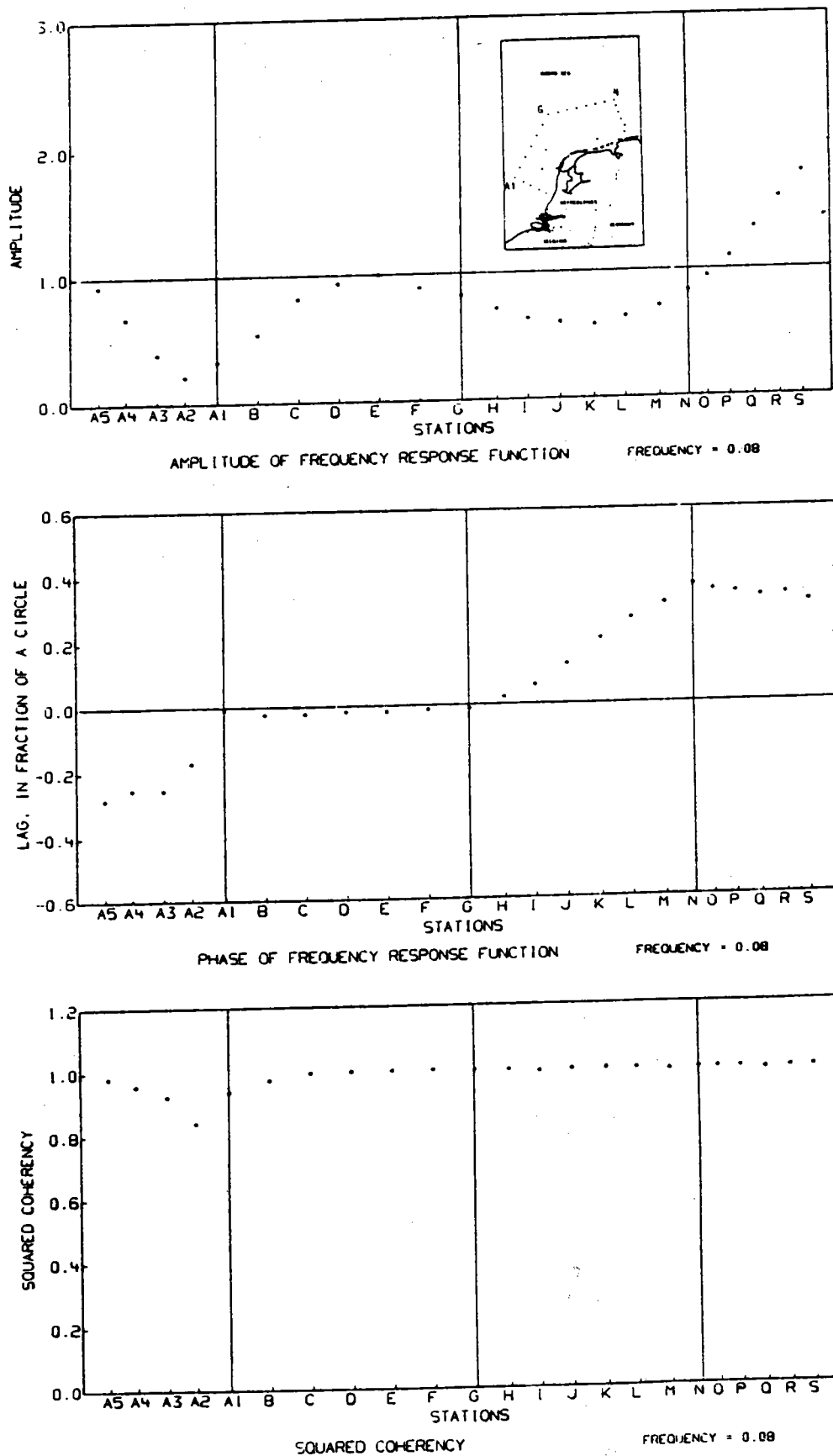


Figure 4 Amplitude, phase, and squared coherency of the computed frequency response function for the semi-diurnal tidal component (frequency = 0.08/hr)

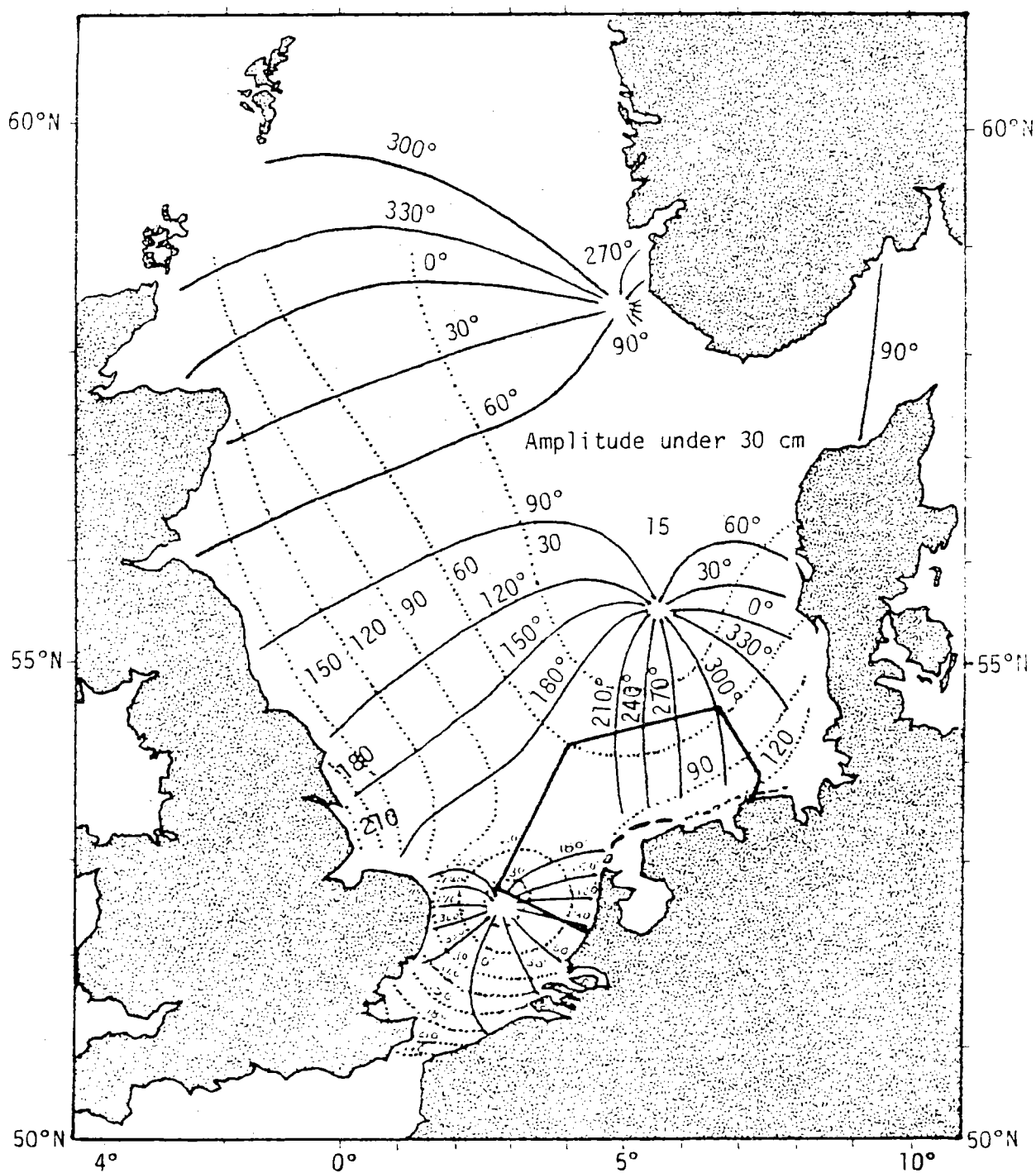


Figure 5 Cotidal times, observed amplitudes, and amphidromic points near the area of the coastal model for the M_2 component (from Proudman and Doodson, 1924)

NORTH SEA TIDE ANALYSIS

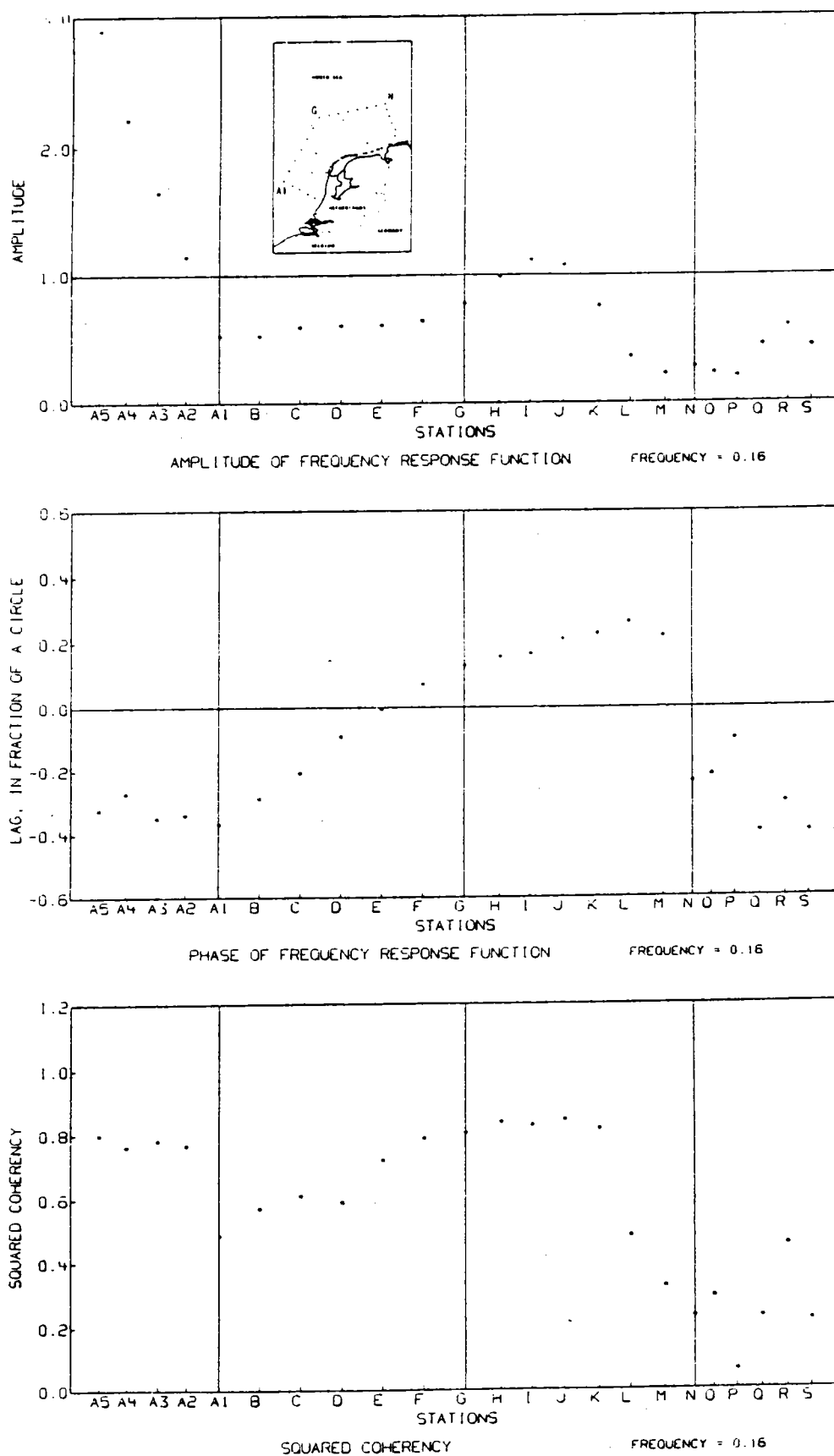


Figure 6 Amplitude, phase, and squared coherency of the computed frequency response function for the quarter-diurnal tidal component (frequency = 0.16/hr)

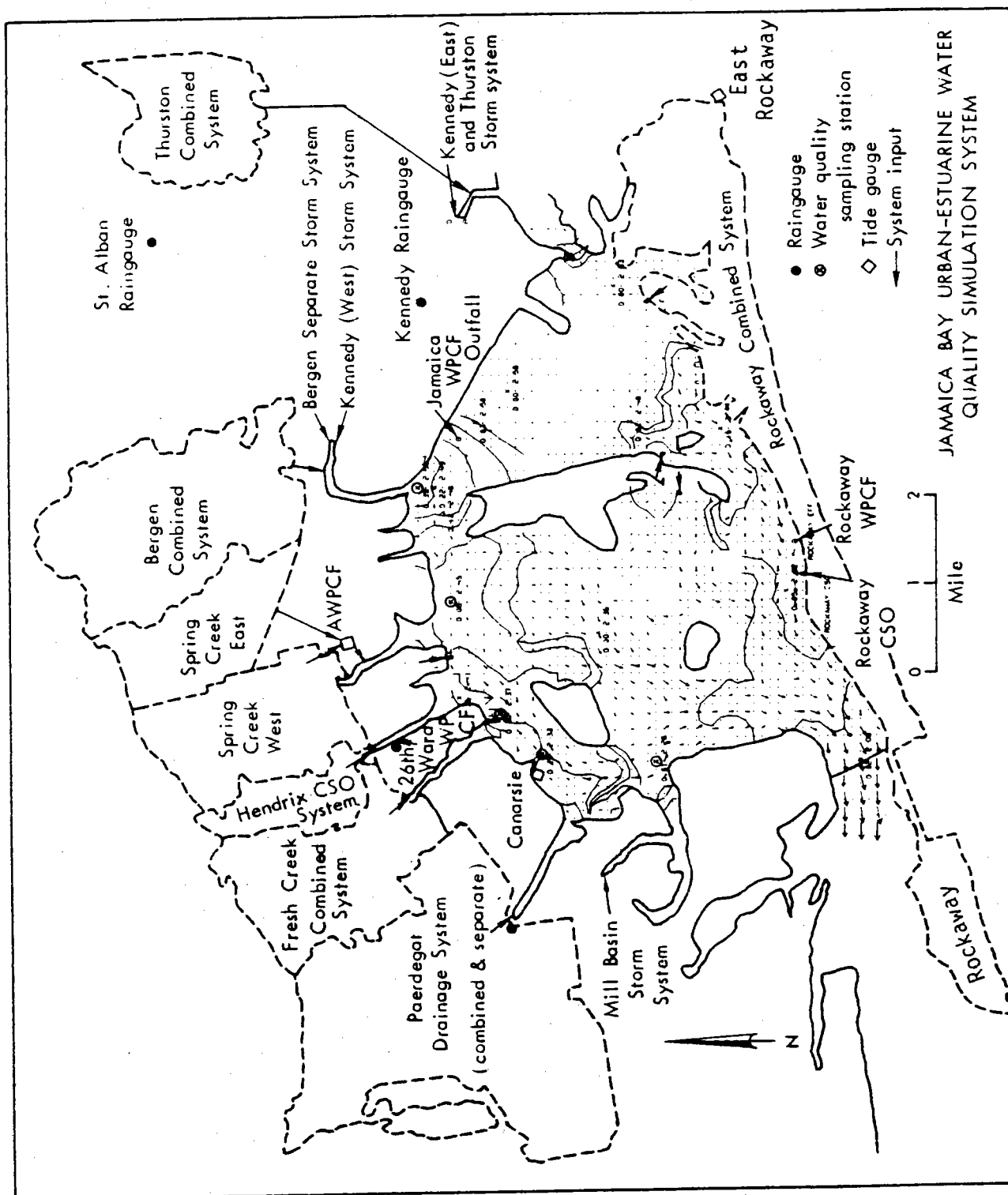


Figure 7 Jamaica Bay Urban-estuarine Water Quality Simulation System (Ref. 2)

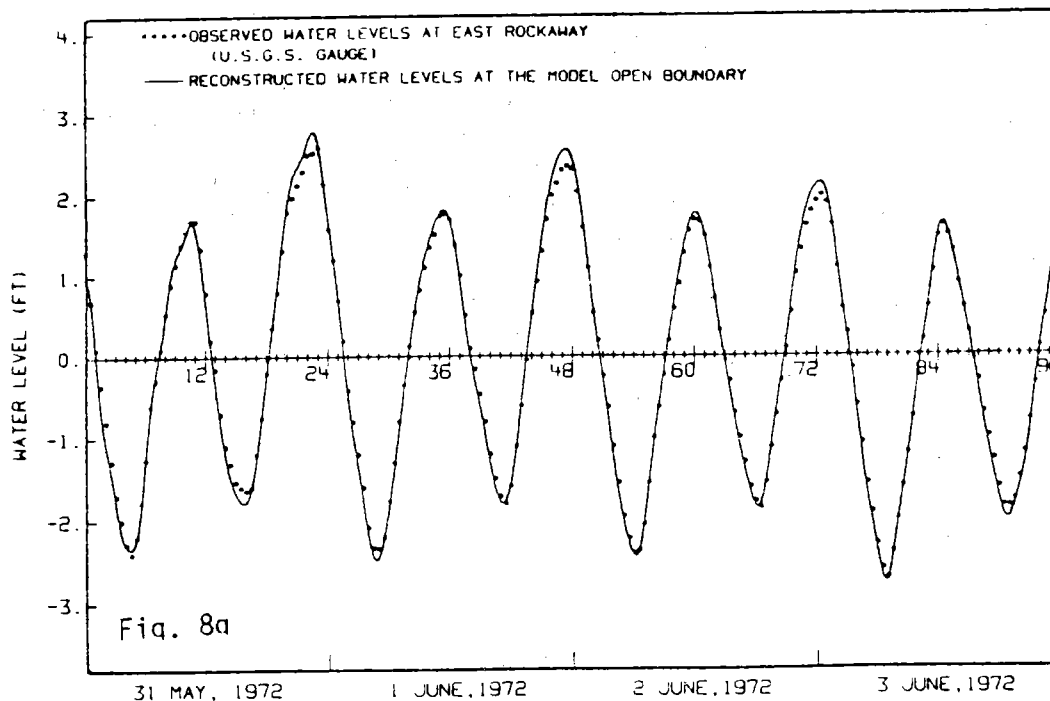
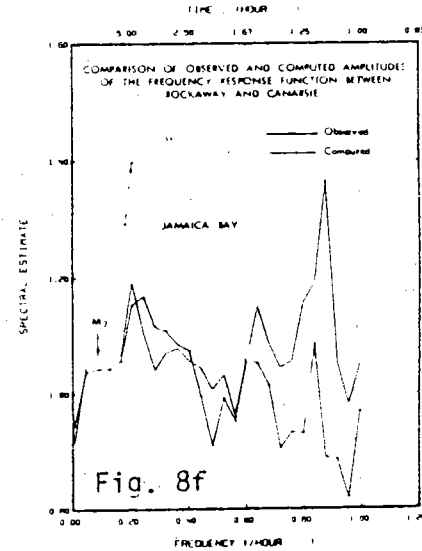
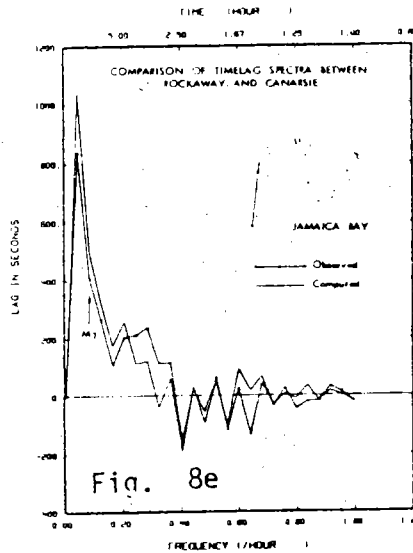
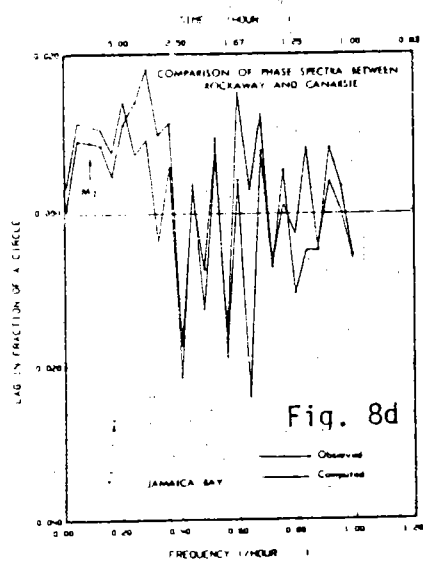
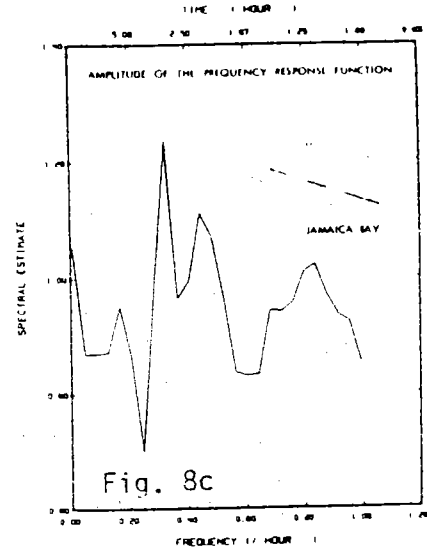
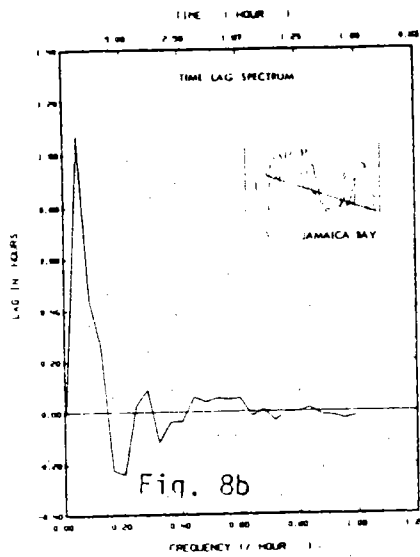
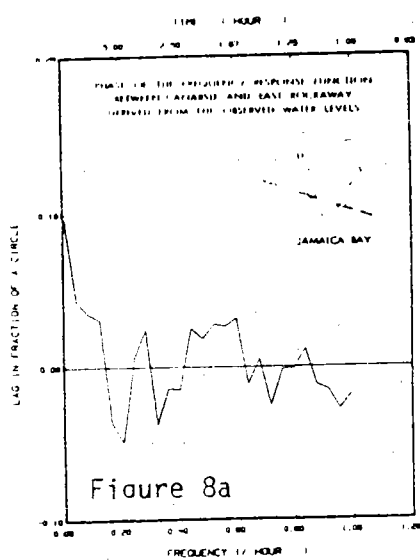
pollutant distribution after a rainstorm, the tide gauge in the bay at Canarsie (northwestern corner of Fig. 7) malfunctioned without being detected until after the entire field operation was completed. It was found that the gauge became inoperative just before the sampling started with a few days of usable data prior to that period. Without tide information, no meaningful simulation could be made. Rather than request another survey, it was decided to reconstruct the missing water level time history. The nearest available tide data covering the entire period was the East Rockaway gauge located on Long Island (southeast corner of map). The only way for making the simulation is to derive the response (transfer) function from East Rockaway to Canarsie with the mutually available data before the experiment. Secondly, response function can be derived between Canarsie and Rockaway (open boundary) with data collected in October 1970 with high accuracy. Once these response functions are determined, the time history of the water levels at the open boundary can be reconstructed (either in the frequency domain by transformation or in the time domain by convolution) for the water quality simulation period.

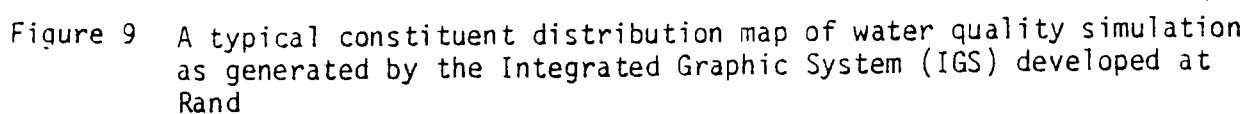
Figures 8a, 8b, and 8c are the computed frequency response functions between Canarsie and East Rockaway using the group of data just prior to May 31, 1972. Figures 8d, 8e, and 8f are the response functions between Rockaway and Canarsie using October 1970 data (dotted lines). The solid lines in this set of graphs are results derived from another numerical simulation of tidal flows between these two stations. Figure 8g is the reconstructed water level (by transformation) at the model boundary using data observed from May 31 through June 3, 1972, at East Rockaway.

With the boundary information determined, the water quality simulation can then be carried out. A typical constituent distribution map is shown in Fig. 9. Detailed discussion of this particular simulation can be found in Leendertse and Liu.(2)

SUMMARY

The usefulness of the network response function in numerical simulation is threefold: (1) future conditions can be simulated using observed water levels at any single location; (2) boundary information for models of different grid size can be obtained by spatial interpolation along the boundary line; and (3) missing data at any location can be estimated optimally (in a least square error sense) using data at neighboring station and the network response functions. The uses of response function and cross-spectral density function to make numerical or hydraulic model adjustment are discussed elsewhere.(2,3)





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APPENDIX 6-B

CROSS-SPECTRAL COMPUTATION FOR SYSTEMS WITH MULTIPLE INPUTS *

The extension of the analyses described in *Appen. 6A* is the multivariate cross-spectral computation. In many cases, a water-quality parameter in a given location may be influenced jointly by many factors. Then the water-quality response relationship between these factors must be analyzed using multivariate formulations. In other words, we want to determine the vector frequency response function $\vec{H}$ between the multiple input series $\vec{X} [X_1(t), X_2(t), \dots, X_n(t)]$ and the output series $Y(t)$.

If one expresses the problem as a vector-valued integral equation similar to Eq. (2), then, by calculus of variation, it can be shown that the optimal estimates of $\vec{h}$ satisfy the following integral equation:

$$\vec{\gamma}_{xy}(u) = \int_{-\infty}^{\infty} \vec{h}(v) \vec{\gamma}_{xx}(u-v) dv \quad (B-1)$$

The solution may be obtained again by Fourier transforms. Because the cross-covariance function matrix and the cross-spectrum matrix form a Fourier transform pair, therefore

$$\vec{P}_{xy}(f) = \vec{H}(f) \cdot \vec{P}_{xx}(f) \quad (B-2)$$

which is equivalent to

$$\begin{bmatrix} P_{x_1 y}(f) \\ P_{x_2 y}(f) \\ \vdots \\ P_{x_n y}(f) \end{bmatrix} = \begin{bmatrix} P_{x_1 x_1}(f) & P_{x_1 x_2}(f) & \dots & P_{x_1 x_n}(f) \\ P_{x_2 x_1}(f) & P_{x_2 x_2}(f) & \dots & P_{x_2 x_n}(f) \\ \vdots & \vdots & & \vdots \\ P_{x_n x_1}(f) & P_{x_n x_2}(f) & \dots & P_{x_n x_n}(f) \end{bmatrix} \begin{bmatrix} H_1(f) \\ H_2(f) \\ \vdots \\ H_n(f) \end{bmatrix} \quad (B-3)$$

The solution requires the inversion of the n-dimensional complex-valued Hermitian matrix $[P_{xx}]$.

* From:

LIU, S. K. (1974). "Stochastic Analysis and Control of Urban Estuarine Water-Quality Systems: Vol. I. Estimation and Prediction." R-1622-NYC. Rand Corp., Santa Monica, California.

$$\begin{bmatrix} H_1(f) \\ H_2(f) \\ \vdots \\ H_n(f) \end{bmatrix} = \begin{bmatrix} P_{x_1 x_1}(f) & P_{x_1 x_2}(f) & \dots & P_{x_1 x_n}(f) \\ P_{x_2 x_1}(f) & P_{x_2 x_2}(f) & \dots & P_{x_2 x_n}(f) \\ \vdots & \vdots & & \vdots \\ P_{x_n x_1}(f) & P_{x_n x_2}(f) & \dots & P_{x_n x_n}(f) \end{bmatrix}^{-1} \begin{bmatrix} P_{x_1 y}(f) \\ P_{x_2 y}(f) \\ \vdots \\ P_{x_n y}(f) \end{bmatrix} \quad (B-4)$$

Each element in the complex Hermitian matrix $\vec{P}_{xx}(f)$ and the column vector $\vec{P}_{xy}(f)$ is the co-spectral and quadrature spectral estimate between the input and the resulting output series, except, of course, the real-valued principal diagonal, which is the auto-spectral density function of the multiple input series. With the basic computational program for cross-spectral computation, the frequency response matrix can then be derived using inversion techniques. For a system of two-inputs and one output, through transforms we obtain

$$P_{x_1 y}(f) = H_{yx_1}(f)P_{x_1 x_1}(f) + H_{yx_2}(f)P_{x_1 x_2}(f) \quad (B-5)$$

$$P_{x_2 y}(f) = H_{yx_1}(f)P_{x_2 x_1}(f) + H_{yx_2}(f)P_{x_2 x_2}(f) \quad (B-6)$$

The frequency response matrix becomes

$$H_{yx_1}(f) = \frac{P_{x_1}(f)P_{x_2 x_2}(f) - P_{x_2 y}(f)P_{x_1 x_2}(f)}{P_{x_1 x_1}(f)P_{x_2 x_2}(f) - |P_{x_1 x_2}(f)|^2} \quad (B-7)$$

$$H_{yx_2}(f) = \frac{P_{x_2 y}(f)P_{x_1 x_1}(f) - P_{x_1 y}(f)P_{x_2 x_1}(f)}{P_{x_1 x_1}(f)P_{x_2 x_2}(f) - |P_{x_1 x_2}(f)|^2} \quad (B-8)$$

The amplification and phase components of the frequency response in terms of co-spectral and quadrature spectral estimates are:

$$\hat{A}_{mx_1y}(f) = \left\{ \left[\frac{\hat{C}_{x_1y}(f)\hat{P}_{x_2x_2}(f) + \hat{Q}_{x_2y}(f)\hat{Q}_{x_1x_2}(f) - \hat{C}_{x_2y}(f)\hat{C}_{x_1x_2}(f)}{[\hat{P}_{x_1x_1}(f)\hat{P}_{x_2x_2}(f) - \hat{A}_{mx_1x_2}(f)^2]} \right]^2 + \left[\frac{\hat{Q}_{x_1y}(f)\hat{P}_{x_2x_2}(f) - \hat{Q}_{x_2y}(f)\hat{C}_{x_1x_2}(f) - \hat{C}_{x_2y}(f)\hat{Q}_{x_1x_2}(f)}{[\hat{P}_{x_1x_1}(f)\hat{P}_{x_2x_2}(f) - \hat{A}_{mx_1x_2}(f)^2]} \right]^2 \right\}^{\frac{1}{2}} \quad (B-9)$$

$$\hat{A}_{mx_2y}(f) = \left\{ \left[\frac{\hat{C}_{x_2y}(f)\hat{P}_{x_1x_1}(f) + \hat{Q}_{x_1y}(f)\hat{Q}_{x_2x_1}(f) - \hat{C}_{x_1y}(f)\hat{C}_{x_2x_1}(f)}{[\hat{P}_{x_1x_1}(f)\hat{P}_{x_2x_2}(f) - \hat{A}_{mx_1x_2}(f)^2]} \right]^2 + \left[\frac{\hat{Q}_{x_2y}(f)\hat{P}_{x_1x_1}(f) - \hat{Q}_{x_1y}(f)\hat{C}_{x_2x_1}(f) - \hat{C}_{x_1y}(f)\hat{Q}_{x_2x_1}(f)}{[\hat{P}_{x_1x_1}(f)\hat{P}_{x_2x_2}(f) - \hat{A}_{mx_1x_2}(f)^2]} \right]^2 \right\}^{\frac{1}{2}} \quad (B-10)$$

where

$$\hat{A}_{mx_1x_2}(f) = |\hat{P}_{x_1x_2}(f)| \quad (B-11)$$

$$\hat{\phi}_{x_1y}(f) = \arctan - \frac{\left[\frac{\hat{Q}_{x_1y}(f)\hat{P}_{x_2x_2}(f) - \hat{Q}_{x_2y}(f)\hat{C}_{x_1x_2}(f) - \hat{C}_{x_2y}(f)\hat{Q}_{x_1x_2}(f)}{[\hat{P}_{x_1x_1}(f)\hat{P}_{x_2x_2}(f) - |\hat{P}_{x_1x_2}(f)|^2]} \right]}{\left[\frac{\hat{C}_{x_1y}(f)\hat{P}_{x_2x_2}(f) + \hat{Q}_{x_2y}(f)\hat{Q}_{x_1x_2}(f) - \hat{C}_{x_2y}(f)\hat{C}_{x_1x_2}(f)}{[\hat{P}_{x_1x_1}(f)\hat{P}_{x_2x_2}(f) - |\hat{P}_{x_1x_2}(f)|^2]} \right]} \quad (B-12)$$

$$\hat{\phi}_{x_1y}(f) = \arctan - \frac{\left[\frac{\hat{Q}_{x_2y}(f)\hat{P}_{x_1x_1}(f) - \hat{Q}_{x_1y}(f)\hat{C}_{x_2x_1}(f) - \hat{C}_{x_1y}(f)\hat{Q}_{x_2x_1}(f)}{[\hat{P}_{x_1x_1}(f)\hat{P}_{x_2x_2}(f) - |\hat{P}_{x_1x_2}(f)|^2]} \right]}{\left[\frac{\hat{C}_{x_2y}(f)\hat{P}_{x_1x_1}(f) + \hat{Q}_{x_1y}(f)\hat{Q}_{x_2x_1}(f) - \hat{C}_{x_1y}(f)\hat{C}_{x_2x_1}(f)}{[\hat{P}_{x_1x_1}(f)\hat{P}_{x_2x_2}(f) - |\hat{P}_{x_1x_2}(f)|^2]} \right]} \quad (B-13)$$

The frequency domain decomposition of the uncorrelated components between three series is the spectral estimate of noise, i.e.,

$$\hat{P}_{nn}(f) = \hat{P}_{yy}(f) - \hat{H}_{yx_1}(f)\hat{P}_{yx_1}(f) - \hat{H}_{yx_2}(f)\hat{P}_{yx_2}(f) \quad (B-14)$$

The degree of linear dependency between one input and the output with the other input removed can be derived from the partial coherency spectra:

$$\hat{\Omega}_{x_2y|x_1}^2(f) = 1 - \left[\frac{1 - \hat{\Omega}_{yx_1x_2}^2(f)}{1 - \hat{\Omega}_{yx_1y}^2(f)} \right] \quad (B-15)$$

$$\hat{\Omega}_{x_1y|x_2}^2(f) = 1 - \left[\frac{1 - \hat{\Omega}_{yx_1x_2}^2(f)}{1 - \hat{\Omega}_{x_2y}^2(f)} \right] \quad (B-16)$$

These partial coherency spectra are ratios between the multiple coherency estimators and the bivariate coherency estimators.

$$\hat{\Omega}_{x_1y}^2(f) = \hat{A}_{mx_1y}(f)^2 / [\hat{P}_{x_1x_1}(f)\hat{P}_{yy}(f)] \quad (B-17)$$

$$\hat{\Omega}_{x_2y}^2(f) = \hat{A}_{mx_2y}(f)^2 / [\hat{P}_{x_2x_2}(f)\hat{P}_{yy}(f)] \quad (B-18)$$

$$\hat{\Omega}_{yx_1x_2}^2(f) = \frac{\hat{P}_{x_2x_2}(f)\hat{A}_{mx_1y}(f)^2 + \hat{P}_{x_1x_1}(f)\hat{A}_{mx_2y}(f)^2 - 2\hat{\gamma}(f)}{\hat{P}_{yy}(f)[\hat{P}_{x_1x_1}(f)\hat{P}_{x_2x_2}(f) - \hat{A}_{mx_1x_2}(f)^2]} \quad (B-19)$$

$$\begin{aligned} \hat{\gamma}(f) = & \hat{C}_{x_1x_2}(f)\hat{C}_{x_2y}(f)\hat{C}_{x_1y}(f) + \hat{C}_{x_1x_2}(f)\hat{Q}_{x_2y}(f)\hat{Q}_{x_1y}(f) \\ & - \hat{Q}_{x_1x_2}(f)\hat{Q}_{x_2y}(f)\hat{C}_{x_1y}(f) + \hat{Q}_{x_1x_2}(f)\hat{Q}_{x_1y}(f)\hat{C}_{x_2y}(f) \end{aligned} \quad (B-20)$$

七、數值模式之印證方法及實例

劉 肖 孔

由於數值模式中含有係數，所以使用時要校正。每個模式的格點大小不同，垂直層次多寡有異，故而同一地區，不同模式需要係數亦有所不同。

十數年前的早期數值模式，尚未包括「紊流的內格點效應計算」，所以在校正過程中所需之實地資料，遠較目前為多。以荷蘭為例，正當三角洲計劃進行初期為校正「蘭德三角洲二度模式」(Rand-Delta model) 之各點水位及流速，荷蘭水政部曾先後建立測站近百個(圖 7.1) 測量船二十餘條同時進行水理分析。

至於三度空間模式，則需測站更多。例如美蘇邊境之白令海模式，僅東南大陸棚一帶，即含垂直斷面一百六十三站，定期測量(圖 7.2) 。

然而，由於數值模式的基本技術日漸進步，同樣的精確度，所需實測資料，則逐漸減少；其中一個主要的原因即是模式中含有「紊流內格點計算」，將變數的計算標準化以後，僅需幾個常數即可。此類常數，亦可在實驗室的可控制情況下，進行實驗。所得結果，數值模式亦可使用。在第八章內，我們將把海洋中的「垂直紊流分佈」，在標準化以後，和銅管中的空氣氣流相比較。

原則上，模式的格點距離愈大，其所需的「動量擴散係數」也愈大，因為動量擴散係數，代表了該計算格點內部的「內格點動能擴散」的情況。格點距離愈小，則計算的「平均流速」解析度增大，因此取代了一部份的「內格點效應」。當格點的密度接近於「雷諾係數平

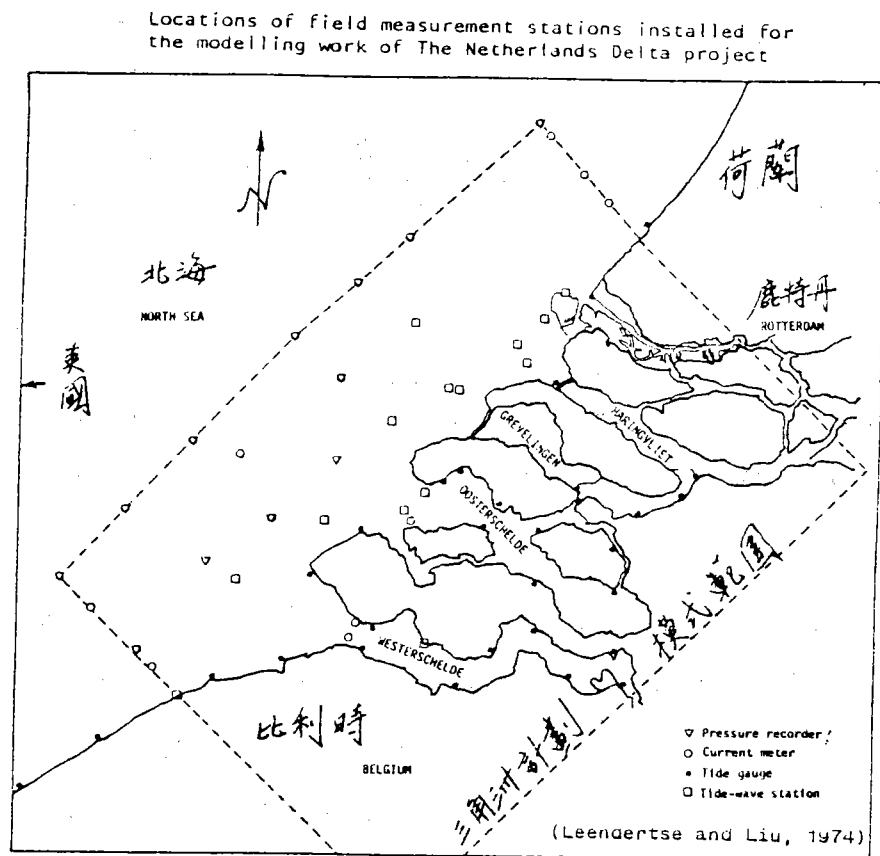


圖 7.1 荷蘭三角洲計劃二度空間模式的水位及流速測量系統圖

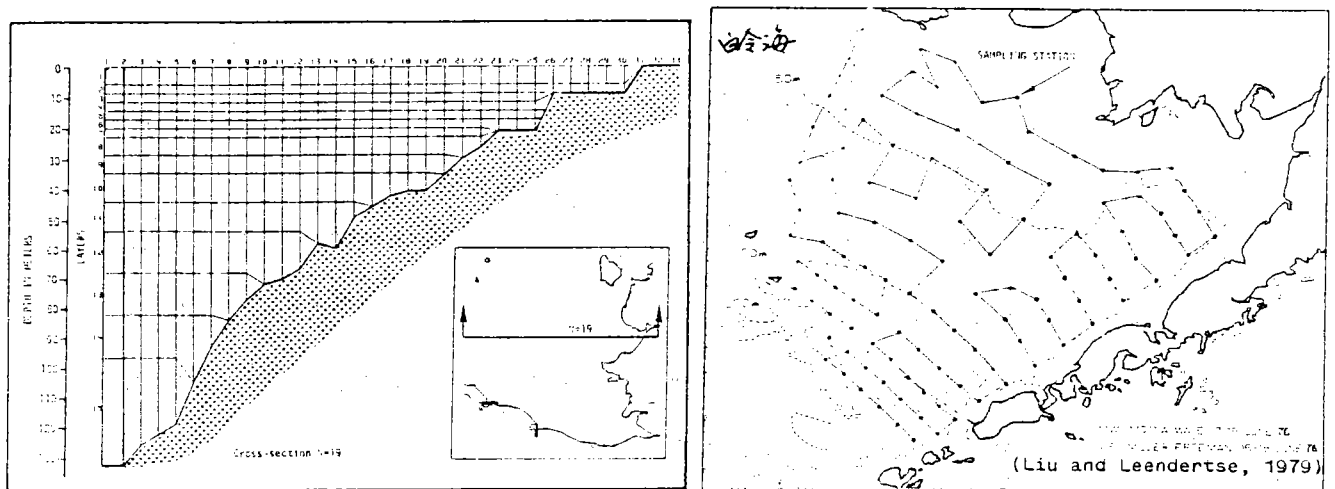


圖 7.2 美國及蘇俄邊境（白令海）三度空間模式測站分佈及採樣順序佈置圖

方」時，當地的流況即為「液體」的函數（Kinematic Viscosity），而非「流況」本身的函數（Eddy Viscosity）。這是一個流體力學的基本問題，也是目前因為電腦不夠大而致力於 turbulent closure 研究工作的一大原因。換言之，模式解析度愈大，其 Eddy Viscosity 愈趨近於 Kinematic Viscosity。前者為「流況之函數」，後者為「流體之函數」。

在海域數值模式的校正工作上，大致有幾個原則：

1. 在同一個海域，使用不同的數值模式，其所需之係數亦不盡相同。例如溫度及鹽度的擴散係數，低階數值模式所需的係數值，較高階模式為大。靜態（Steady-State）模式大於一度動態模式，又大於二度模式，再大於三度模式。
2. 計算同一個海域，所需的「內摩擦係數」（動量擴散），與模式的空間解析度成反比。
3. 在模式的校正方法和率定過程上，頻率分析法（Frequency-domain analysis）較時間分析法（Time-domain analysis）效率高，並易於追蹤模式誤差的來源。頻率分析法中亦以「交絡法」及「反應函數法」最普遍，最有效。其理論背影請參閱附錄 6A。

由於海域根據基本流體力學方程式的數值解析，並包括溫度、鹽度、紊流及污染物等項，故在工程及經建方面有其實用價值。其使用範圍即可包括，但並不限於下列十項：

1. 潮位推算。
2. 海流推算。
3. 計算溫度及鹽度的分佈。
4. 都市防洪及河海交互作用。
5. 氣象潮的計算（包括颱風暴潮）。
6. 港灣的盪漾計算。

7. 電廠熱污染及冷却進水口的設計。

8. 船塢設計。

9. 水污染的分析及防治。

10. 上湧流的計算及漁業研究等等。

以上所述實用範圍及實例，筆者已在「中國海域三度空間數值模式」報告的第六章中已有敘述，在此不再重覆。在本章內，擬以模式實例（case study）方式，由建立工作至印證方法，提供淺見。

因爲三度模式所包含的計算項目多爲非線性項，對於計算大陸棚內的複雜水理（淺水化的動力學）最爲有效。世界上最大面積的大陸棚，位於美國、蘇俄及加拿大三國之間（圖 7.3），其他爲北海地區及中國海域。因爲筆者正受美國大氣海洋總署及內政部委託，建立該地區的三度空間模式，所以即用該模式爲例，將建立模式的一些經驗及印證方法（adjustment）提供諸位同仁參考。

通常建立模式，最好由大陸棚的邊緣開始計算，因爲該地區的天文潮是「大洋潮」，比較「線性」，易於處理。然後再以「次模式」的方式，增加解析度（見圖 7.3 及 7.4）。利用大模式的內點，做爲次模式的邊界條件。

當模式建成之後，先行從事數次計算，決定潮位變化的重要地區，放置水位及流速儀，以便校正模式（見圖 7.5 及 7.6）。

模式初步校正以後，再利用 Tidal filter* 將潮流部份除去，估算「殘餘流」（圖 7.7），因爲殘餘流有時在近岸地區可用浮標觀測，比流速儀經濟。在污染物的計算方面，浮標資料頗有實用價值。

紊流的擴散率最難校正，亦難實測。所以頗爲依賴模式的「計算精確度」。第八章所討論的典型三度模式，其中紊流計算常數均已校

* 三十五小時的 moving-average.

Fig 7-3 Due to the interactive aspects, the western portion of the Beaufort Sea Model (shaded) is overlapped with the Chukchi Sea Model. This technique not only minimizes the boundary's effects on the computed residual currents but also would allow oil spill trajectory computations to be continued into the neighboring models. Submodels improve the near-shore resolution.

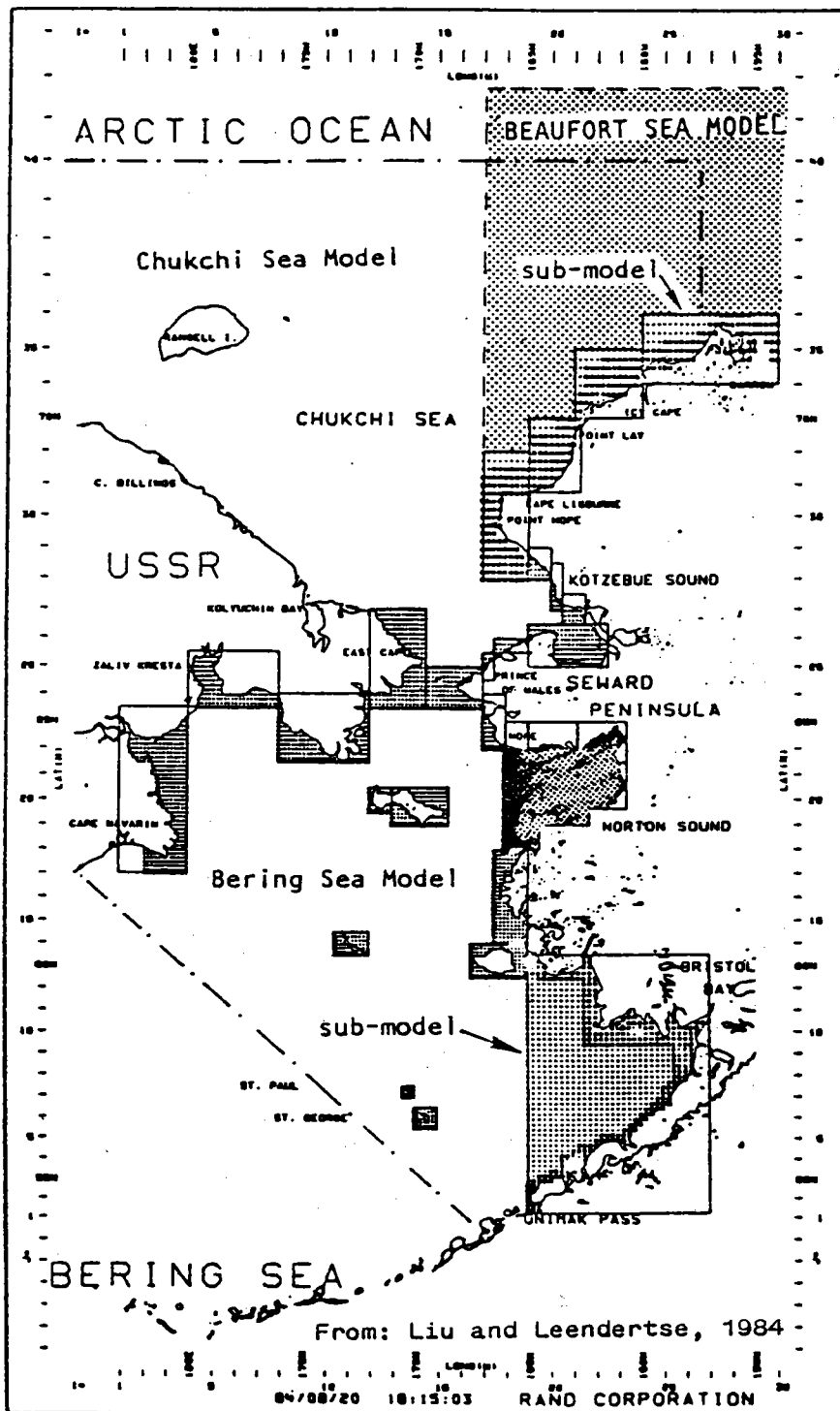
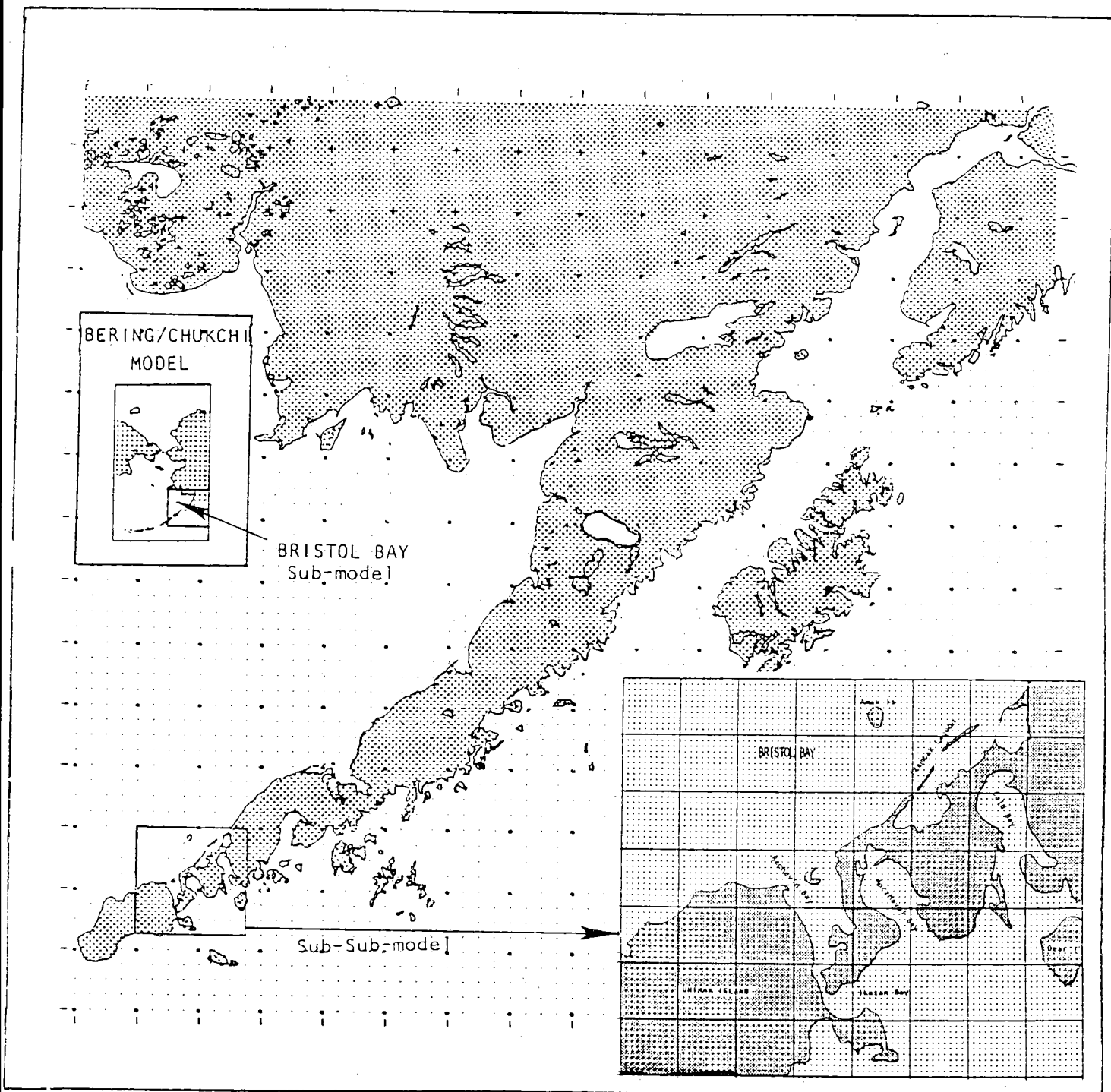


圖 7.3 一個典型的三度模式及「次模式」兩者格點的相關圖，以主模式來求次模式的邊界條件。

sub-model of Bering Sea covering the area of Bristol Bay and a portion of the Gulf of Alaska. Insert map at the lower left corner is another sub-model of this one, covering the area of Izembak Lagoon.



From: Liu and Leendertse, 1983

圖 7.4 前圖右下方之次模式又可分為「次次模式」，以此種方法求得次次模式的邊界條件。注意兩模式在計算點位上的共同網點。

Computed co-tidal chart for the semi-diurnal component (primarily M2). Each ten degrees in phase represents approximately 20 minutes lag relative to the Greenwich mean phase. The maximum tidal amplitudes are found in the Cook Inlet reaching 250 cm.

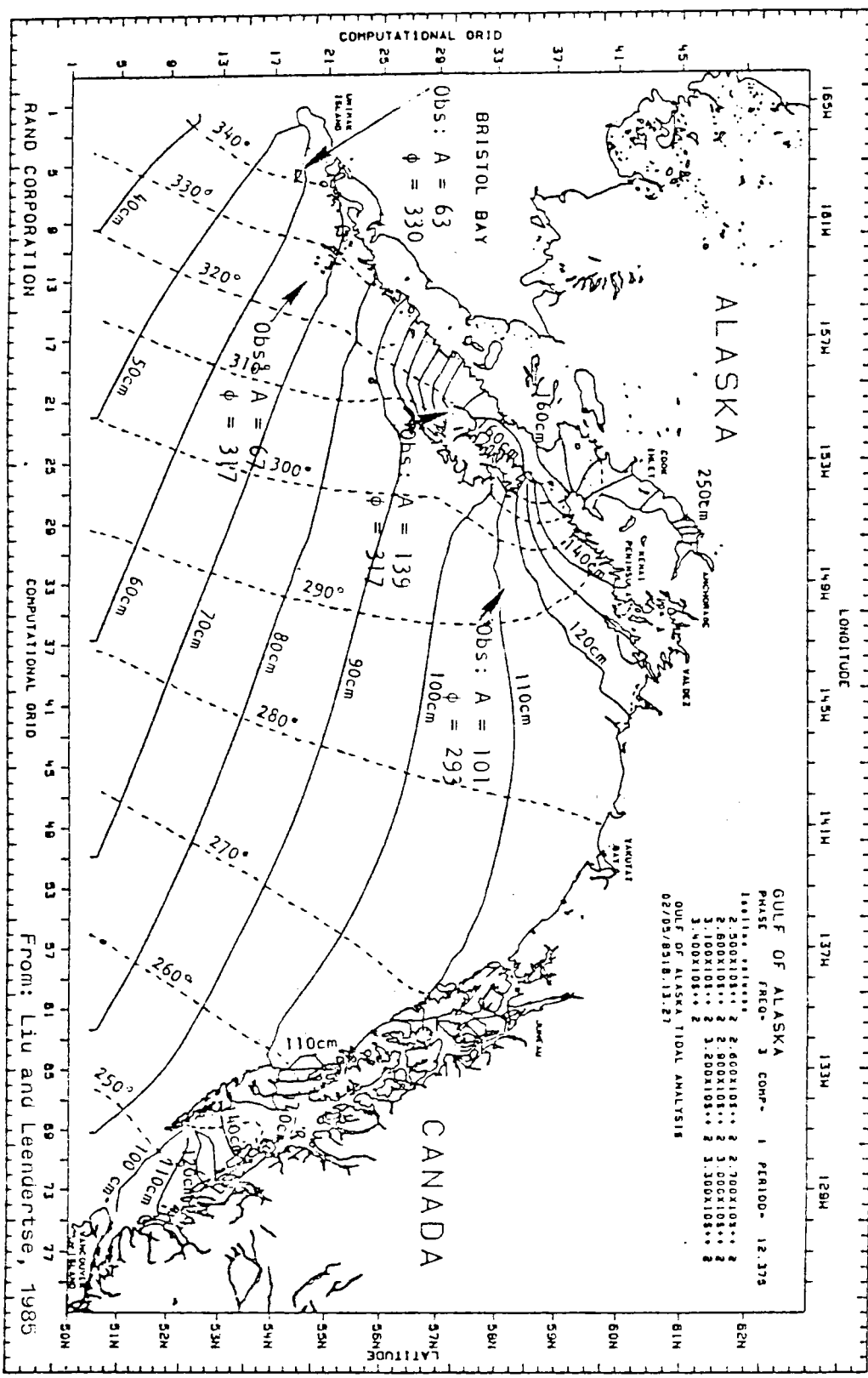
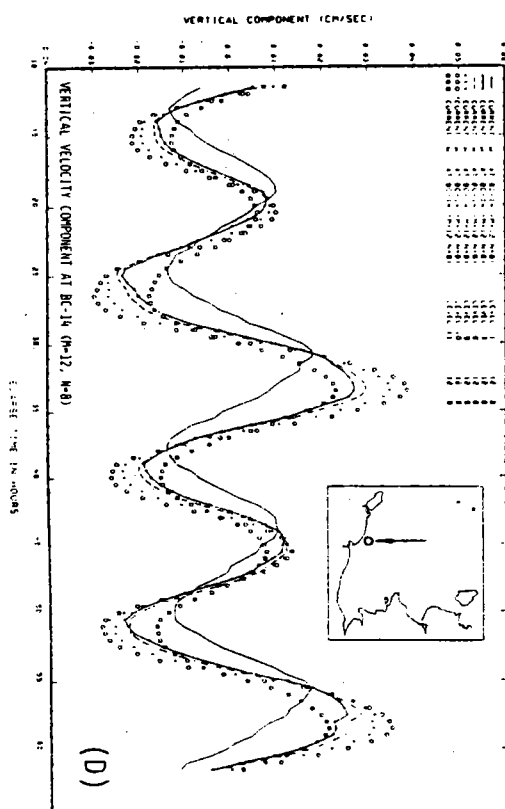
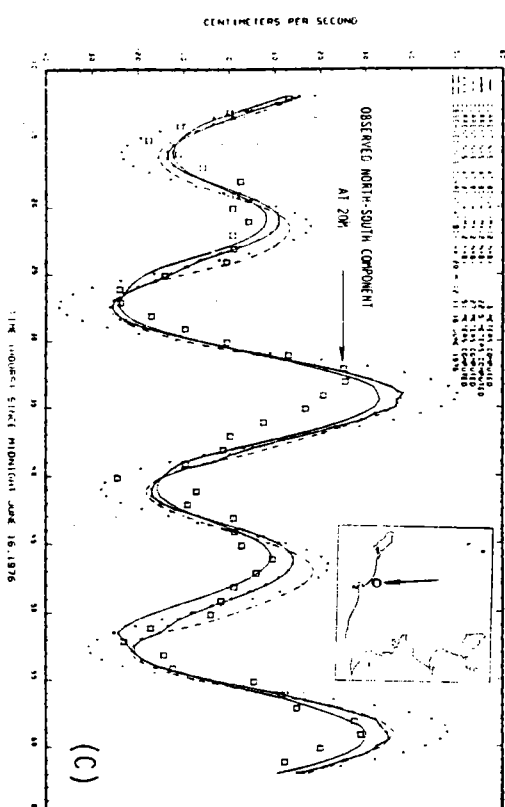
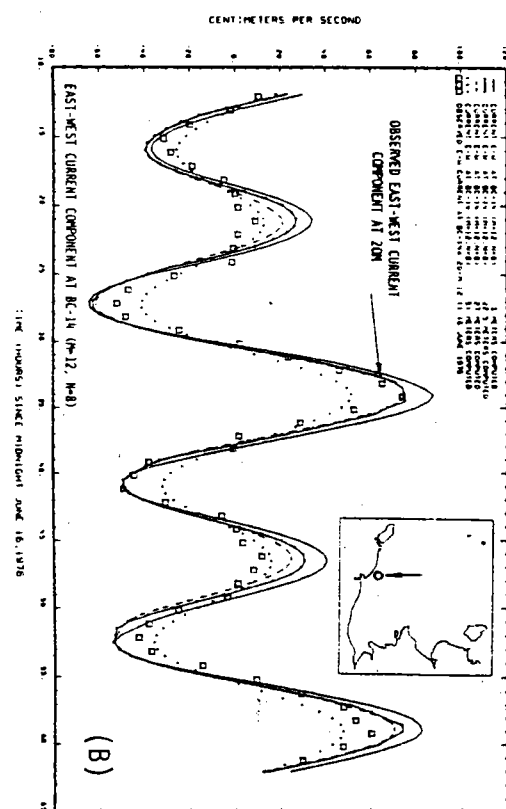
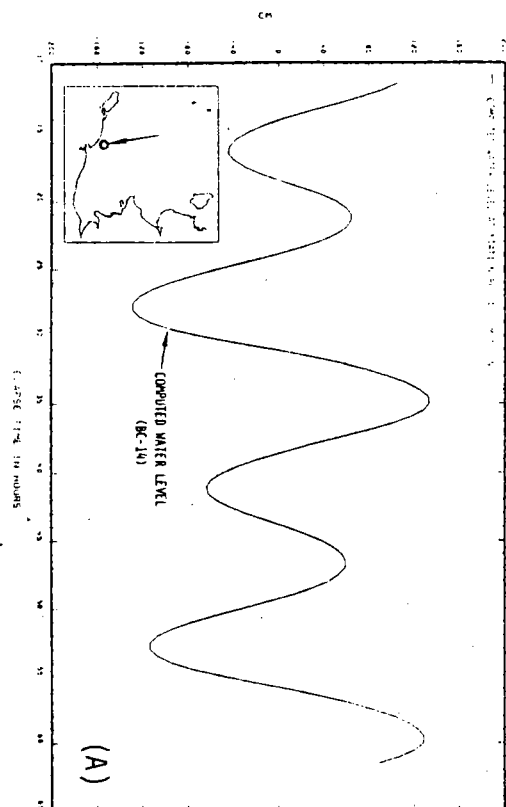


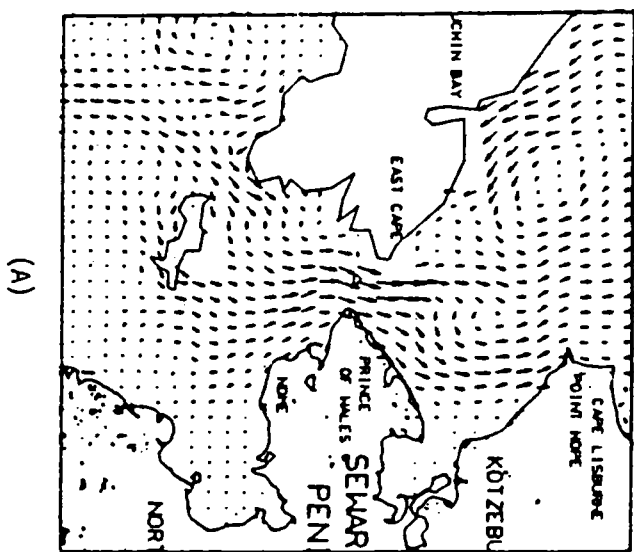
圖 7.5 水位比較法的實例。圖中的潮相，潮位係由模式計算值經「二度富立業」轉移後與實測值相比較。

Typical magnitude of tidal currents near the St. George Basin showing also comparison between the computed and observed currents at a point over the shelf area.

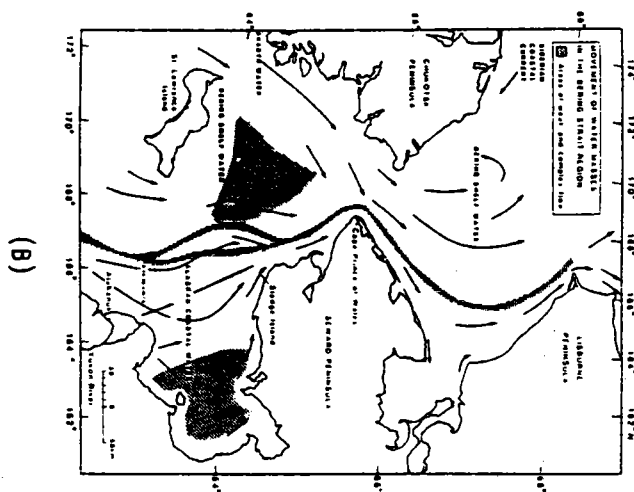


From: Liu and Leendertse 1979

圖 7.6 流速比較法之實例，圖中計算值係事先推算，並非倒推。



(A)



(B)

From: Liu and Leendertse, 1982

圖 7.7 「殘餘流」與長期觀測比較法，上圖為白令海峽由三度模式所計算的「殘餘流」與衛星觀測 (Drogue Study) 兩者間的比較。

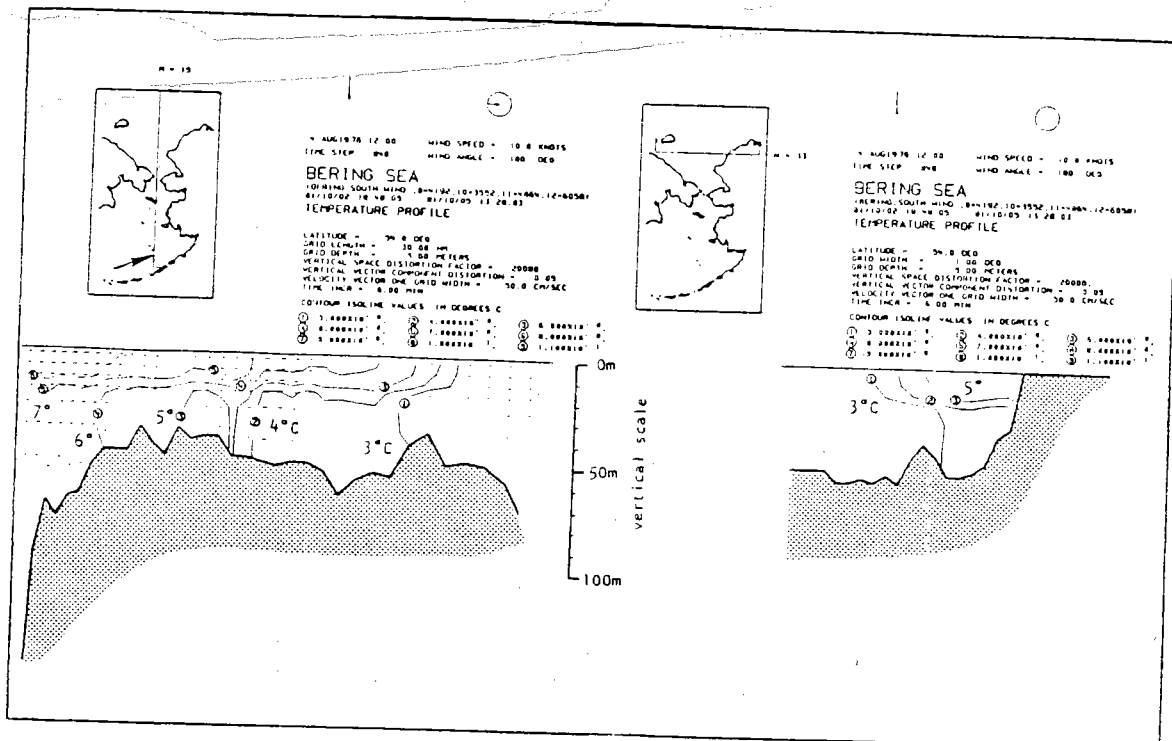


圖 7.11 垂直溫度的變化

Vertical temperature distributions through two cross-sections of the model

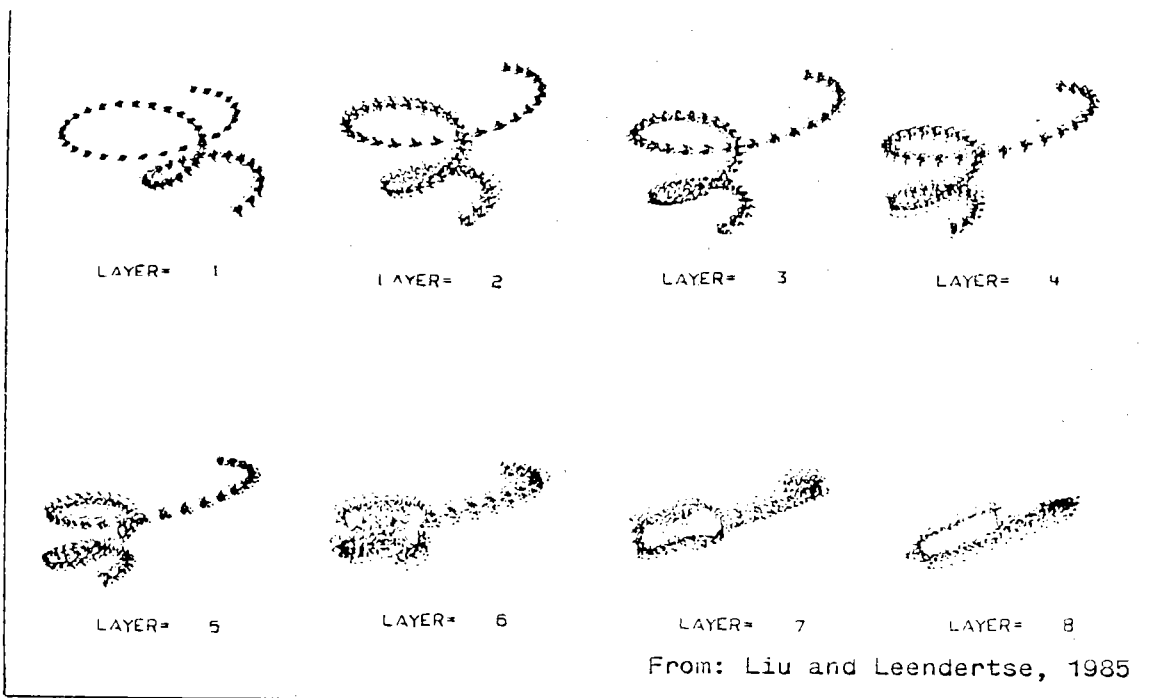


圖 7.8 由於垂直紊流強弱的不同而導至擴散量的層次分別

Partially insulated by the pycnocline, 24-hour trajectories of particle groups released in the upper five layers experience less turbulent diffusion than the lower layers when the system is forced only with tidal energies

正（見圖 8.3），圖 7.8 為垂直紊流強弱的不同而導至擴散量的層次分別。

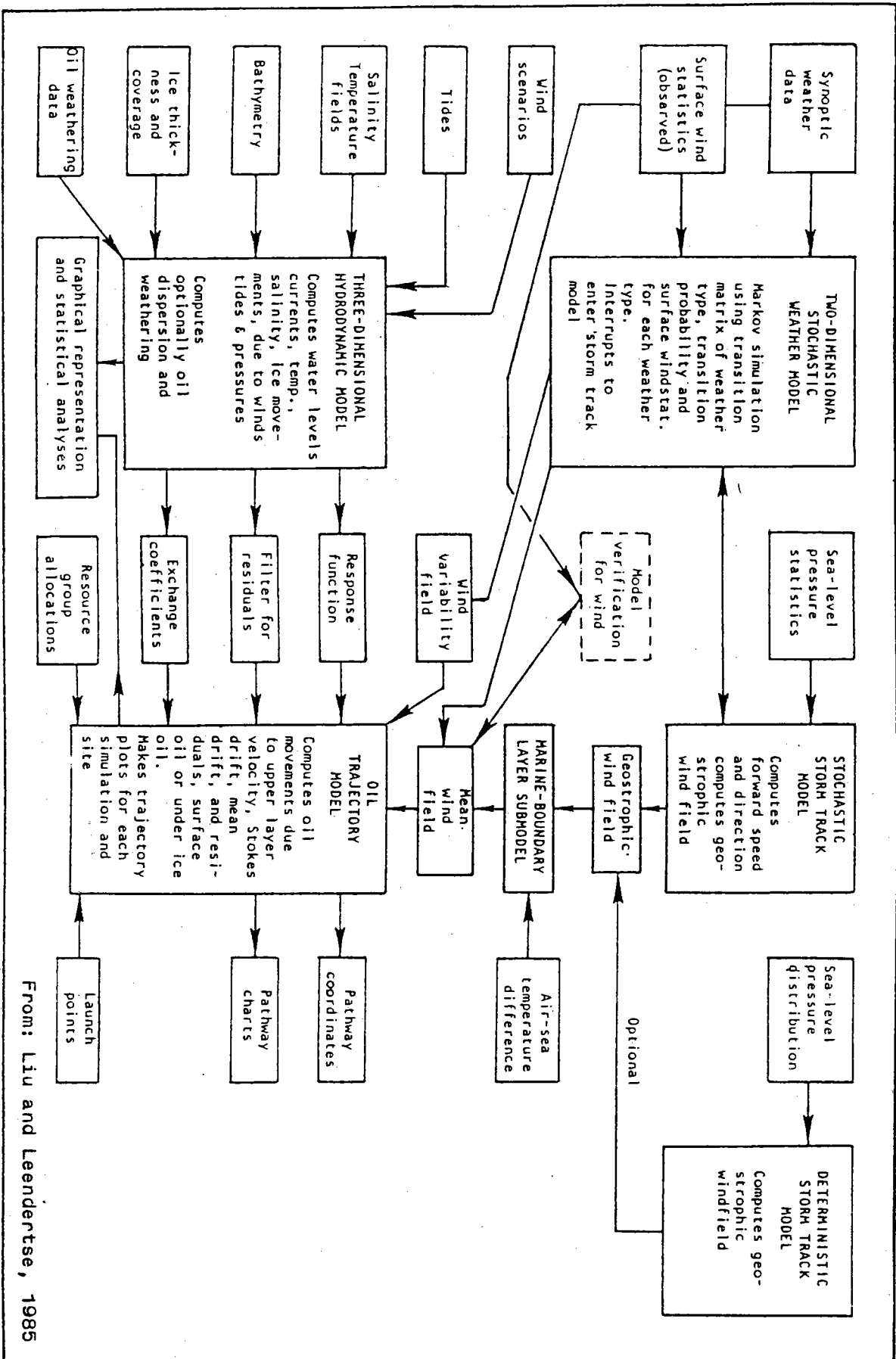
在計算擴散的時候，風的影響不可忽略。所以在大區域的海面計算時，有時要與氣象模式聯合運算（圖 7.9，其中細節，容口頭報告）。

由於在海域內，各點的「對流」及「擴散」作用均因地而異。所以污染物在海域內的位移、擴散及旋轉方向均各不相同，圖 7.10 中的污染物為「瞬時性發放」（Instantaneous release）。圖 7.11 中的情況為「連續發放」（Continuous release）。注意圖中海岸線對污染物擴散與位移的影響。

以上的例子為「保守性污染物（Conservative pollutant）」的計算。一般常用的「大腸菌濃度指引數」為非保守性污染物，其計算除上述各項之外，尚需包括大腸菌的 Die-away process。

海域模式與都市污水處理系統時常因為「次臨界」流的關係，必須聯合運用（圖 7.12 及 7.13），因為篇幅的關係，其他細節，容口頭報告。

Information flow chart showing that both the three-dimensional hydrodynamic and the weather model provides important parameters needed by the oil-spill trajectories model. These parameters are difficult and expensive to measure over the entire Alaskan waters. Observed wind roses, if available, can still be used to drive the trajectories model in its simplest mode.



From: Liu and Leendertse, 1985

圖 7.9 海域模式在使用時常需與其他模式混合運算，上圖為海域模式與二度風場模式，污染物擴散模式交互運算時，資料交換的過程。

Displacements of the one-part-per billion concentration envelop every five days for an instantaneous release of 700000 bbl crude oil from five hypothetical launch points during the summer oceanic season. The variability of local advective and diffusive mechanisms are being illustrated by the changing speed and direction of the oil's movements.

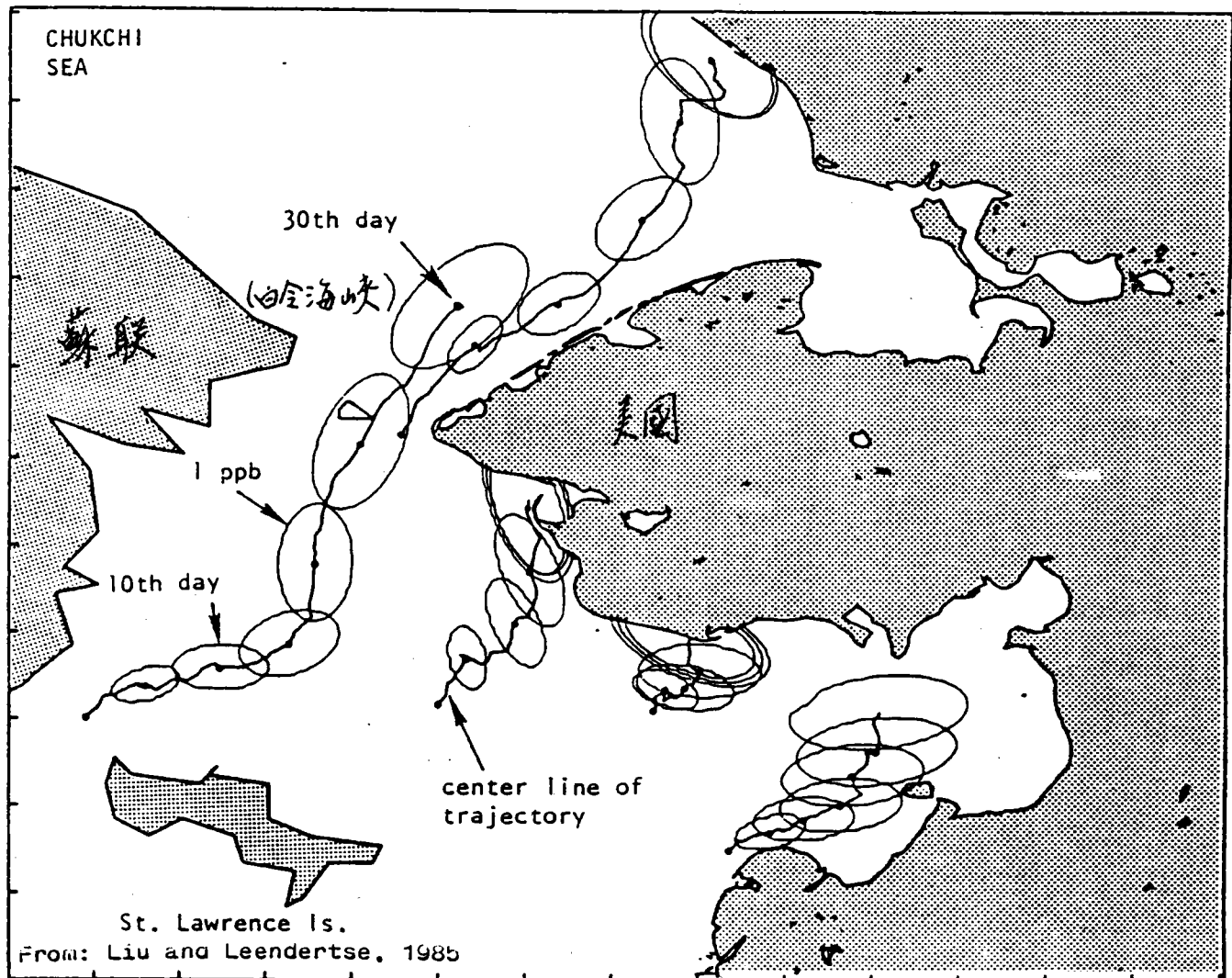
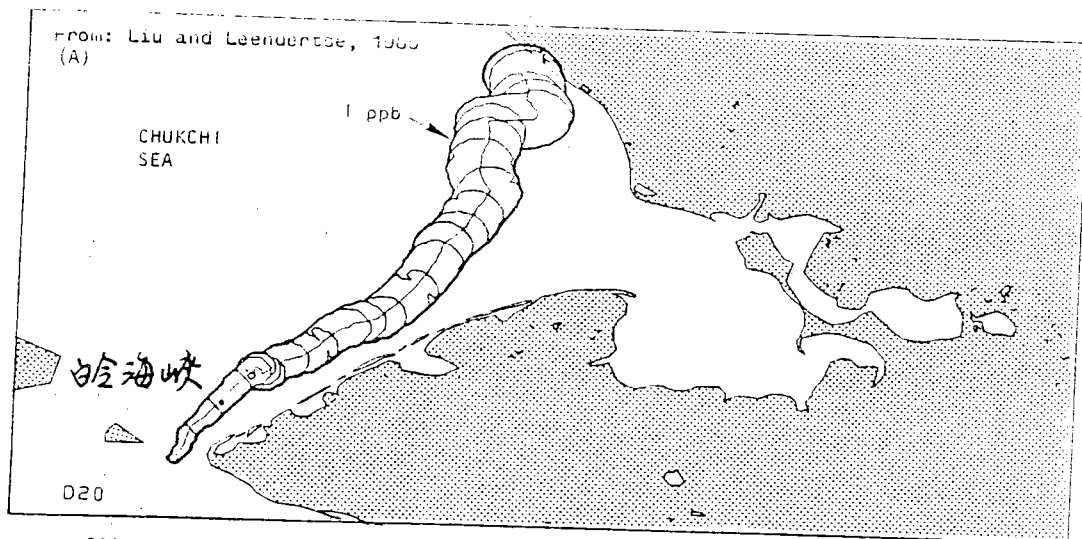


圖 7.10 前述所有模式組合運算後的結果。在同一氣象情況下，由於各點「殘餘流」的方向不同，「紊流擴散量」亦不一樣，所以由五個不同地點所發放的污染物在三十天以後，其擴散大小及方向均不相同。



Oil spill trajectory launched in the middle of the Bering Strait under a 30-day stochastic weather scenario during the summer oceanic period. Figure A illustrates the progressive daily displacements of the 1 part per billion concentration envelop for the continuous discharge of 2000 bbl of crude oil. The traveling speed of oil is governed by the evolutionary weather state as well as the local circulation pattern.

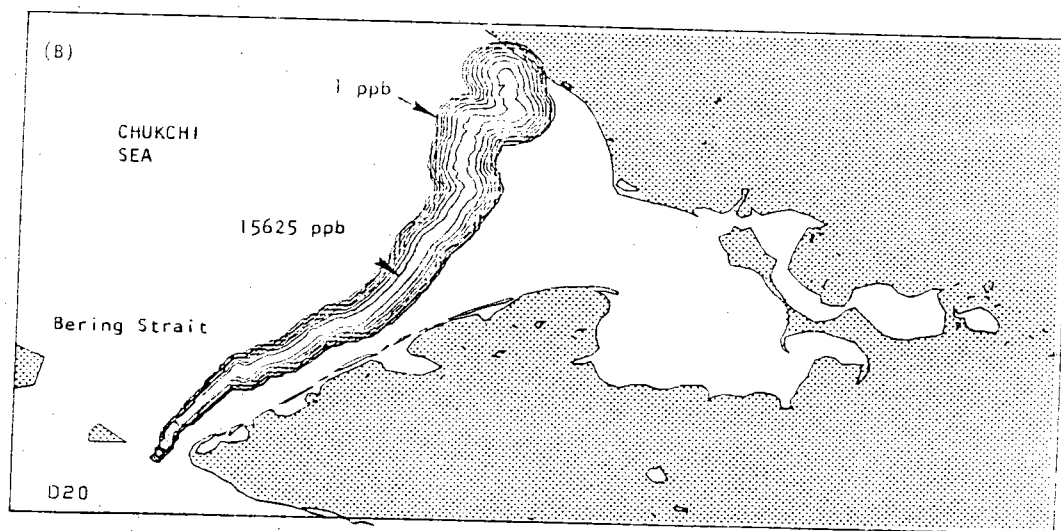


Figure B illustrates the concentration contour of the oil. Notice the cumulation effects when the oil reaches the coastal area where the on-shore current components drop while the along-shore currents strengthened. This near-shore effect tends to redirect the oil while slowing it down somewhat.

圖 7.11 由白令海峽中點發放的污染物，因各點海域流速不同，所以每日的位移差別頗大。注意上方海岸對污染物「對流」與「擴散」的影響。

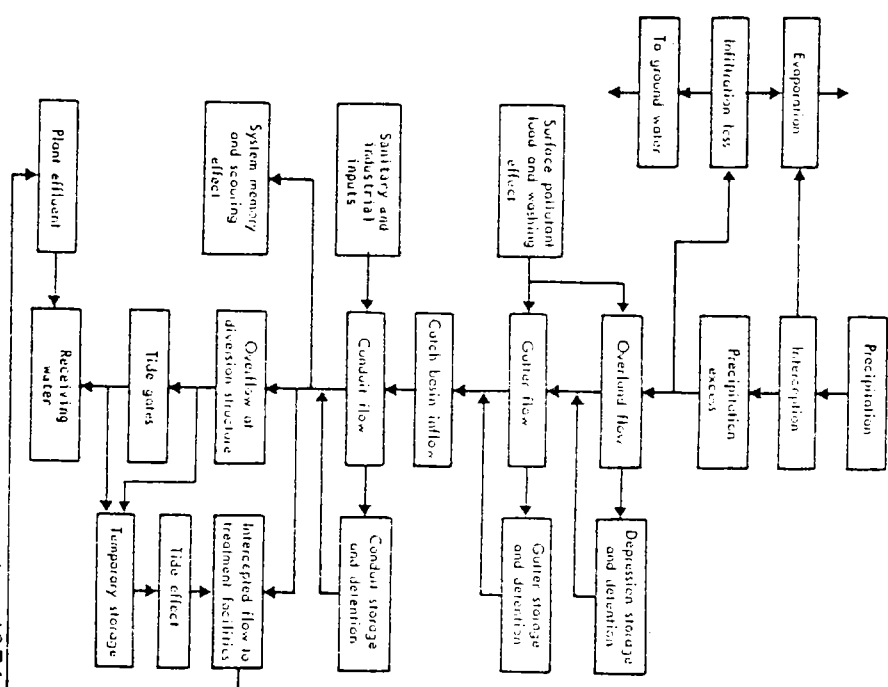
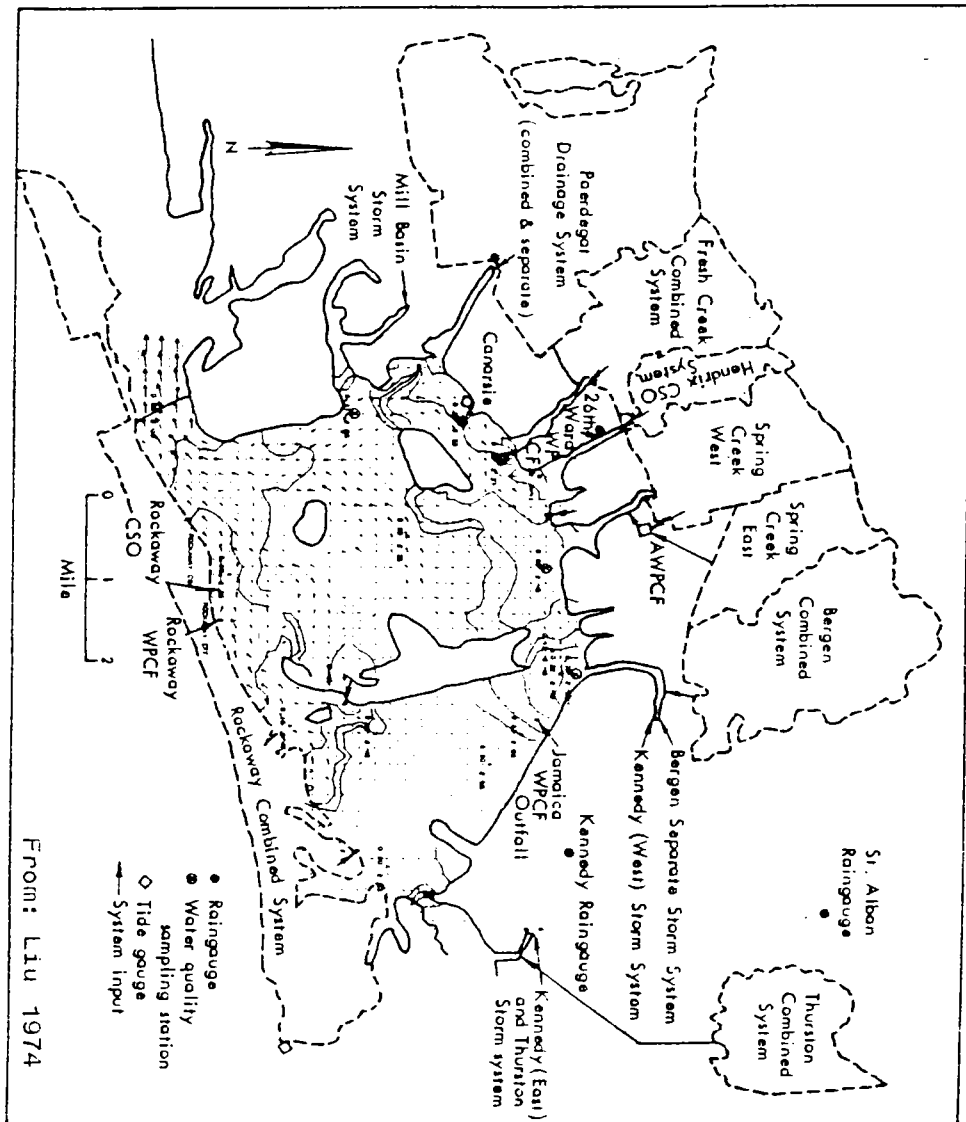
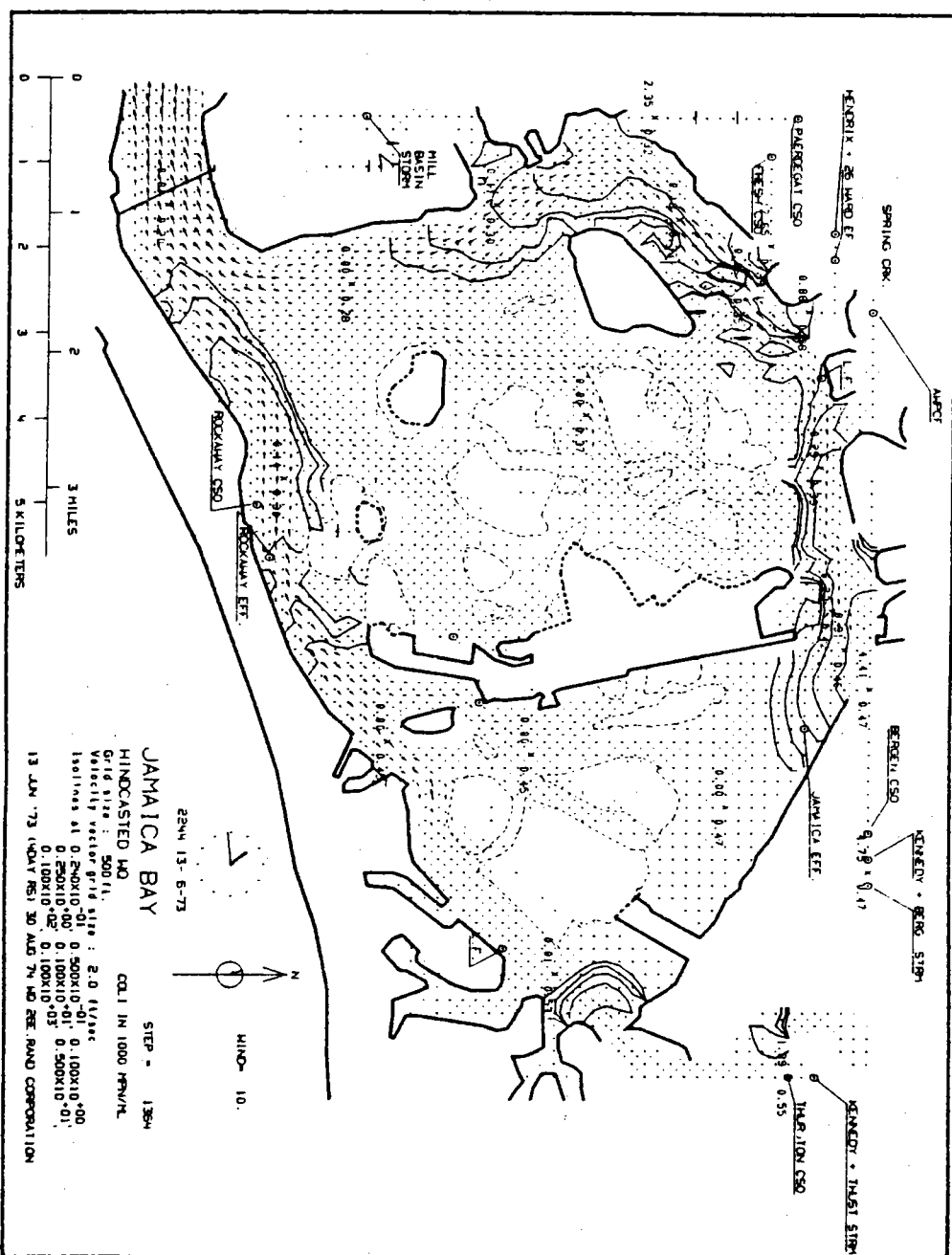


圖 7.12 海域數值模式與都市水文模式聯合運用的實例，左圖為紐約市八個都市洪水及下水道系統。

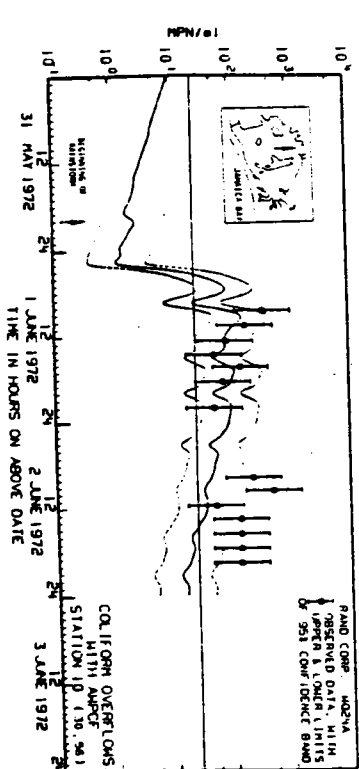
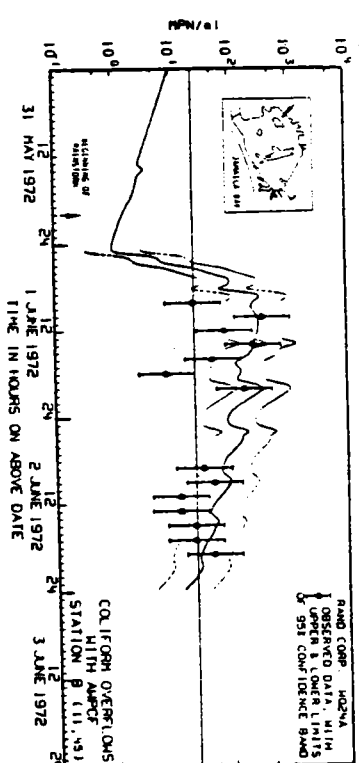
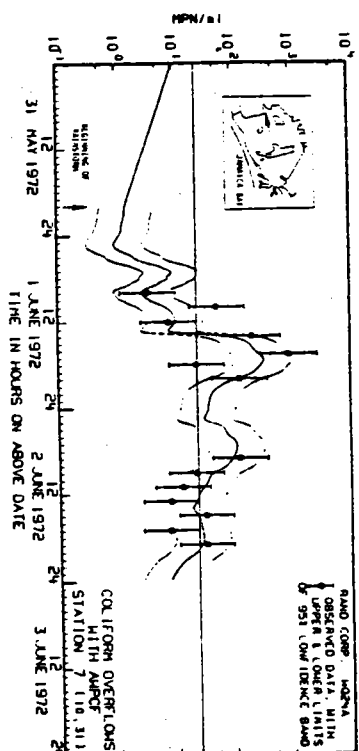
A typical pollutant constituent map from the four-day hindcast (June 13, 1973) showing synoptic tidal current field, pollutant distribution field, numerical values of bacterial concentration, water levels, and wind speed and direction at a particular time, which is 22:44 on June 13, 1973, as indicated by the clock.
(Dotted lines show tidal marshes.)



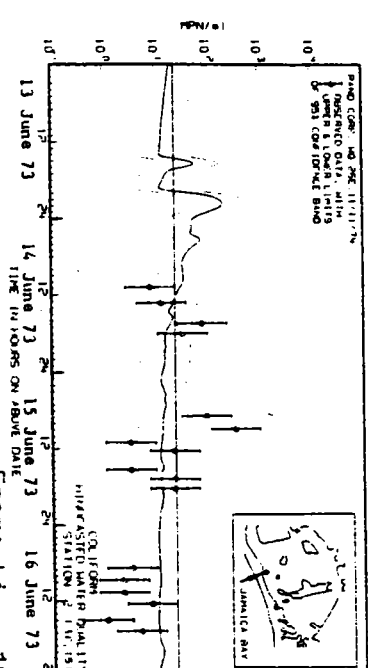
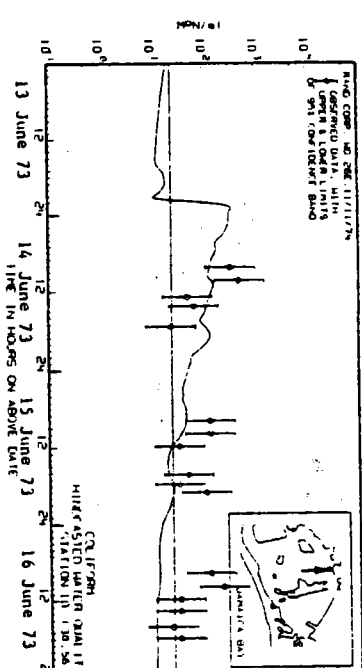
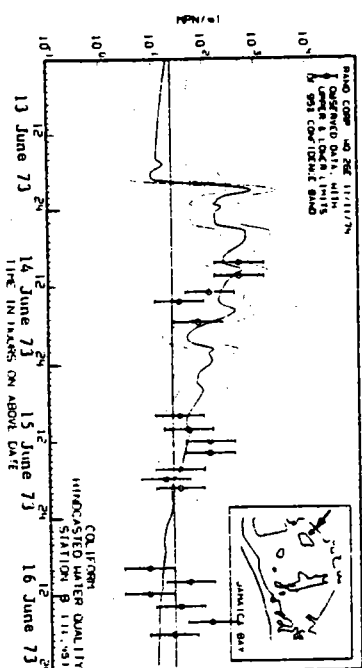
From: Liu, 1974

圖 7.13 紐約八個都市防洪及下水道系統模式由降雨過程線推算逕流及水質，然後再由海域模式接應運算。上圖為暴雨後海灣內之潮流推算及大腸菌密度推算。

Computed and observed time histories of coliform concentration and their associated 95-percent confidence bands at three localities in Jamaica Bay after the storm occurred on May 31, 1972



Indicated and observed time histories of coliform concentration and their associated 95-percent confidence bands at three localities in Jamaica Bay after the storm occurred June 13, 1973



From: Liu, 1974

圖 7.14 紐約模式由降雨推算海城內數點大腸菌濃度及實測濃度之比較

八、典型三度空間海域模式 及電腦程式

劉 肖 孔

8.1 理論根據，計算方法及程式結構

在本章內，我們介紹一個目前相當通用的三度空間海域數值模式，其電腦程式自從公開之後*，因為主程式與副程式在結構上各自獨立，所以使用人很容易配合當時情況，增加或減少。其基本程式自從發表之後，又經過多次增修，納入新發表的研究結果。

這個模式所根據的九個聯立偏微分方程式計有水平及垂直方向的動能平衡方程式：

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial(uu)}{\partial x} + \frac{\partial(uv)}{\partial y} + \frac{\partial(uw)}{\partial z} + fv + \frac{1}{\rho} \frac{\partial p}{\partial x} \\ - \frac{1}{\rho} \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \right) = 0 \end{aligned} \quad (1)$$

$$\begin{aligned} \frac{\partial v}{\partial t} + \frac{\partial(vu)}{\partial x} + \frac{\partial(vv)}{\partial y} + \frac{\partial(vw)}{\partial z} + fu + \frac{1}{\rho} \frac{\partial p}{\partial y} \\ - \frac{1}{\rho} \left(\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} \right) = 0 \end{aligned} \quad (2)$$

$$\frac{\partial p}{\partial z} + \rho g = 0 \quad (3)$$

水的質量不滅定律：

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (4)$$

* Liu, S. K. and A. B. Nelson, (1977) : A three dimensional model of estuaries and coastal seas, Vol. V, Turbulent Energy Program, RAND R-2188. (Listing of the computer programs).

塩度の平衡方程式：

$$\begin{aligned} \frac{\partial s}{\partial t} + \frac{\partial(us)}{\partial x} + \frac{\partial(vs)}{\partial y} + \frac{\partial(ws)}{\partial z} - \frac{\partial[D_x(\partial s / \partial x)]}{\partial x} \\ - \frac{\partial[D_y(\partial s / \partial y)]}{\partial y} - \frac{\partial[k(\partial s / \partial z)]}{\partial z} = 0 \end{aligned} \quad (5)$$

溫度的平衡方程式：

$$\begin{aligned} \frac{\partial T}{\partial t} + \frac{\partial(uT)}{\partial x} + \frac{\partial(vT)}{\partial y} + \frac{\partial(wT)}{\partial z} - \frac{\partial[D_x(\partial T / \partial x)]}{\partial x} \\ - \frac{\partial[D_y(\partial T / \partial y)]}{\partial y} - \frac{\partial[k'(\partial T / \partial z)]}{\partial z} = 0 \end{aligned} \quad (6)$$

代表溫度、塩度與密度三者間關係的形態方程式：

$$\rho = \bar{\rho} + \rho'(s, T) \quad (7)$$

代表「內格點」效應的亂流動能平衡方程式：

$$\begin{aligned} \frac{\partial e}{\partial t} + \frac{\partial(ue)}{\partial x} + \frac{\partial(ve)}{\partial y} + \frac{\partial(we)}{\partial z} - \frac{\partial D_x(\partial e / \partial x)}{\partial x} \\ - \frac{\partial[D_y(\partial e / \partial y)]}{\partial y} - \frac{\partial[k(\partial e / \partial z)]}{\partial z} + hD \pm S_e = 0 \end{aligned} \quad (8)$$

以及可溶性污染物的質量平衡方程式：

$$\begin{aligned} \frac{\partial P}{\partial t} + \frac{\partial(uP)}{\partial x} + \frac{\partial(vP)}{\partial y} + \frac{\partial(wP)}{\partial z} - \frac{\partial D_x(\partial P / \partial x)}{\partial x} \\ - \frac{\partial[D_y(\partial P / \partial y)]}{\partial y} - \frac{\partial[k(\partial P / \partial z)]}{\partial z} + hK_p' \pm S_p = 0 \end{aligned} \quad (9)$$

在垂直方面，將海域以有限的層次來代表分層積分：

$$\langle \rangle_k = \int_{k+\frac{1}{2}}^{k-\frac{1}{2}} () dz$$

$$z = \zeta(x, y, t)$$

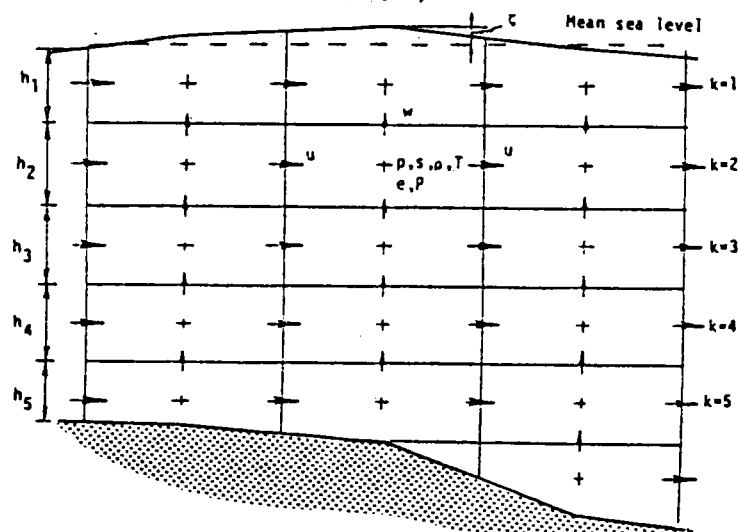


圖 8.1

即可寫出每一層的動力平衡方程式：溫度、鹽度、動能及污染物的平衡方程式：

$$\begin{aligned} & \frac{\partial \langle v \rangle}{\partial t} + \frac{\partial \langle v u \rangle}{\partial x} + \frac{\partial \langle v v \rangle}{\partial y} + (v w)_{k-1/2} - (v w)_{k+1/2} + f \langle u \rangle \\ & + \frac{1}{\bar{\rho}} \left\langle \frac{\partial p}{\partial y} \right\rangle - \frac{1}{\bar{\rho}} \left[(\tau_{k-1/2}^{yz} - \tau_{k+1/2}^{yz}) + \left\langle \frac{\partial \tau_{yx}}{\partial x} \right\rangle + \left\langle \frac{\partial \tau_{yy}}{\partial y} \right\rangle \right] \\ & = 0 \end{aligned} \quad (11)$$

$$\begin{aligned} & \frac{\partial(hu)}{\partial t} + \frac{\partial(huu)}{\partial x} + \frac{\partial(hvu)}{\partial y} + (uw)_{k-1/2} - (uw)_{k+1/2} - fhv \\ & + \frac{h}{\rho} \frac{\partial p}{\partial x} + \left(\frac{1}{\rho} \tau^{xx} \right)_{k+1/2} - \left(\frac{1}{\rho} \tau^{xx} \right)_{k-1/2} - \frac{1}{\rho} \frac{\partial [hA_x (\partial u / \partial x)]}{\partial x} \\ & - \frac{1}{\rho} \frac{\partial [hA_y (\partial u / \partial y)]}{\partial y} = 0 \end{aligned} \quad (12)$$

$$\begin{aligned}
& \frac{\partial(hv)}{\partial t} + \frac{\partial(hvu)}{\partial x} + \frac{\partial(hvv)}{\partial y} + (vw)_{k-1/2} - (vw)_{k+1/2} + fhu \\
& + \frac{h}{\rho} \frac{\partial p}{\partial y} + \left(\frac{1}{\rho} \tau^{yz} \right)_{k+1/2} - \left(\frac{1}{\rho} \tau^{yz} \right)_{k-1/2} - \frac{1}{\rho} \frac{\partial[hA_x(\partial v/\partial x)]}{\partial x} \\
& - \frac{1}{\rho} \frac{\partial[hA_y(\partial v/\partial y)]}{\partial y} = 0
\end{aligned} \tag{13}$$

$$\begin{aligned}
& \frac{\partial(hs)}{\partial t} + \frac{\partial(hus)}{\partial x} + \frac{\partial(hvs)}{\partial y} + (ws)_{k-1/2} - (ws)_{k+1/2} \\
& - \frac{\partial[hD_x(\partial s/\partial x)]}{\partial x} - \frac{\partial[hD_y(\partial s/\partial y)]}{\partial y} + \left(k \frac{\partial s}{\partial z} \right)_{k+1/2} \\
& - \left[k(\partial s/\partial z) \right]_{k-1/2} = 0
\end{aligned} \tag{14}$$

$$\begin{aligned}
& \frac{\partial(hT)}{\partial t} + \frac{\partial(huT)}{\partial x} + \frac{\partial(hvT)}{\partial y} + (wT)_{k-1/2} - (wT)_{k+1/2} \\
& - \frac{\partial[hD_x(\partial T/\partial x)]}{\partial x} - \frac{\partial[hD_y(\partial T/\partial y)]}{\partial y} + \left(k' \frac{\partial T}{\partial z} \right)_{k+1/2} \\
& - \left(k' \frac{\partial T}{\partial z} \right)_{k-1/2} = 0
\end{aligned} \tag{15}$$

$$\begin{aligned}
& \frac{\partial(he)}{\partial t} + \frac{\partial(hue)}{\partial x} + \frac{\partial(hve)}{\partial y} + (we)_{k-1/2} - (we)_{k+1/2} \\
& - \frac{\partial[hD_x(\partial e/\partial x)]}{\partial x} - \frac{\partial[hD_y(\partial e/\partial y)]}{\partial y} + \left(E \cdot \frac{\partial e}{\partial z} \right)_{k+1/2} \\
& - \left(E \cdot \frac{\partial e}{\partial z} \right)_{k-1/2} + S_e - D_e h = 0
\end{aligned} \tag{16}$$

$$\begin{aligned}
& \frac{\partial(hP)}{\partial t} + \frac{\partial(huP)}{\partial x} + \frac{\partial(hvP)}{\partial y} + (wP)_{k-1/2} - (wP)_{k+1/2} \\
& - \frac{\partial[hD_x(\partial P/\partial x)]}{\partial x} - \frac{\partial[hD_y(\partial P/\partial y)]}{\partial y} + \left(k \frac{\partial P}{\partial z} \right)_{k+1/2} \\
& - \left(k \frac{\partial P}{\partial z} \right)_{k-1/2} - D_p h = 0
\end{aligned} \tag{17}$$

依照物質不滅定律，由下至上，分層積分，倒算水位變化；

$$\frac{\partial \zeta}{\partial t} + \sum_{i=1}^{i=b} \left[\frac{\partial(hu)}{\partial x} + \frac{\partial(hv)}{\partial y} \right]_i = 0 \quad (18)$$

再以同樣原理計算每層之間的垂直流速；

$$w_{k-1/2} = - \sum_{i=k}^{i=b} \left[\frac{\partial(hu)}{\partial x} + \frac{\partial(hv)}{\partial y} \right]_i \quad (19)$$

表面層的平面壓力坡度爲：

$$\begin{aligned} \frac{\partial p_1}{\partial x} &= g \rho_1 \frac{\partial \zeta}{\partial x} + \frac{1}{2} g h_1 \frac{\partial \rho_1}{\partial x} \\ \frac{\partial p_1}{\partial y} &= g \rho_1 \frac{\partial \zeta}{\partial y} + \frac{1}{2} g h_1 \frac{\partial \rho_1}{\partial y} \end{aligned} \quad k = 1 \quad (20)$$

依次再算其他各層的平面壓力坡度：

$$\begin{aligned} \frac{\partial p_k}{\partial x} &= \frac{\partial p_{k-1}}{\partial x} + g h_{k-1/2} \frac{\partial \rho_{k-1/2}}{\partial x} \\ \frac{\partial p_k}{\partial y} &= \frac{\partial p_{k-1}}{\partial y} + g h_{k-1/2} \frac{\partial \rho_{k-1/2}}{\partial y} \end{aligned} \quad k = 2, 3, \dots, b \quad (21)$$

$$\rho_{k-1/2} = \frac{1}{2} (\rho_{k-1} + \rho_k)$$

ρ_k = layer average density of layer k

依照以前所討論過的有限差分方式，以簡化縮寫法，模式方程式即爲

：

$$\overline{\delta_1 \zeta^1} = - \sum_k [\delta_x (\bar{h}^x u) + \delta_y (\bar{h}^y v)] \quad \text{at } i, j, n \quad (22)$$

$$\begin{aligned} \overline{\delta_1 (\bar{h}^x u)}^1 &= - \delta_x (\bar{h}^x \bar{u}^x \bar{u}^x) - \delta_y (\bar{h}^y \bar{v}^x \bar{u}^y) - \bar{h}^x o_z (\bar{u}^z \bar{w}^x) \\ &\quad + f \bar{h}^x \bar{v}^{xy} - \frac{1}{\rho^x} \bar{h}^x \delta_x p \\ &\quad + \frac{1}{\rho^x} [h \delta_z E_x \delta_z \bar{u}^{zt} + \delta_x (h A_x \delta_x u) - + \delta_y \bar{h}^x \bar{A}_y^{xy} \delta_y u -] \end{aligned}$$

$$\text{at } i + \frac{1}{2}, j, k, n \quad (23)$$

$$\begin{aligned}
\overline{\delta_1(\bar{h}^y v)}' &= -\delta_x(\bar{h}^x u^y \bar{v}^x) - \delta_y(\bar{h}^y u^y \bar{v}^y) - \bar{h}^y \delta_z(\bar{v}^x \bar{w}^y) \\
&\quad - f \bar{h}^y \bar{u}^{xy} - \frac{1}{\rho^y} \bar{h}^y \delta_y p \\
&\quad + \frac{1}{\rho^y} [h \delta_z E_y \delta_z \bar{v}^{2x} + \delta_x(\bar{h}^y A_x^y \delta_x v)_- + \delta_y(h A_y \delta_y v)_-] \\
&\quad \text{at } i, j + \frac{1}{2}, k, n \quad (24)
\end{aligned}$$

$$\begin{aligned}
\overline{\delta_1(\bar{h} s)}' &= -\delta_x(\bar{h}^x u \bar{s}^x) - \delta_y(\bar{h}^y v \bar{s}^y) - h \delta_z(w \bar{s}^z) \\
&\quad + \delta_x(\bar{h}^x D_x \delta_x s)_- + \delta_y(\bar{h}^y D_y \delta_y s)_- \\
&\quad - h \delta_z(k \delta_z \bar{s}^{2x}) \quad \text{at } i, j, k, n \quad (25)
\end{aligned}$$

$$\begin{aligned}
\overline{\delta_1(\bar{h} T)}' &= -\delta_x(\bar{h}^x u \bar{T}^x) - \delta_y(\bar{h}^y v \bar{T}^y) - h \delta_z(w \bar{T}^z) \\
&\quad + \delta_x(\bar{h}^x D_x \delta_x T)_- + \delta_y(\bar{h}^y D_y \delta_y T)_- \\
&\quad + h \delta_z(k' \delta_z \bar{T}^{2x}) \quad \text{at } i, j, k, n \quad (26)
\end{aligned}$$

$$\begin{aligned}
\overline{\delta_1(\bar{h} e)}' &= -\delta_x(\bar{h}^x u \bar{e}^x) - \delta_y(\bar{h}^y v \bar{e}^y) - h \delta_z(w \bar{e}^z) \\
&\quad + \delta_x(\bar{h}^x D_x \delta_x e)_- + \delta_y(\bar{h}^y D_y \delta_y e)_- \\
&\quad + h \delta_z(E_e \delta_z \bar{e}^{2x}) + S - D \quad \text{at } i, j, k, n \quad (27)
\end{aligned}$$

$$\begin{aligned}
\overline{\delta_1(\bar{h} P)}' &= -\delta_x(\bar{h}^x u \bar{P}^x) - \delta_y(\bar{h}^y v \bar{P}^y) - h \delta_z(w \bar{P}^z) \\
&\quad + \delta_x(\bar{h}^x D_x \delta_x P)_- + \delta_y(\bar{h}^y D_y \delta_y P)_- \\
&\quad + h \delta_z(k \delta_z \bar{P}^{2x}) + S \quad \text{at } i, j, k, n \quad (28)
\end{aligned}$$

其中表面壓力項包括自由表面水位及空氣氣壓兩項。

垂直方向每層的壓力尚包括溫度鹽度的變化、海水的密度比重，係由「形態方程式」求：

$$\begin{aligned}
\rho &= (5890 + 38T - 0.375 T^2 + 3s) / [(1779.5 + 11.25 T \\
&\quad - 0.0745 T^2) - (3.8 + 0.01T) s + 0.698 (5890 \\
&\quad + 38T - 0.375 T^2 + 3s)] \quad \text{at } i, j, k, n + 1 \quad (29)
\end{aligned}$$

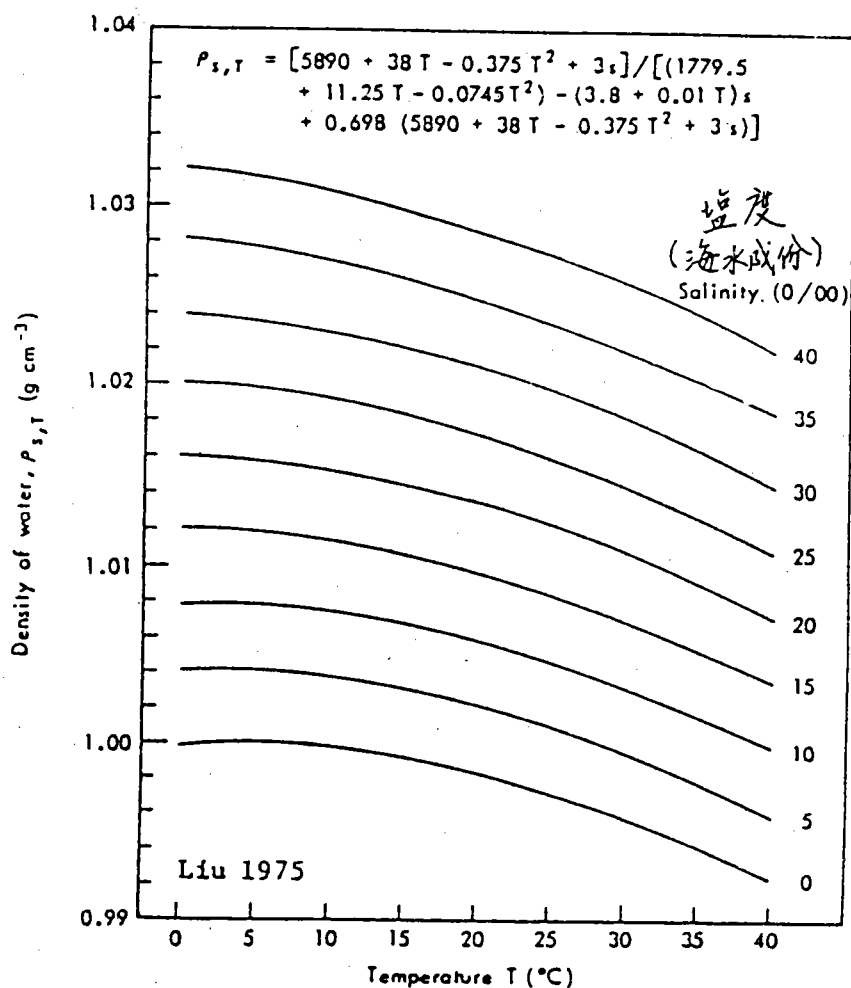


圖 8.2

由上圖可以看出，淡水的密度大約在攝氏四度時最大。

每層之間的垂直流速則由底層向上計算；

$$\delta_z w = -\delta_x (\bar{h}^x u) - \delta_y (\bar{h}^y v) \quad \text{at } i, j, k, n+1 \quad (30)$$

反之，每一層的平面壓力坡度則因水面變化與颱風氣壓的關係，由上向下積分；

$$\delta_x p = g \bar{\rho}^x \delta_x \zeta + \frac{1}{2} \bar{h}^x \delta_x \rho + \delta_x \rho_u \quad \text{at } i + \frac{1}{2}, j, l, n+1$$

(31)

$$\delta_z p = g \bar{\rho}' \delta_z \zeta + \frac{1}{2} \bar{h}' \delta_z \rho + \delta_z p_a \quad \text{at } i, j + \frac{1}{2}, l, n+1 \quad (32)$$

$$\delta_z (\delta_x p) = g \delta_x \bar{\rho}' \quad \text{at } i + \frac{1}{2}, j, k + \frac{1}{2}, n+1 \quad (33)$$

$$\delta_z (\delta_y p) = g \delta_y \bar{\rho}' \quad \text{at } i, j + \frac{1}{2}, k + \frac{1}{2}, n+1 \quad (34)$$

然後，計算水平方向的流速變化；

$$\begin{aligned} \overline{\delta_1(\bar{h}^x u)}' &= -\delta_x(\bar{h}^x u^x \bar{u}^x) - \delta_y(\bar{h}^y v^x \bar{u}^y) - \bar{h}^x \delta_z(\bar{u}^x \bar{w}^x) \\ &\quad + f \bar{h}^x \bar{v}^{xy} - \frac{1}{\rho^x} \bar{h}^x \delta_x p + \frac{1}{\rho^x} [\theta \rho_a w_a^z \sin \phi \\ &\quad - (E_x \delta_z \bar{u}^{zt})_{k=3/2} + \delta_x(h A_x \delta_x u) - \\ &\quad + \delta_y(\bar{h}^x \bar{A}_x^y \delta_y u) -] \quad \text{at } i + \frac{1}{2}, j, l, n \quad (35) \\ \overline{\delta_1(\bar{h}^y v)}' &= -\delta_x(\bar{h}^x u^y \bar{v}^x) - \delta_y(\bar{h}^y v^y \bar{v}^y) - \bar{h}^y \delta_z(\bar{v}^y \bar{w}^y) \\ &\quad - f \bar{h}^y \bar{u}^{xy} - \frac{1}{\rho^y} \bar{h}^y \delta_y p + \frac{1}{\rho^y} [\theta \rho_a w_a^z \cos \phi \\ &\quad - (E_y \delta_z \bar{v}^{zt})_{k=3/2} + \delta_x(\bar{h}^y \bar{A}_x^y \delta_x v) - \\ &\quad + \delta_y(h A_y \delta_y v) -] \quad \text{at } i, j + \frac{1}{2}, l, n \quad (36) \end{aligned}$$

以上兩式為表面層的動力平衡式，其中風的剪力係數（ θ ），為當地、當時海面溫度（模式計算）與空氣溫度的函數，由表一及表二中可以看出，當空氣溫度高於海面溫度時，水面的摩擦係數與波浪高度稍低於無溫差時。反之，如果空氣的溫度低於水面的溫度，其摩擦係數反較無溫差時為大。因此同樣的風速可產生不一樣高度的暴潮與氣象潮。

這個基本原理，非但在大氣中如此，在垂直方向有「溫差」的海

表 8.1

SEA SURFACE WIND STRESS COEFFICIENTS FOR VARIOUS
WIND SPEED AND AIR-SEA TEMPERATURE DIFFERENCES

Wind Speed in M/S	STRESS COEFFICIENTS		
	Air-Sea Temperature Difference (C)		
	0.0	5.0	-5.0
5.0	0.00110	0.00029	0.00148
10.0	0.00137	0.00113	0.00160
15.0	0.00166	0.00145	0.00176
20.0	0.00191	0.00174	0.00202
25.0	0.00220	0.00205	0.00227
30.0	0.00250	0.00232	0.00257
35.0	0.00274	0.00291	0.00286
40.0	0.00303	0.00291	0.00313
45.0	0.00330	0.00320	0.00339
50.0	0.00360	0.00349	0.00366
55.0	0.00389	0.00376	0.00393
60.0	0.00412	0.00408	0.00421
65.0	0.00443	0.00435	0.00447
70.0	0.00470	0.00462	0.00475

表 8.2

RELATIONSHIP BETWEEN AIR-SEA TEMPERATURE DIFFERENCE
AND THE FULLY DEVELOPED OCEAN WAVES (H 1/3) UNDER
VARVARIOUS WIND SPEEDS

10-Meter Wind Speed (M/S)	SIGNIFICANT WAVE HEIGHTS IN METERS		
	Air-Sea Temperature Difference (C)		
	0.0	5.0	-5.0
0.0	0.0	0.0	0.0
2.5	0.241	0.093	0.500
5.0	0.624	0.335	0.900
7.5	1.510	0.710	1.820
10.0	2.550	1.520	2.800
12.5	3.750	2.900	4.380
15.0	5.500	4.380	6.330
17.5	7.490	6.350	8.400
20.0	9.800	8.700	10.650
22.5	12.320	11.250	13.100
25.0	14.700	13.650	15.450
27.5	17.300	16.370	17.900

水中亦復如此。熱水密度低，易浮於上層，上下溫差越大的，其「穩定性」愈高。

在海域模式中，計算穩定性，常常使用「李氏係數」(Richardson number)， $Ri = -\frac{g(\partial\rho/\partial z)}{\rho(\partial u/\partial z)^2}$ 。在上下密度相等的「混合層」中， $Ri = 0$ ， Ri 愈大，則流況愈穩定。如果 Ri 小於零，則呈上重下輕的不穩定現象。層次間的摩擦作用(動量轉移，vertical momentum transfer)也因此增加。

台海及中國海域模式所採用的方法是根據最新發表的研究結果，將動能平衡方程式中的最後兩項(見Eq. 8)展開。

$$\begin{aligned} \bar{S}_e^* - D_e = & \underbrace{a_3 \overline{L \sqrt{e}}^2 \{ (\delta_z \bar{u}^*)^2 + (\delta_z \bar{v}^*)^2 \}}_{(1)} \\ & + \underbrace{a_3 \overline{L \sqrt{e}}^2 \frac{g}{\bar{h}^* \rho} (\delta_z \bar{\rho}^*)}_{(2)} - \underbrace{a_2 e^{3/2} / L}_{(3)} \end{aligned}$$

此種動能平衡計算方法的原理(Liu & Leendertse, 1981, 82, 83, 84, 85)是將垂直方向流體混合時所損失的「次格點動能」(SGS energy)在當地的「位能項」中加入。因為不同密度的流體在垂直方向混合以後，改變了當地的「平均位能」。

上式中的第一項為當地、當時的動能增加率，第二項為位能增加率，最後一項為動能損失率。

這種方法將原先所需要的一些時空變數改為兩個流體力學中相當可靠的「常數」，因而較為準確。同時因為採用流力中的通用常數，所以在海域中計算出來的結果與類似的大氣模式，甚至於管中的「氣流」都可以。

8.2 垂直方向的計算法則

早期的數值模式，在垂直方向的動量轉移計算，多半採用含有李氏係數的經驗公式。

$$E_z = \bar{\rho}^{xz} \overline{L \sqrt{e_-}}^{xz} \exp \left[m \frac{g}{\bar{\rho}^{xz}} \frac{(\bar{L}^z)^2 \delta_z(\bar{\rho}^z)}{\bar{e_-}^{xz}} \right]$$

$$E_y = \bar{\rho}^{yz} \overline{L \sqrt{e_-}}^{yz} \exp \left[m \frac{g}{\bar{\rho}^{yz}} \frac{(\bar{L}^z)^2 \delta_z(\bar{\rho}^y)}{\bar{e_-}^{yz}} \right]$$

$$k = a_4 \overline{L \sqrt{e_-}}^z \exp \left[r \frac{g}{\bar{\rho}^z} (\bar{L}^z)^2 \frac{\delta_z \rho}{\bar{e_-}^z} \right]$$

$$E_e = a_1 \overline{L \sqrt{e_-}}^z \exp \left[m \frac{g}{\bar{\rho}^z} (\bar{L}^z) \frac{\delta_z \rho}{\bar{e_-}^z} \right]$$

以上四個式子是計算「垂直動能轉移」及「溫度塩度溫合」的定量公式，括弧內的計算項目即含有「李氏係數」的效應*。在1977年以前所發表的模式，在亂流的計算方面，多半採用此類方法。其中的係數m, r要靠實地測量，因此是一個很大的缺點，因為敏感度要求高，所以準確性低*。

相互比較，以下圖為例，(Liu, 1983)。圖中實線為海域三度模式所計算的層次流，圓點為修正後的非層次「混合流」，點線為銅管中的實測氣流，三種完全不同流體及流況，但經過標準化**以後，在紊流、主流以及邊界層三者間的相對關係上是一致的。由數值模式所計算的海流與實驗室中所測出的銅管中氣流，兩者相同，代表數值模式中「紊流封閉性問題」***處理的正確性。

* Liu, S. K., and J. J. Leendertse, (1978) :Multi-Dimensional Numerical Modeling of Estuaries and Coastal Seas, in Advances in Hydroscience, Academic Press, New York.

** 「邊界層距離」的 normalization

*** Turbulent-closure schome.

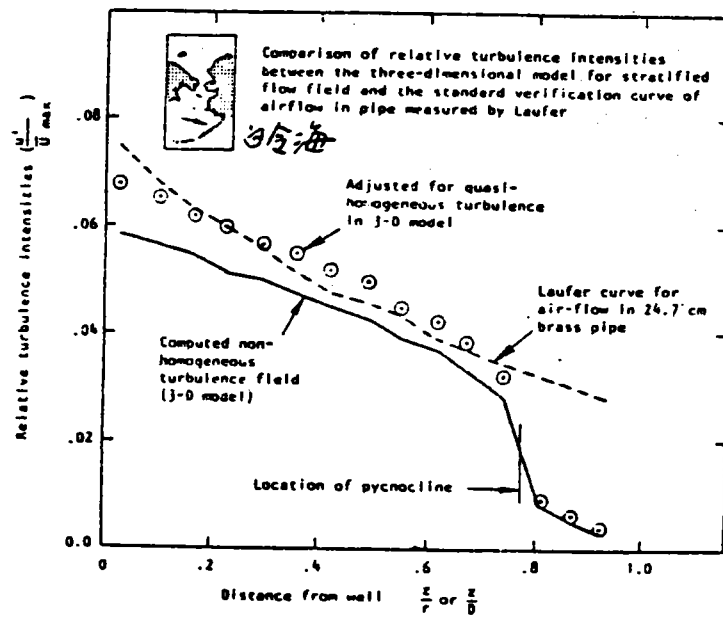


圖 8.3

8.3 數值模式變數定義

MAJOR SYMBOLS

Note: Other symbols are defined in the text.

a_1, a_2, a_3, a_4	Constants	K	Bottom layer
A_x	Horizontal momentum exchange coefficient in the x direction	L	Length scale
A_y	Horizontal momentum exchange coefficient in the y direction	m	Index for the number of constituent
C	Chezy coefficient	n	Time level
D_x	Horizontal diffusion coefficient in the x direction	p	Pressure
D_y	Horizontal diffusion coefficient in the y direction	P	Concentration of any dissolved constituents
e	Subgridscale kinetic energy density	q_x, q_y	Tidal flow
E_e	Vertical exchange coefficient for subgridscale turbulent energy	r	A constant
E_p	Vertical exchange coefficient for dissolved pollutant concentration	s	Salinity
E_x	Vertical momentum exchange coefficient in the x direction	t	Time
E_y	Vertical momentum exchange coefficient in the y direction	T	Temperature
f	Coriolis parameter	u, v, w	Respective components of velocity
g	Acceleration of gravity	U, V	Respective components of horizontal mean velocity
h	Water depth measured downward from the reference plane to the bottom	w_a	Wind velocity
i	A location index	x, y, z	Cartesian coordinates: positive eastward, northward, and upward, respectively
j	A location index	Δt	Time step size
k	Index for layer or level	Δl	Grid distance
$\tau_{xx}, \tau_{xy}, \tau_{yx}$	Components of the stress tensor	θ	Wind stress coefficient
$\tau_{yy}, \tau_{xz}, \tau_{yz}$	Component of the wind stress in the x direction	κ	Vertical mass diffusion coefficient
τ_x^s	Component of the wind stress in the x direction	κ'	Vertical thermal diffusion coefficient
τ_y^s	Component of the wind stress in the y direction	γ	A constant
		ρ	Water density
		$\bar{\rho}$	Reference density, a constant
		ρ'	Departure from $\bar{\rho}$, depending on salinity and temperature
		ρ_a	Atmospheric density
		ψ	Angle between the wind direction and the y axis
		ζ	Water level, vertical distance from the reference plane

8.4 三度空間海域模式電腦程式結構

PROGRAM DESCRIPTION

The computational program consists of a main program and a set of subroutines written in FORTRAN IV. It operates on IBM 360/370 series machines.

The main program reads and prints all control parameters and input data. It also administers the output display, and controls saving and reading of data for the restart of simulation. The subroutines are listed and described in Table 1. Table 2 gives dimensions for the arrays in the blank COMMON block, and Table 3 for the arrays in program MAIN.

Table 8.3
DESCRIPTION OF SUBROUTINES

DSVP	Computes local density of water from most recent values of temperature and salinity according to equation of state.
PRNT1— PRNT6	Control the format of output reports according to user's specification.
RESAL	Specifies manner in which initial values of salinity field are to be read at beginning of simulation. (If a constant salinity field is specified, a "RETURN" instruction should be inserted at the beginning of this subroutine to bypass its execution.)
SESUV	Computes new water level. Sets up conditions for computation of velocity and constituent concentration at closed and open boundaries. Computes constituent concentrations at upper time level for entire field. Computes velocity components u and v at upper time level for entire field.
SETCON	Sets up initial conditions for time integration and defines constants used frequently in computation.
SGSLOS	Computes constituent concentrations across open boundaries.
TIDAL	Specifies water level information at open boundary of model. In the sample program, specifies angular frequencies of, and Fourier coefficients associated with, major tidal components. (If model is used for an enclosed body of water, such as a lake, this subroutine should be bypassed.)
TRANS	Handles output of computed results to a sequence of time series data at user-specified locations.

Table 8-4
 DIMENSIONS OF ARRAYS IN BLANK
 COMMON BLOCK

Arrays	Dimensions
U, V, S, T, E, P	N, M, KMAX, 3
W, TAUX, TAUZ, EPSILON	N, M, KMAX+1
PGRAX, PGRAY, RHO	N, M, KMAX
SE	N, M, 3
KH, C, RHOZ, HZ, NSE	N, M
AHUV	N, 3, KMAX
DEPTH	KMAX
HS	KMAX+1
YY, ZL	KMAX, KMAX

Table 8-5
 DIMENSIONS OF ARRAYS IN
 PROGRAM MAIN

Arrays	Dimensions
UG1, UG2, UG3, VG1, VG2, VG3, SG1, SG2, SG3, TG1, TG2, TG3, EG1, EG2, EG3, PG1, PG2, PG3	N, M, KMAX
SEG1, SEG2, SEG3, PRINT	
	N, M

8.5 輸入資料之處理

INPUT PARAMETERS

Making a numerical simulation using the model presented in this report requires the insertion of several groups of data for the following purposes:

- To specify control parameters for the execution of machine computation.
- To describe physical characteristics of the system to be simulated.
- To set up initial conditions for the numerical integration.
- To provide boundary conditions during the simulation.
- To select coefficients and constants for the computation.
- To specify time intervals and locations for displaying results.

To describe the requirements and conditions for model setup and data insertion, we use a sample data sequence involving a rectangular basin that has a time-varying water level input at the open boundary and a river inflow at the closed end. Appendix C contains a table showing the location, format, name, and description of the input data, as well as a listing of the sample run's actual data deck. Results of the sample simulation are presented in App.

8.6 計算機容量之需求

IMPLEMENTATION AND OPERATIONAL ASPECTS

The program presented here can be implemented on computer systems with an extended FORTRAN compiler that accepts variables with four subscripts. The core size and central processing unit (CPU) time requirements and the input/output (I/O) requirements for model implementation depend on the grid dimensions of the model schematization, the size of the time integration steps, the user-specified frequency for output display, and the computer installation.

Experience with the FORTRAN H compiler and the IBM 370/158 system has shown that in the sample case, a rectangular model with a grid dimension of $10 \times 26 \times 6$ uses approximately 348,000 bytes of core memory, and each simulation time step requires about 4.4 seconds of CPU time. The compilation time was approximately 115 seconds. The Long Island Sound sample case described in Leendertse and Liu [1] has a grid dimension of $24 \times 79 \times 9$. With the FORTRAN H compiler on an IBM 360/91 machine, the model uses approximately 2 million bytes of core memory, and each simulation time step requires approximately 3 seconds of CPU time.

In addition to the core space required for making a simulation with the model, the system may require up to four data files for accumulating data for later processing or restart. These unformatted files contain the following data:

File 1 - TITLE is written as the first record, and KMAX, LMAX, M, N, DX, DY, TD, HS, KH, C are written as the second record. Then arrays of the state variables are saved at time level L1. The following records are written at intervals of LGRPH:

Rec. 1 - L, SE
Rec. 2 - W
Rec. 3 - U
Rec. 4 - V
Rec. 5 - S
Rec. 6 - T
Rec. 7 - E
Rec. 8 - P

Files 2 and 3 - Data required for restarting the simulation are saved at all three time levels. The following records are written at intervals of LRST:

Rec. 1 - L, L1, L2, L3, IB, JB, KB, UVB, KH, HS
Rec. 2 - CP0, CLAM, CLAM0, CP1, CLAM1, RHOT, RMM1, RMM2
Rec. 3 - M, MM1, MM2, N, NM1, NM2, DX, DY, RDX, RDY, RDXS, RDYS
Rec. 4 - DT, DT2, DT4, CNU, GRAV, KMAX, CKA, AHX, AHY, AMX, AMY, F
Rec. 5 - RHOM, RHOM2, ALPH0, TINV, THETA, RHOA, WSPD, PHI, WINCH

Rec. 6 - C, RHOZ, HZ, UFLOS, UFLOT, UFLOE, UFLOP, LTURN,
 STURN, TTURN, ETURN, PTURN
 Rec. 7 - W, TAUX, TAUY
 Rec. 8 - PGRAX, PGRAY, RHO
 Rec. 9 - U
 Rec. 10 - V
 Rec. 11 - S
 Rec. 12 - T
 Rec. 13 - E
 Rec. 14 - P
 Rec. 15 - SE

File 4 - Data are saved at specified locations and used for time histories. This file is written using subroutine TRANS and may be either sequential or direct access. Fixed size records are written when a temporary buffer is filled. Initially, TITLE, DATE, DT, LMAX, LHIST, MHIST, NOWL3, NOCUR3, MWL3, NWL3, TITWL3, MC3, NC3, KC3, TITC3, and some flags are saved. At intervals of LHIST, time (L) and the values of water level (SE) and of current (u and v) at the designated stations are saved. When the run terminates, the maximum and minimum values of water level and current are saved.

If one or more of the above files is not desired, a dummy output unit must be specified in the job control language (JCL). For a non-IBM computer, the JCL describing the job and data file must be converted to appropriate form.

8.7 電腦程式變數定義

DEFINITION OF FORTRAN VARIABLES (* indicates variable array)

FORTRAN Symbol	Definition and Units
A	Temporary variable for wind speed (cm/sec).
B	Temporary variable for wind-generated kinetic energy (ergs/unit mass).
*C	Values for Chezy coefficient (square-root of cm/sec).
*E	Variables for kinetic energy density (ergs/unit mass).
F	Coriolis parameter.
H	Temporary value for layer thickness (cm).
I	Index for spatial differencing in x-direction.
J	Index for spatial differencing in y-direction.
K	Index for vertical grid.
L	Index for time level.
M	Index used for max number of grid points in x-direction.
N	Index for the max grid number in y-direction.
*P	Variables for pollutant concentration in a user-specified unit.
*S	Variables for salinity (gr/kg, or approx gr/l).
*T	Variables for temperature in Celsius.
*U	Velocity components in x-direction (cm/sec).
*V	Velocity components in y-direction in (cm/sec).
*W	Velocity components in z-direction in (cm/sec).
AE	Recursion coef of turbulent energy diffusion in z-direction.
AP	Recursion coef for vertical implicit computation of pollutant constituent diffusion in z-direction.
AS	Recursion coef for vertical implicit computation of salt diffusion in z-direction.
AT	Recursion coef for vertical implicit computation of heat diffusion in z-direction.
AU	Recursion coef for vertical implicit computation of momentum transfer in x-direction.
AV	Recursion coef for vertical implicit computation of momentum transfer in y-direction.
A1	Coef relating transfer coef of vertical exchange of momentum to transfer coef for vertical exchange of kinetic energy.
A2	Coef for estimating decay of kinetic energy density.
A3	Coef for turbulent energy generation at bottom.

A4	Coef relating vertical exchange coef of mass and heat to universal vertical exchange coef as derived from local subgridscale turbulent energy intensity.
BE	Recursion coef for vertical implicit computation of turbulent energy diffusion in z-direction.
BP	Recursion coef for vertical implicit computation of pollutant constituent diffusion in z-direction.
BS	Recursion coef for vertical implicit computation of salt transfer.
BT	Recursion coef for vertical implicit computation of heat transfer.
BU	Recursion coef for vertical implicit computation of momentum transfer in x-direction.
BV	Recursion coef for vertical implicit computation of momentum transfer in y-direction.
CB	Averaged value of Chezy coef in x-direction (square-root of cm/sec).
CP	Recursion coef for vertical implicit computation of pollutant constituents transport in z-direction.
CS	Recursion coef for vertical implicit computation of salt transport in z-direction.
CT	Input value for Chezy coef (square-root of cm/sec).
CU	Recursion coef for vertical implicit computation of momentum transfer in x-direction.
CV	Recursion coef for vertical implicit computation of momentum transfer in y-direction.
DG	Recursion coef for vertical computation of turbulent energy transfer in z-direction.
DP	Recursion coef for vertical implicit computation of pollutant transfer in z-direction.
DS	Recursion coef for vertical implicit computation of salt transfer in z-direction.
DT	Value of time step for integration (sec).
DU	Recursion coef for vertical implicit computation of momentum transfer in x-direction.
DV	Recursion coef for vertical implicit computation of momentum transfer in y-direction.
DX	Horizontal grid size in x-direction (cm).
DY	Horizontal grid size in y-direction (cm).
EM	Total value of computed $e(j,i,k,l)$ in the field.
EN	Variable for SGS energy density (ergs/unit mass).
HB	Averaged value of layer thickness in x-direction (cm).
HC	Surface layer thickness for time level L3 (cm).
HL	Temporary variable.
HQ	Averaged value of layer thickness in z-direction (cm).

HR	Average surface layer thickness in y-direction (cm).
*HS	Thickness of each layer in z-direction.
HT	Temporary variable.
HZ	Unused parameter.
H2	Temporary variable for layer thickness (cm).
*IB	Location indices for open boundary in x.
IH	Location index for water level station in x.
II	Index.
IJ	Index.
*JB	Location indices for open boundary in y.
JH	Temporary variable.
J1	Index.
*KB	Input parameters used to specify type of opening.
*KH	Number of layers at location x,y.
KK	Index.
KR	Number of layers at location i + 1.
KT	Number of layers at location j + 1.
K1	Maximum number of layers.
K2	Index.
K3	Index.
LB	Unused parameter.
LC	Control parameter for frequency of time smoothing.
LF	Index.
L1	Time level index.
L2	Time level index.
L3	Time level index.
NH	Index.
PI	3.1415926
PM	Cumulated value for pollutant concentration.
Q1	Temporary variable similar to Q2, Q3, Q4, etc.
RU	Bottom stress for vertical implicit computation in x-direction.
RV	Bottom stress for vertical implicit computation in y-direction.
*SE	Water surface elevations (cm) from reference plane.
TD	Size of time step (sec).
T1	Temporary variable similar to T2, T3, T4, etc.
UB	Averaged velocity component in x-direction.
UU	Temporary variable.
VB	Averaged velocity component in y-direction.
VV	Temporary variable.
WL	Temporary variable.
WO	Temporary variable.
XE	Temporary variable.
XL	Floating point value of time step number.

XP	Recursion coef for implicit computation of pollutant diffusion in z-direction.
XS	Temporary variable.
XT	Temporary variable.
XU	Recursion coef for implicit computation in x-direction.
XV	Recursion coef for implicit computation in y-direction.
YP	Temporary variable.
YT	Temporary variable.
YU	Recursion coef for implicit computation in x-direction.
YV	Recursion coef for implicit computation in y-direction.
*YY	Distances from bottom to point of concern.
*ZL	Length scales in turbulent energy computation.
ABS	Absolute value of variable.
AHX	Horizontal mass-heat diffusion coef in x.
AHY	Horizontal mass-heat diffusion coef in y.
AMX	Horizontal momentum diffusion coef in x.
AMY	Horizontal momentum diffusion coef in y.
*BUF	Indices for time history data output.
CKA	Coef of vertical diffusion of mass.
CKE	Coef of vertical diffusion of kinetic energy.
CKH	Coef of vertical diffusion of heat.
CKP	Coef of vertical diffusion of pollutant.
CNU	Coef of vertical diffusion of momentum.
CP0	Constant in the equation of state.
CP1	Constant in the equation of state.
CTT	Temporary variable.
DEN	Temporary variable.
DHE	Temporary variable.
DHP	Temporary variable.
DHS	Temporary variable.
DHT	Temporary variable.
DHU	Temporary variable.
DHV	Temporary variable.
DTT	Temporary variable.
DT2	Temporary variable.
DT4	Temporary variable.
EBZ	Temporary variable.
EEM	Temporary variable.
EEP	Temporary variable.
*EG1	Values of energy for time level L1.
*EG2	Values of energy for time level L2.
*EG3	Values of energy for time level L3.
EJI	Temporary variable.
GEN	Temporary variable.

HBL	Temporary variable.
HBR	Temporary variable.
HBT	Temporary variable.
HB3	Temporary variable.
HEK	Temporary variable.
HPK	Temporary variable.
HSB	Temporary variable.
HSK	Temporary variable.
HSL	Temporary variable.
HSR	Temporary variable.
HST	Temporary variable.
HS1	Temporary variable.
HTD	Temporary variable.
HTK	Temporary variable.
HTM	Temporary variable.
HTS	Temporary variable.
HTX	Temporary variable.
HTY	Temporary variable.
IAH	Temporary variable.
IBT	Location index.
IM1	Temporary index.
IP1	Temporary index.
JBT	Location index.
JM1	Temporary index.
JP1	Temporary index.
K1U	Temporary variable.
K1V	Temporary variable.
KBT	Temporary variable.
KBT	Type of opening.
*KC3	Number of layers at current station.
KC3	Number of layers at specified current station.
KHT	Number of layer if entire model has same number of layers.
KI1	Index.
KI2	Index.
KI3	Index.
KI4	Index.
KKK	Index.
KM1	Temporary index.
KN1	Index.
KP1	Temporary index.
KP1	Index.
KQ1	Index.
LP1	Print-control parameter.
LP2	Print-control parameter.
LP3	Print-control parameter.
LP4	Print-control parameter.
LST	Time step to restart computation.

*MC3	Location indices for current station.
MM1	Temporary location index.
MM2	Temporary location index.
MOD	Modulus.
*NC3	Location indices for current station.
NM1	Location index.
NM2	Location index.
*NSE	Temporary variable.
NWD	Index.
*PG1	Temporary variable.
*PG2	Temporary variable.
*PG3	Temporary variable.
PHI	Clockwise angle (deg) between model's y-axis and direction into which wind is blowing.
PJI	Temporary variable.
PPM	Temporary variable.
PPP	Temporary variable.
Q2A	Temporary variable.
Q4A	Temporary variable.
Q6A	Temporary variable.
Q8A	Temporary variable.
RDX	Temporary constant.
RDY	Temporary constant.
*RHO	Density anomaly from mean density (RHOM).
RIX	Richardson number for momentum transfer in x.
RIY	Richardson number for momentum transfer in y.
SAL	Cumulative value of salinity.
SEM	Cumulative value of computed water level.
SET	Unused parameter.
*SG1	Values of salinity for time level L1.
*SG2	Values of salinity for time level L2.
*SG3	Values of salinity for time level L3.
SJI	Temporary variable.
SSM	Temporary variable.
SSP	Temporary variable.
TD2	Temporary constant.
TD4	Temporary constant.
TEM	Cumulative value of computed temperature of entire field.
*TG1	Values of temperature for time level L1.
*TG2	Values of temperature for time level L2.
*TG3	Values of temperature for time level L3.
TJI	Temporary variable.
TOM	Cumulative value of computed water level.
TTM	Temporary variable.
TTP	Temporary variable.
UBK	Temporary variable.
UBM	Temporary variable.

UBP	Temporary variable.
*UG1	Values of u-component for time level L1.
*UG2	Values of u-component for time level L2.
*UG3	Values of u-component for time level L3.
UUM	Temporary variable.
UUP	Temporary variable.
*UVB	Specified velocity components at open boundary.
VBK	Temporary variable.
VBM	Temporary variable.
VBP	Temporary variable.
VEL	Temporary variable.
*VG1	Values of v-component for time level L1.
*VG2	Values of v-component for time level L2.
*VG3	Values of v-component for time level L3.
VVM	Temporary variable.
VVP	Temporary variable.
WBX	Temporary variable.
WBY	Temporary variable.
ZLB	Temporary variable.
*AHUV	Horizontal diffusion coef derived from deformation of velocity field.
AHXP	Fixed portion of horizontal diffusion coef for transport of pollutant in x-direction.
*AHXT	Fixed portion of horizontal diffusion coef for transport of heat in x-direction.
AHYP	Fixed portion of horizontal diffusion coef for transport of pollutant in y-direction.
AHYT	Fixed portion of horizontal diffusion coef for transport of heat in y-direction.
BND1	Temporary control parameter.
BND2	Temporary control parameter.
BND3	Temporary control parameter.
BND4	Temporary control parameter.
BND5	Temporary control parameter.
CLAM	Constant in the equation of state.
DHEI	Temporary variable.
DISP	Temporary variable.
DRHO	Temporary variable.
*EINI	Specified initial values at open boundary for estimating concentration recovery of SGS turbulent energy density.
EINV	Initial value for turbulent energy density specified for entire field.
GRAV	Gravitational acceleration.
HBB3	Temporary variable for average layer thickness.
HBLB	Temporary variable for average layer thickness.
HBL3	Temporary variable for average layer thickness.
HBRB	Temporary variable for average layer thickness.

HBRT	Temporary variable for average layer thickness.
HBR3	Temporary variable for average layer thickness.
HBTL	Temporary variable for average layer thickness.
HBTR	Temporary variable for average layer thickness.
HBT3	Temporary variable for average layer thickness.
HBU1	Temporary variable for average layer thickness.
HBV1	Temporary variable for average layer thickness.
HESQ	Temporary variable in turbulent energy computation.
IAHM	Temporary index for horizontal diffusion coefficient.
IAHP	Temporary index for horizontal diffusion coefficient.
IMC3	Temporary index.
INC3	Temporary index.
KBST	Temporary constant denoting layer number at open boundary point.
KCUR	Layer number at current station.
KLP2	Print-control parameter.
KMAP	Print-control parameter.
KMAX	Maximum number of layers assigned for vertical grid dimension.
LMAX	Number of time steps specified for simulation.
LRST	Frequency to write restart file.
*MWL3	Location indices for water-level station in x-direction.
NREC	Temporary index for data management.
*NWL3	Location indices for water-level station in y-direction.
*PINI	Specified initial values at open boundary for estimating concentration recovery of pollutant constituent.
PINV	Initial value of pollutant concentration field.
RDXS	Reciprocal of square of x-grid spacing.
RDYS	Reciprocal of square of y-grid spacing.
RHOA	Density of air.
RHOM	Mean value for computing density anomaly, usually set to 1.0.
RHOT	Actual value of water density.
RHOZ	Twice the vertically averaged value of density.
RICH	Richardson number.
RMM1	Coef used for Richardson number-related vertical exchange coef.
RMM2	Coef used for Richardson number-related vertical exchange coef.
*SEG1	Values of computed water level for time level L1.
*SEG2	Values of computed water level for time level L2.
*SEG3	Values of computed water level for time level L3.

*SINI	Specified initial values for salinity at open boundary.
STMP	Initially specified value of salinity.
*TAUX	Inter-layer shear-stress terms in x-direction.
*TAUY	Inter-layer shear-stress terms in y-direction.
TEMP	Unused parameter.
*TINI	Specified initial values at open boundaries.
TINV	Initially specified value of temperature array.
UBAR	Temporary variable.
VBAR	Temporary variable.
WBXM	Temporary variable.
WBXP	Temporary variable.
WBYM	Temporary variable.
WBYP	Temporary variable.
WSPD	Input windspeed (knots).
WSPS	Converted wind speed (cm/sec).
ZEXP	Temporary variable.
*ZWL3	Water levels at water-level stations.
AHXBB	Temporary variable for averaged values of horizontal diffusion coefficient.
AHXLB	Temporary variable for averaged values of horizontal diffusion coefficient.
AHXPL	Temporary variable for averaged values of horizontal diffusion coef.
AHXPR	Temporary variable for averaged values of horizontal diffusion coef.
AHXRB	Temporary variable for averaged values of horizontal diffusion coef.
AHXSL	Temporary variable for averaged values of horizontal diffusion coef.
AHXSR	Temporary variable for averaged values of horizontal diffusion coef.
AHXTB	Temporary variable for averaged values of horizontal diffusion coef.
AHXTL	Temporary variable for averaged values of horizontal diffusion coef.
AHXTR	Temporary variable for averaged values of horizontal diffusion coef.
AHYBB	Temporary variable for averaged values of horizontal diffusion coef.
AHYLB	Temporary variable for averaged values of horizontal diffusion coef.
AHYPB	Temporary variable for averaged values of horizontal diffusion coef.
AHYPT	Temporary variable for averaged values of horizontal diffusion coef.
AHYRB	Temporary variable for averaged values of horizontal diffusion coef.

AHYST	Temporary variable for averaged values of horizontal diffusion coef.
AHYTB	Temporary variable for averaged values of horizontal diffusion coef.
AHYTT	Temporary variable for averaged values of horizontal diffusion coef.
ALPH0	Constant used in equation of state.
CLAM0	Constant used in equation of state.
CLAM1	Transitional variable in equation of state.
CWMAX	Upper or lower limit for output.
DECAY	Decay coef for pollutant concentration (1/sec).
*DEPTH	Local depths in (cm).
EGRAD	SGS turbulent energy gradient across open boundary.
*ETURN	Stored values of SGS turbulent energy at open boundary for computing energy transport across open boundary.
GENN1	Energy generation term from interface located above the layer of concern.
GENN2	Energy generation term from interface located below layer of concern.
HBLB3	Temporary variable for averaged layer thickness.
HBLLT	Temporary variable for averaged layer thickness.
HBLT3	Temporary variable for averaged layer thickness.
HBRT3	Temporary variable for averaged layer thickness.
HBTR3	Temporary variable for averaged layer thickness.
LGRPH	Frequency to write an output file.
LHIST	Frequency to write time-history file.
*LTURN	Time step at which most recent change of current direction took place at open boundary.
MHIST	Buffer size for time history output file.
NAVDA	Constant for time history output file.
NHIST	Constant for time history output file.
NOWL3	Number of water level stations.
NUNIT	Output unit for time history file.
PGRAD	Pollutant concentration gradient across open boundary.
*PGRAX	Pressure gradient terms in x-direction.
*PGRAY	Pressure gradient terms in y-direction.
*PTURN	Stored values of pollutant concentration when current changes its direction from ebb to flood at open boundary.
SEINV	Initial specified value of water level for the entire computational field.
SGRAD	Salinity gradient across open boundary.
SKIPT	Variable not used in computation.
*STURN	Stored values of salinity when current changes its direction from ebb to flood at open boundary.

TGRAD	Temperature gradient across open boundary.
THETA	Coef of momentum transfer across air-water interface.
*TITC3	Names of current stations.
*TITLE	Title of simulation.
*TTURN	Stored values of temperature at open boundary when current changes its direction from ebb to flood.
*UFLOE	Kinetic energy density of inflow at outside of open boundary.
*UFLOP	Pollutant concentrations of inflow.
*UFLOS	Salinity of inflow at outside of open boundary.
*UFLOT	Temperature of inflow.
WINCH	Flag for changing wind speed during computation restart.
*WLMAX	Maximum values of water level at selected stations.
*WLMIN	Minimum values of water level at selected stations.
*ZCUR3	Value of computed value of w-component of current.
ANGLAT	Latitude (deg).
*CURMAX	Maximum values of current at selected stations.
*CURMIN	Minimum values of current at selected stations.
DUVDZ1	Temporary variable for velocity derivatives.
DUVDZ2	Temporary variable for velocity derivatives.
*EPSLON	Computed vertical exchange coef from SGS turbulent energy.
ERETIM	Recovery time (hrs) for SGS energy at open boundary.
ESQRT1	Temporary variable in energy computation.
ESQRT2	Temporary variable in energy computation.
HBLLT3	Temporary variable for averaged layer thickness.
NOCON3	Constant for time history output file.
NOCUR3	Number of current stations.
PRETIM	Recovery time (hrs) for pollutant concentration at open boundary.
RHOBXZ	Temporary variable for density derivatives.
RHOBYZ	Temporary variable for density derivatives.
SALINV	Specified initial value for salinity in field.
SETIDE	Convolutd value of water level from tidal constituents.
SRETIM	Recovery time (hrs) of salinity at open boundary.
*TITWL3	Names of specified water level stations.
TRETIM	Recovery time (hrs) of temperature at open boundary.
WSCALE	Multiplier for linear increase of wind speed in time.

8.8 電腦程式

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C
CTHREE-DIMENSIONAL MODEL (MODIFIED FROM RAND-2188, LIU AND NELSON BY SKL)
C
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COMMON CPO,CLAM,CLAM0,CP1,CLAM1,RHOT
COMMON M,MM1,MM2,N,NM1,NM2,DX,DY,RDX,RDY,RDXS,RDYS
COMMON DT,DT2,DT4,CNU,L,L1,L2,L3,GRAV,KMAX,CKA,AHX,AHY,AMX,AMY,F
COMMON RHOM,RHOM2,ALPH0,TEMP,THETA,RHOA,WSPD,PHI,WINCH,DEPMIN
COMMON PI,KMAP,TD,TD2,TD4,KHT,CT,CKH,AHXT,AHYT
COMMON RMM1,RMM2,CKP,AHXP,AHYP,PRETIM,PINV,DECAY,LMAX,CONVER
COMMON U(10,26,6,3),V(10,26,6,3),S(10,26,6,3),T(10,26,6,3)
COMMON W(10,26,7),TAUX(10,26,7),TAUY(10,26,7),EPSLON(10,26,7)
COMMON PGRAX(10,26,6),PGRAY(10,26,6),RHO(10,26,6),HK(10,26,6)
COMMON SE(10,26,3),KH(10,26),C(10,26),HS(7),YY(6),DEPTH(10,26)
COMMON RHOZ(10,26),HZ(10,26),NSE(10,26),ZL(6)
COMMON IB(100),JB(100),KB(100),UVB(100),AHUV(10,3,6)
COMMON UFLOS(100),UFLOT(100),UFLOP(100),PTURN(10,100)
COMMON STMP,LTURN(10,100),STURN(10,100),SET,TTURN(10,100)
COMMON SINI(10,100),TINI(10,100),SRETIM,TRETIM
COMMON ETURN(10,100),EINI(10,100),UFLOE(100),E(10,26,6,3)
COMMON ERETIM,EINV,TINV,SALINV,P(10,26,6,3),PINI(10,100)
DOUBLE PRECISION CPO,CLAM,CLAM0,CP1,CLAM1,RHOT
DIMENSION PRINT(10,26)
DIMENSION UG1(10,26,6),UG2(10,26,6),UG3(10,26,6)
DIMENSION VG1(10,26,6),VG2(10,26,6),VG3(10,26,6)
DIMENSION SG1(10,26,6),SG2(10,26,6),SG3(10,26,6)
DIMENSION TG1(10,26,6),TG2(10,26,6),TG3(10,26,6)
DIMENSION EG1(10,26,6),EG2(10,26,6),EG3(10,26,6)
DIMENSION PG1(10,26,6),PG2(10,26,6),PG3(10,26,6)
DIMENSION SEG1(10,26),SEG2(10,26),SEG3(10,26)
DIMENSION TITLE(20),DATE(4)
DIMENSION BUF(100)
DIMENSION WLMAX(20),WLMIN(20),CURMAX(3,9,20),CURMIN(3,9,20)
DIMENSION MWL3(20),NWL3(20),TITWL3(5,20),ZWL3(20)
DIMENSION MC3(20),NC3(20),TITC3(5,20),ZCUR3(3,20),KC3(20)
EQUIVALENCE (UG1(1,1,1),U(1,1,1,1))
EQUIVALENCE (UG2(1,1,1),U(1,1,1,2))
EQUIVALENCE (UG3(1,1,1),U(1,1,1,3))
EQUIVALENCE (VG1(1,1,1),V(1,1,1,1))
EQUIVALENCE (VG2(1,1,1),V(1,1,1,2))
EQUIVALENCE (VG3(1,1,1),V(1,1,1,3))
EQUIVALENCE (SG1(1,1,1),S(1,1,1,1))
EQUIVALENCE (SG2(1,1,1),S(1,1,1,2))
EQUIVALENCE (SG3(1,1,1),S(1,1,1,3))
EQUIVALENCE (TG1(1,1,1),T(1,1,1,1))
EQUIVALENCE (TG2(1,1,1),T(1,1,1,2))
EQUIVALENCE (TG3(1,1,1),T(1,1,1,3))
EQUIVALENCE (EG1(1,1,1),E(1,1,1,1))
EQUIVALENCE (EG2(1,1,1),E(1,1,1,2))
EQUIVALENCE (EG3(1,1,1),E(1,1,1,3))
EQUIVALENCE (PG1(1,1,1),P(1,1,1,1))
EQUIVALENCE (PG2(1,1,1),P(1,1,1,2))
EQUIVALENCE (PG3(1,1,1),P(1,1,1,3))
EQUIVALENCE (SEG1(1,1),SE(1,1,1))
EQUIVALENCE (SEG2(1,1),SE(1,1,2))
EQUIVALENCE (SEG3(1,1),SE(1,1,3))
DIMENSION KCOL1(30),KCOL2(30)

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DATA DEND/'END '/
DATA KCOL1/'0000','0000','0111','1111','1112','2222','2222',
* '2333','3333','3334','4444','4444','4555','5555','5556','6666',
* '6666','6777','7777','7778','8888','8888','8999','9999','9990',
* '0000','0000','0111','1111','1112'/
DATA KCOL2/'1234','5678','9012','3456','7890','1234','5678',
* '9012','3456','7890','1234','5678','9012','3456','7890','1234',
* '5678','9012','3456','7890','1234','5678','9012','3456','7890',
* '1234','5678','9012','3456','7890'/
1 FORMAT (1X,'NU=',1PE12.5,' F=',1PE12.5,
* 1X,'AMX=',1PE12.5,' AMY=',1PE12.5)
2 FORMAT (20A4)
3 FORMAT (4I10,4E10.0)
4 FORMAT (7E10.0)
5 FORMAT (4I10,5(1X,1PE12.5))
6 FORMAT (7(1X,1PE12.5))
7 FORMAT (1X,7I11)
8 FORMAT (1X,72I1)
9 FORMAT (20F4.0/20F4.0)
10 FORMAT (7E10.0)
11 FORMAT (7(1X,1PE10.3))
12 FORMAT (2I10,5A4)
13 FORMAT (3I10,5A4)
15 FORMAT (4I5,5E10.0)
16 FORMAT (72I1)
17 FORMAT (1H0,'CONSTANT DEPTHS KHT=',I3)
18 FORMAT (1X,'LOCATIONS TERMINATED BY A BLANK CARD ',
* I5,' LOCATIONS READ')
19 FORMAT (1H1,'MODEL:',4A4)
21 FORMAT (1H0,5X,'LMAX',9X,'M',9X,'N',6X,'KMAX',2X,'DT',11X,'DX',
* 11X,'DY')
22 FORMAT (1H0,2X,'ANGLAT',7X,'AHX',10X,'AHY',10X,'AMX',10X,'AMY',
* 10X,'NU',11X,'KAPA')
23 FORMAT (1H0,2X,'RHO',10X,'RHOA',9X,'TINV',9X,'THETA',8X,'PHI',
* 10X,'WSPD')
24 FORMAT (1H0,6X,'LP1',7X,'LP2',7X,'LP3',7X,'LP4')
25 FORMAT (1H0,6X,'KHT',8X,'LC',5X,'LGRPH',6X,'KLP2')
26 FORMAT (1H0,2X,'CT',11X,'SALINV',9X,'SEINV',8X,'RMM1',9X,'RMM2',
* 9X,'EINV')
27 FORMAT (1H0,5X,'DEPTHS OF LEVELS 1 THROUGH KMAX')
28 FORMAT (1H0,3X,'DEPTH ARRAY OF ESTUARY')
29 FORMAT (1H0,5X,'LOCATION & TYPE OF OPENINGS',/,
* 9X,'KB=(1,2,3,4)=(W,S,E,N)',/,
* 9X,'KB.GT.0=TIDAL OPENING',/,
* 9X,'KB.LT.0=WATER INPUT',/,
* 8X,'IB',8X,'JB',8X,'KB',13X,'UV',8X,'UFLOS',8X,'UFLOT',
* 8X,'UFLOE',8X,'UFLOP')
30 FORMAT (1H0,2X,'RESTART FREQUENCY',5X,'LHIST')
31 FORMAT (1H0,'SALINITY AT LEVEL',I3,' IS BEING READ')
32 FORMAT (1H0,2X,'CKH',10X,'AHXT',6X,'AHYT',4X,'SKIPT',
* 6X,'SRETIM',5X,'TRETIM',5X,'ERETIM')
33 FORMAT (1H0,5X,'MWL3',6X,'NWL3',4X,'TITWL3')
34 FORMAT (1H0,6X,'MC3',7X,'NC3',9X,'K',4X,'TITC3')
35 FORMAT (1H0,'RUN TERMINATING UNDER PROGRAM CONTROL',I8,
* ' INTEGRATION STEPS MAXIMUM WATER LEVEL=',1PE12.5)
36 FORMAT (1H0,'DEPTH K=',I2,9(10X,I2))
37 FORMAT (7X,10(1X,1PE11.4))
38 FORMAT (1H0,2X,'YY K=',I2,9(10X,I2))
39 FORMAT (1H0,2X,'ZL K=',I2,9(10X,I2))
40 FORMAT (1X,'K1=',I2,2X,10(1X,1PE11.4))
41 FORMAT (1H0,2X,'CKP',8X,'AHXP',7X,'AHYP',7X,'PRETIM',

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*      5X, 'PINV', 7X, 'DECAY', 5X, 'CONVER')
49 FORMAT(30F2.0)
51 FORMAT (1H0,40X, 'VARIABLE DEPTHS')
59 FORMAT (1X,I3,2X,10F10.1,3(/,6X,10F10.1))
55 FORMAT (1H0,10X, 'DEPTH ARRAY')
56 FORMAT (6X,30A4)
57 FORMAT (4X, 'M=', 30A4)
67 FORMAT (3X, 'N')
60 FORMAT (1X,I3,1X,35(1X,I2))
  NIN=5
  NOUT=6
  READ (5,2) TITLE
  WRITE (6,2) TITLE
  READ (5,3) LMAX,M,N,KMAX,TD,DX,DY
  WRITE (6,21)
  WRITE (6,5) LMAX,M,N,KMAX,TD,DX,DY
  READ (5,3) LST,LRST,LHIST
  WRITE (6,30)
  WRITE (6,5) LST,LRST,LHIST
  READ (5,4) ANGLAT,AHX,AHY,AMX,AMY,CNU,CKA
  WRITE (6,22)
  WRITE (6,6) ANGLAT,AHX,AHY,AMX,AMY,CNU,CKA
  F=.0001458*SIN(ANGLAT/57.2957795)
  READ(5,10) CKH,AHXT,AHYT,SKIPT,SRETIM,TRETIM,ERETIM
  WRITE (6,32)
  WRITE (6,11) CKH,AHXT,AHYT,SKIPT,SRETIM,TRETIM,ERETIM
  READ (5,4) RHOM,RHOA,TINV,THETA,PHI,WSPD
  WRITE (6,23)
  WRITE (6,6) RHOM,RHOA,TINV,THETA,PHI,WSPD
  READ (5,3) LP1,LP2,LP3,LP4
  WRITE (6,24)
  WRITE (6,5) LP1,LP2,LP3,LP4
  READ (5,3) KHT,LC,LGRPH,KLP2
  WRITE (6,25)
  WRITE (6,5) KHT,LC,LGRPH,KLP2
  READ (5,4) CT,SALINV,SEINV,RMM1,RMM2,EINV
  WRITE (6,26)
  WRITE (6,6) CT,SALINV,SEINV,RMM1,RMM2,EINV
  READ (5,10) CKP,AHXP,AHYP,PRETIM,PINV,DECAY,CONVER
  WRITE (6,41)
  WRITE (6,11) CKP,AHXP,AHYP,PRETIM,PINV,DECAY,CONVER
  READ (5,4) (HS(K),K=1,KMAX)
  WRITE (6,27)
  WRITE (6,6) (HS(K),K=1,KMAX)
  WRITE (NOUT,51)
  DO 100 J=1,N
    READ (NIN,49) (DEPTH(J,I), I=1,M)
    DO 100 I=1,M
      DEPTH(J,I)=DEPTH(J,I)*CONVER
    DO 100 K=1,KMAX
      HK(J,I,K)=0.0
100 CONTINUE
    DO 105 J1=1,N
      J=N+1-J1
      WRITE(NOUT,59) J, (DEPTH(J,I),I=1,M)
105 CONTINUE
    DO 210 I=1,M
    DO 210 J=1,N
    DEPMIN=.10
    DEPMIN=DEPMIN*CONVER
    IF(DEPTH(J,I).LE.DEPMIN) GO TO 208

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```

      IF(DEPTH(J,I).LT.(0.5*HS(1))) GO TO 209
      TEMP1=DEPTH(J,I)
      DO 206 K=1,KMAX
      IF (K.EQ.KMAX) GO TO 207
      IF(TEMP1.GE.(0.5*HS(K)).AND.TEMP1.LT.(HS(K)+0.5*HS(K+1)))
*   GO TO 207
      HK(J,I,K)=HS(K)
206  TEMP1=TEMP1-HS(K)
207  HK(J,I,K)=TEMP1
      KH(J,I)=K
      GOTO 210
208  HK(J,I,1)=0.0
      KH(J,I)=0
      GO TO 210
209  HK(J,I,1)=DEPTH(J,I)
      KH(J,I)=1
210  CONTINUE
      GO TO 250
245  CONTINUE
      DO 246 I=1,M
      DO 246 J=1,N
      KH(J,I)=KHT
      DO 246 K=1,KHT
      HK(J,I,K)=HS(K)
246  CONTINUE
      WRITE (NOUT,17) KHT
250  CONTINUE
      DO 252 I=1,M
      KH(1,I)=0
      KH(N-1,I)=0
      KH(N,I)=0
252  CONTINUE
      DO 253 J=1,N
      KH(J,1)=0
      KH(J,M-1)=0
      KH(J,M)=0
253  CONTINUE
      MCOL=(M+3)/4
      WRITE (NOUT,55)
      WRITE (NOUT,56) (KCOL1(I),I=1,MCOL)
      WRITE (NOUT,57) (KCOL2(I),I=1,MCOL)
      WRITE (NOUT,67)
      DO 255 J1=1,N
      J=N+1-J1
      WRITE (NOUT,60) J,(KH(J,I),I=1,M)
255  CONTINUE
      DO 270 K=1,KMAX
      CALL PRNT3 (HK,'HK ',M,N,KMAX,K,L)
270  CONTINUE
      NOWL3=0
      WRITE (6,33)
110  CONTINUE
      NOWL3=NOWL3+1
      READ (5,12) MWL3(NOWL3),NWL3(NOWL3),(TITWL3(K,NOWL3),K=1,5)
      WRITE (6,12) MWL3(NOWL3),NWL3(NOWL3),(TITWL3(K,NOWL3),K=1,5)
      IF (MWL3(NOWL3)+NWL3(NOWL3).NE.0) GO TO 110
      NOWL3=NOWL3-1
      NOCUR3=0
120  CONTINUE
      NOCUR3=NOCUR3+1
      READ (5,13) MC3(NOCUR3),NC3(NOCUR3),KC3(NOCUR3),

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*          (TITC3(K,NOCUR3),K=1,5)
  IF (MC3(NOCUR3)+NC3(NOCUR3).NE.0) GO TO 120
  NOCUR3=NOCUR3-1
  NHIST=0
  NAVDA=0
  MHIST=100
  NUNIT=11
  NWD=1
  NREC=1
  MONE=-1
  CWMAX=1.0E50
  CALL SETCON
  IF (LHIST.EQ.0) GO TO 1253
  DO 1252 I=1,NOCUR3
    IMC3=MC3(I)
    INC3=NC3(I)
    KC3(I)=KH(INC3,IMC3)
    DO 1252 II=1,3
      DO 1252 K=1,KMAX
        CURMAX(II,K,I)=-CWMAX
        CURMIN(II,K,I)=CWMAX
1252  CONTINUE
    DO 1001 I=1,NOWL3
      WLMAX(I)=-CWMAX
      WLMIN(I)=CWMAX
1001  CONTINUE
    NOCON3=0
    CALL TRANS (BUF,MHIST,TITLE,18,NREC,NWD,NUNIT,NAVDA,0)
    CALL TRANS (BUF,MHIST,DATE,4,NREC,NWD,NUNIT,NAVDA,0)
    CALL TRANS (BUF,MHIST,TD,1,NREC,NWD,NUNIT,NAVDA,0)
    CALL TRANS (BUF,MHIST,LMAX,1,NREC,NWD,NUNIT,NAVDA,0)
    CALL TRANS (BUF,MHIST,LHIST,1,NREC,NWD,NUNIT,NAVDA,0)
    CALL TRANS (BUF,MHIST,MHIST,1,NREC,NWD,NUNIT,NAVDA,0)
    CALL TRANS (BUF,MHIST,MONE,1,NREC,NWD,NUNIT,NAVDA,0)
    CALL TRANS (BUF,MHIST,NOWL3,1,NREC,NWD,NUNIT,NAVDA,0)
    CALL TRANS (BUF,MHIST,NOCUR3,1,NREC,NWD,NUNIT,NAVDA,0)
    CALL TRANS (BUF,MHIST,NOCON3,1,NREC,NWD,NUNIT,NAVDA,0)
    CALL TRANS (BUF,MHIST,MONE,1,NREC,NWD,NUNIT,NAVDA,0)
    CALL TRANS (BUF,MHIST,MWL3,NOWL3,NREC,NWD,NUNIT,NAVDA,0)
    CALL TRANS (BUF,MHIST,NWL3,NOWL3,NREC,NWD,NUNIT,NAVDA,0)
    CALL TRANS (BUF,MHIST,TITWL3,5*NOWL3,NREC,NWD,NUNIT,NAVDA,0)
    CALL TRANS (BUF,MHIST,MC3,NOCUR3,NREC,NWD,NUNIT,NAVDA,0)
    CALL TRANS (BUF,MHIST,NC3,NOCUR3,NREC,NWD,NUNIT,NAVDA,0)
    CALL TRANS (BUF,MHIST,KC3,NOCUR3,NREC,NWD,NUNIT,NAVDA,0)
    CALL TRANS (BUF,MHIST,TITC3,5*NOCUR3,NREC,NWD,NUNIT,NAVDA,0)
    CALL TRANS (BUF,MHIST,NWD,1,NREC,NWD,NUNIT,NAVDA,0)
1253  CONTINUE
    WRITE (6,34)
    DO 1000 I=1,NOCUR3
1000  WRITE (6,13) MC3(I),NC3(I),KC3(I),(TITC3(K,I),K=1,5)
      WRITE (6,28)
      DO 1255 J1=1,N
        J=N+1-J1
1255  WRITE (6,8) (KH(J,I),I=1,M)
      DO 1275 I=1,M
      DO 1275 J=1,N
      DO 1275 L=1,3
      IF (KH(J,I).EQ.0) GO TO 1275
      SE(J,I,L)=SEINV
1275  CONTINUE
      CALL RESAL

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DO 1260 IJ=1,100
  IB(IJ)=0
  JB(IJ)=0
  KB(IJ)=0
  UVB(IJ)=0.0
DO 1260 K=1,KMAX
  LTURN(K,IJ)=-1
  STURN(K,IJ)=0.0
  TTURN(K,IJ)=0.0
  ETURN(K,IJ)=0.0
  PTURN(K,IJ)=0.0
1260 CONTINUE
  WRITE (6,29)
  DO 310 IJ=1,100
    READ (5,15) IB(IJ),JB(IJ),KB(IJ),LB,UV,UFLOS(IJ),UFLOT(IJ),
    *   UFLOE(IJ),UFLOP(IJ)
    WRITE (6,5) IB(IJ),JB(IJ),KB(IJ),LB,UV,UFLOS(IJ),UFLOT(IJ),
    *   UFLOE(IJ),UFLOP(IJ)
    IF (IB(IJ).EQ.0) GO TO 320
    IBT=IB(IJ)
    JBT=JB(IJ)
    UVB(IJ)=UV
    KI1=0
    KI2=0
    KI3=0
    KI4=0
    KBT=IABS(KB(IJ))
    IF (MOD(KBT,2).EQ.1) KI1=1
    IF (MOD(KBT,2).EQ.0) KI2=1
    IF (KBT.LT.3) GO TO 1265
    KI1=-KI1
    KI2=-KI2
    KI3=KI1
    KI4=KI2
1265 CONTINUE
    IF (KB(IJ).GT.0) GO TO 305
    DO 1270 LF=1,3
      S(JBT,IBT,1,LF)=-S(JBT+KI2,IBT+KI1,1,LF)+UFLOS(IJ)+UFLOS(IJ)
      T(JBT,IBT,1,LF)=-T(JBT+KI2,IBT+KI1,1,LF)+UFLOT(IJ)+UFLOT(IJ)
      E(JBT,IBT,1,LF)=-E(JBT+KI2,IBT+KI1,1,LF)+UFLOE(IJ)+UFLOE(IJ)
      P(JBT,IBT,1,LF)=-P(JBT+KI2,IBT+KI1,1,LF)+UFLOP(IJ)+UFLOP(IJ)
      SE(JBT,IBT,LF)=-SE(JBT+KI2,IBT+KI1,LF)
      IF (KI2.EQ.0) U(JBT+KI4,IBT+KI3,1,LF)=UVB(IJ)
      IF (KI1.EQ.0) V(JBT+KI4,IBT+KI3,1,LF)=UVB(IJ)
1270 CONTINUE
    GO TO 309
305 CONTINUE
309 CONTINUE
    KBST = KH(JBT,IBT)
    DO 308 K=1,KBST
      SINI(K,IJ)=S(JBT,IBT,K,1)
      TINI(K,IJ)=T(JBT,IBT,K,1)
      EINI(K,IJ)=E(JBT,IBT,K,1)
      PINI(K,IJ)=P(JBT,IBT,K,1)
308 CONTINUE
310 CONTINUE
320 CONTINUE
    L=0
    L3=MOD(L,3)+1
    L2=MOD(L3,3)+1
    L1=MOD(L2,3)+1

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WRITE (8) TITLE
WRITE (8) KMAX, LMAX, M, N, DX, DY, TD, HS, KH, C, HK, DEPTH
WINCH=1.0
SEM=0.0
IF (LST.EQ.0) GO TO 300
I=10
301 CONTINUE
READ (I) L, L1, L2, L3, IB, JB, KB, UVB, KH, HS, HK, DEPTH
READ (I) CP0, CLAM, CLAM0, CP1, CLAM1, RHOT, RMM1, RMM2
READ (I) M, MM1, MM2, N, NM1, NM2, DX, DY, RDX, RDY, RDXS, RDYS
READ (I) DT, DT2, DT4, CNU, GRAV, KMAX, CKA, AHX, AHY, AMX, AMY, F
READ (I) RHOM, RHOM2, ALPHO, TINV, THETA, RHOA, WSPD, PHI, WINCH, DEPMIN
READ (I) C, RHOZ, HZ, UFLOS, UFLOT, UFLOE, UFLOP,
*          LTURN, STURN, TTURN, ETURN, PTURN
READ (I) W, TAUX, TAUY
READ (I) PGRAX, PGRAY, RHO
READ (I) U
READ (I) V
READ (I) S
READ (I) T
READ (I) E
READ (I) P
READ (I) SE
IF (L.LT.LST) GO TO 301
READ (5,4) ANGLAT, WSPD, PHI
WRITE (6,6) ANGLAT, WSPD, PHI
F=.0001458*SIN(ANGLAT/57.2957795)
PHI=PI*PHI/180.
READ (5,4) RMM1, RMM2
WRITE (6,6) RMM1, RMM2
GO TO 620
300 CONTINUE
WRITE (6,5) L
DT=TD
DT2=TD2
DT4=TD4
IF (L.LE.0) GO TO 325
IF (MOD(L,LC).NE.0) GO TO 375
325 CONTINUE
DT2=TD
DT4=TD2
L3=L2
375 CONTINUE
CALL DSVP
IF (L.EQ.0) GO TO 500
IF (MOD(L,LP3).EQ.0) CALL PRNT5 (SE, 'ZETA', M, N, L1, L)
DO 376 KPR=1, KMAX
IF (MOD(L,LP3).EQ.0) CALL PRNT6 (V, 'V', M, N, KMAP, L1, KPR, L)
IF (MOD(L,LP3).EQ.0) CALL PRNT6 (U, 'U', M, N, KMAP, L1, KPR, L)
IF (MOD(L,LP3).EQ.0) CALL PRNT6 (W, 'W', M, N, KMAP, 1, KPR, L)
376 CONTINUE
C IF (MOD(L,LP3).EQ.0) CALL PRNT6 (V, 'V', M, N, KMAP, L1, 1, L)
C IF (MOD(L,LP3).EQ.0) CALL PRNT6 (V, 'V', M, N, KMAP, L1, 2, L)
C IF (MOD(L,LP3).EQ.0) CALL PRNT6 (V, 'V', M, N, KMAP, L1, 3, L)
C IF (MOD(L,LP3).EQ.0) CALL PRNT6 (V, 'V', M, N, KMAP, L1, 4, L)
C IF (MOD(L,LP3).EQ.0) CALL PRNT6 (V, 'V', M, N, KMAP, L1, 5, L)
C IF (MOD(L,LP3).EQ.0) CALL PRNT6 (V, 'V', M, N, KMAP, L1, 6, L)
C IF (MOD(L,LP3).EQ.0) CALL PRNT6 (U, 'U', M, N, KMAP, L1, 1, L)
C IF (MOD(L,LP3).EQ.0) CALL PRNT6 (U, 'U', M, N, KMAP, L1, 2, L)
C IF (MOD(L,LP3).EQ.0) CALL PRNT6 (U, 'U', M, N, KMAP, L1, 3, L)
C IF (MOD(L,LP3).EQ.0) CALL PRNT6 (U, 'U', M, N, KMAP, L1, 4, L)

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C      IF (MOD(L,LP3).EQ.0) CALL PRNT6 (U,'U      ',M,N,KMAP,L1,5,L)
C      IF (MOD(L,LP3).EQ.0) CALL PRNT6 (U,'U      ',M,N,KMAP,L1,6,L)
      SEM=0.0
      TOM=0.0
      SAL=0.0
      TEM=0.0
      EM=0.0
      PM=0.0
      DO 400 I=1,M
      DO 400 J=1,N
      DO 390 K=1,KMAX
      TEM=TEM+T(J,I,K,L1)
      EM=EM+E(J,I,K,L1)
      PM=PM+P(J,I,K,L1)
390    SAL=SAL+S(J,I,K,L1)
      IF (ABS(SE(J,I,L1)).GT.SEM) SEM=ABS(SE(J,I,L1))
      TOM=TOM+SE(J,I,L1)
400    NSE(J,I)=SE(J,I,L1)
      WRITE (6,6) TOM,SAL,TEM,EM,PM
      IF (MOD(L,LP2).NE.0) GO TO 425
      DO 422 KK=1,KLP2
      DO 421 J=1,N
      DO 421 I=1,M
421    PRINT(J,I)=P(J,I,KK,L1)
      CALL PRNT4 (PRINT,'POLLUTANT',M,N,KK,L)
422    CONTINUE
      DO 415 KK=1,KLP2
      DO 416 J=1,N
      DO 416 I=1,M
416    PRINT(J,I)=E(J,I,KK,L1)
      CALL PRNT4 (PRINT,'ENERGY',M,N,KK,L)
415    CONTINUE
      DO 420 KK=1,KLP2
      DO 410 J=1,N
      DO 410 I=1,M
410    PRINT(J,I)=S(J,I,KK,L1)
      CALL PRNT4 (PRINT,'SALINITY',M,N,KK,L)
420    CONTINUE
425    CONTINUE
500    CONTINUE
      IF (LHIST.EQ.0) GO TO 560
      IF (MOD(L,LHIST).NE.0) GO TO 560
      CALL TRANS (BUF,MHIST,L,1,NREC,NWD,NUNIT,NAVDA,0)
      DO 510 NH=1,NOWL3
      IH=MWL3(NH)
      JH=NWL3(NH)
      ZWL3(NH)=SE(JH,IH,L1)
      IF (ZWL3(NH).GT.WLMAX(NH)) WLMAX(NH)=ZWL3(NH)
      IF (ZWL3(NH).LT.WLMIN(NH)) WLMIN(NH)=ZWL3(NH)
510    CONTINUE
      CALL TRANS (BUF,MHIST,ZWL3,NOWL3,NREC,NWD,NUNIT,NAVDA,0)
      DO 530 NH=1,NOCUR3
      IH=MC3(NH)
      JH=NC3(NH)
      K1=KC3(NH)
      ZCUR3(1,1)=W(JH,IH,1)
      IF (ZCUR3(1,1).GT.CURMAX(1,1,NH)) CURMAX(1,1,NH)=ZCUR3(1,1)
      IF (ZCUR3(1,1).LT.CURMIN(1,1,NH)) CURMIN(1,1,NH)=ZCUR3(1,1)
      DO 520 K=1,K1
      ZCUR3(2,K)=U(JH,IH,K,L1)
      ZCUR3(3,K)=V(JH,IH,K,L1)

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ZCUR3(1,K+1)=W(JH,IH,K+1)
IF (ZCUR3(2,K).GT.CURMAX(2,K,NH)) CURMAX(2,K,NH)=ZCUR3(2,K)
IF (ZCUR3(2,K).LT.CURMIN(2,K,NH)) CURMIN(2,K,NH)=ZCUR3(2,K)
IF (ZCUR3(3,K).GT.CURMAX(3,K,NH)) CURMAX(3,K,NH)=ZCUR3(3,K)
IF (ZCUR3(3,K).LT.CURMIN(3,K,NH)) CURMIN(3,K,NH)=ZCUR3(3,K)
IF (ZCUR3(1,K+1).GT.CURMAX(1,K+1,NH))
*   CURMAX(1,K+1,NH)=ZCUR3(1,K+1)
IF (ZCUR3(1,K+1).LT.CURMIN(1,K+1,NH))
*   CURMIN(1,K+1,NH)=ZCUR3(1,K+1)
520 CONTINUE
CALL TRANS (BUF,MHIST,ZCUR3,3*K1+1,NREC,NWD,NUNIT,NAVDA,0)
530 CONTINUE
WRITE (6,3) NREC,NWD
560 CONTINUE
IF (MOD(L,LGRPH).NE.0) GO TO 600
IF (L1.NE.1) GO TO 570
WRITE (8) L,SEG1
DO 561 K=1,KMAX
WRITE (8) ((W(J,I,K),J=1,N),I=1,M)
WRITE (8) ((UG1(J,I,K),J=1,N),I=1,M)
WRITE (8) ((VG1(J,I,K),J=1,N),I=1,M)
WRITE (8) ((SG1(J,I,K),J=1,N),I=1,M)
WRITE (8) ((TG1(J,I,K),J=1,N),I=1,M)
WRITE (8) ((EG1(J,I,K),J=1,N),I=1,M)
WRITE (8) ((PG1(J,I,K),J=1,N),I=1,M)
561 CONTINUE
KK=KMAX+1
WRITE (8) ((W(J,I,KK),J=1,N),I=1,M)
GO TO 600
570 CONTINUE
IF (L1.NE.2) GO TO 580
WRITE (8) L,SEG2
DO 571 K=1,KMAX
WRITE (8) ((W(J,I,K),J=1,N),I=1,M)
WRITE (8) ((UG2(J,I,K),J=1,N),I=1,M)
WRITE (8) ((VG2(J,I,K),J=1,N),I=1,M)
WRITE (8) ((SG2(J,I,K),J=1,N),I=1,M)
WRITE (8) ((TG2(J,I,K),J=1,N),I=1,M)
WRITE (8) ((EG2(J,I,K),J=1,N),I=1,M)
WRITE (8) ((PG2(J,I,K),J=1,N),I=1,M)
571 CONTINUE
KK=KMAX+1
WRITE (8) ((W(J,I,KK),J=1,N),I=1,M)
GO TO 600
580 CONTINUE
WRITE (8) L,SEG3
DO 581 K=1,KMAX
WRITE (8) ((W(J,I,K),J=1,N),I=1,M)
WRITE (8) ((UG3(J,I,K),J=1,N),I=1,M)
WRITE (8) ((VG3(J,I,K),J=1,N),I=1,M)
WRITE (8) ((SG3(J,I,K),J=1,N),I=1,M)
WRITE (8) ((TG3(J,I,K),J=1,N),I=1,M)
WRITE (8) ((EG3(J,I,K),J=1,N),I=1,M)
WRITE (8) ((PG3(J,I,K),J=1,N),I=1,M)
581 CONTINUE
KK=KMAX+1
WRITE (8) ((W(J,I,KK),J=1,N),I=1,M)
600 CONTINUE
L=L+1
IF (LRST.EQ.0) GO TO 620
IF (MOD(L,LRST).NE.0) GO TO 620

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WRITE (9) L,L1,L2,L3,IB,JB,KB,UVB,KH,HS,HK,DEPTH
WRITE (9) CPO,CLAM,CLAMO,CP1,CLAM1,RHOT,RMM1,RMM2
WRITE (9) M,MM1,MM2,N,NM1,NM2,DX,DY,RDX,RDY,RDXS,RDYS
WRITE (9) DT,DT2,DT4,CNU,GRAV,KMAX,CKA,AHX,AHY,AMX,AMY,F
WRITE (9) RHOM,RHOM2,ALPHO,TINV,THETA,RHOA,WSPD,PHI,WINCH,DEPMIN
WRITE (9) C,RHOZ,HZ,UFLOS,UFL0T,UFL0E,UFL0P,
*      LTURN,STURN,TTURN,ETURN,PTURN
WRITE (9) W,TAUX,TAUY
WRITE (9) PGRAX,PGRAY,RHO
WRITE (9) U
WRITE (9) V
WRITE (9) S
WRITE (9) T
WRITE (9) E
WRITE (9) P
WRITE (9) SE
620 CONTINUE
  IF (L.GT.LMAX) GO TO 700
  L3=MOD(L,3)+1
  L2=MOD(L3,3)+1
  L1=MOD(L2,3)+1
  IF (SEM.GT.1.0E3) GO TO 700
  CALL SESUV
  WINCH=0.0
  WINCH=1.0
  GO TO 300
700 CONTINUE
  WRITE (6,35) L,SEM
  IF (LHIST.EQ.0) STOP
  CALL TRANS (BUF,MHIST,WLMAX,NOWL3,NREC,NWD,NUNIT,NAVDA,0)
  CALL TRANS (BUF,MHIST,WLMIN,NOWL3,NREC,NWD,NUNIT,NAVDA,0)
  KCUR=3*KMAX*NOCUR3
  CALL TRANS (BUF,MHIST,CURMAX,KCUR,NREC,NWD,NUNIT,NAVDA,0)
  CALL TRANS (BUF,MHIST,CURMIN,KCUR,NREC,NWD,NUNIT,NAVDA,0)
  IF (NAVDA.EQ.0) WRITE (NUNIT) BUF
  IF (NAVDA.NE.0) WRITE (NUNIT'NAVDA) BUF
  WRITE (6,6) BUF
  STOP
  END
  SUBROUTINE TRANS (A,LREC,B,LTM,NREC,NWD,NUNIT,NAVDA,IO)
  DIMENSION A(LREC),B(LTM)
  DEFINE FILE 50 (50,100,L,NAVDA)
1  FORMAT (10(1X,1PE12.5))
  IF (IO.EQ.0) GO TO 400
  IF (NREC.NE.0) GO TO 25
  IF (NAVDA.EQ.0) GO TO 10
  READ (NUNIT'NAVDA) A
  NREC=NAVDA
  GO TO 50
10 CONTINUE
  READ (NUNIT,END=800) A
  NREC=2
  GO TO 50
25 CONTINUE
  IF (NAVDA.EQ.0) GO TO 50
  IF (NAVDA.EQ.NREC) GO TO 50
  READ (NUNIT'NAVDA) A
50 CONTINUE
  DO 300 I=1,LTM
  B(I)=A(NWD)
  NWD=NWD+1

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      IF (NWD.LE.LREC) GO TO 200
      NWD=1
      NREC=NREC+1
      IF (NAVDA.EQ.0) GO TO 100
      READ (NUNIT,NAVDA) A
      GO TO 200
100  CONTINUE
      READ (NUNIT,END=800) A
200  CONTINUE
300  CONTINUE
      IF (NAVDA.NE.0) NREC=NAVDA
      RETURN
400  CONTINUE
      DO 700 I=1,LTM
      A(NWD)=B(I)
      NWD=NWD+1
      IF (NWD.LE.LREC) GO TO 600
      NWD=1
      NREC=NREC+1
      IF (NAVDA.EQ.0) GO TO 500
      WRITE (NUNIT,NAVDA) A
      GO TO 600
500  CONTINUE
      WRITE (NUNIT) A
600  CONTINUE
700  CONTINUE
      IF (NAVDA.NE.0) NREC=NAVDA
      RETURN
800  CONTINUE
      IO=-1
      RETURN
      END
      SUBROUTINE SETCON
      COMMON CPO,CLAM,CLAMO,CP1,CLAM1,RHOT
      COMMON M,MM1,MM2,N,NM1,NM2,DX,DY,RDX,RDY,RDXS,RDYS
      COMMON DT,DT2,DT4,CNU,L,L1,L2,L3,GRAV,KMAX,CKA,AHX,AHY,AMX,AMY,F
      COMMON RHOM,RHOM2,ALPHO,TEMP,THETA,RHOA,WSPD,PHI,WINCH,DEPMIN
      COMMON PI,KMAP,TD,TD2,TD4,KHT,CT,CKH,AHXT,AHYT
      COMMON RMM1,RMM2,CKP,AHXP,AHYP,PRETIM,PINV,DECAY,LMAX,CONVER
      COMMON U(10,26,6,3),V(10,26,6,3),S(10,26,6,3),T(10,26,6,3)
      COMMON W(10,26,7),TAUX(10,26,7),TAUY(10,26,7),EPSLON(10,26,7)
      COMMON PGRAX(10,26,6),PGRAY(10,26,6),RHO(10,26,6),HK(10,26,6)
      COMMON SE(10,26,3),KH(10,26),C(10,26),HS(7),YY(6),DEPTH(10,26)
      COMMON RHOZ(10,26),HZ(10,26),NSE(10,26),ZL(6)
      COMMON IB(100),JB(100),KB(100),UVB(100),AHUV(10,3,6)
      COMMON UFLOS(100),UFLOT(100),UFLOP(100),PTURN(10,100)
      COMMON STMP,LTURN(10,100),STURN(10,100),SET,TTURN(10,100)
      COMMON SINI(10,100),TINI(10,100),SRETIM,TRETIM
      COMMON ETURN(10,100),EINI(10,100),UFLOE(100),E(10,26,6,3)
      COMMON ERETIM,EINV,TINV,SALINV,P(10,26,6,3),PINI(10,100)
      DOUBLE PRECISION CPO,CLAM,CLAMO,CP1,CLAM1,RHOT
      PI=3.14159265
      ALPHO=.698
      GRAV=980.
      PHI=PI*PHI/180.
      L1=1
      L2=2
      L3=3
      MM1=M-1
      MM2=M-2
      NM1=N-1

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KMAP=KMAX+1
KMAP=KMAX
NM2=N-2
RDX=1.0/DX
RDY=1.0/DY
RDXS=RDX*RDX
RDYS=RDY*RDY
RHOM2=RHOM+RHOM
TD2=TD+TD
TD4=TD2+TD2
A1=.68
CP0=5890.+38.*TINV-.375*TINV*TINV
CLAM=1779.5+11.25*TINV-.0745*TINV*TINV
CLAM0=3.8+.01*TINV
CLAM0=3.8+.01*TINV
CP1=CP0+3.0*SALINV
CLAM1=CLAM-CLAM0*SALINV
RHOT=CP1/(CLAM1+ALPHO*CP1)-RHOM
DO 200 I=1,M
DO 200 J=1,N
IF (KHT.NE.0) KH(J,I)=KHT
IF (I.EQ.1) KH(J,I)=0
IF (I.EQ.MM1) KH(J,I)=0
IF (I.EQ.M) KH(J,I)=0
IF (J.EQ.1) KH(J,I)=0
IF (J.EQ.NM1) KH(J,I)=0
IF (J.EQ.N) KH(J,I)=0
C(J,I)=CT
RHOZ(J,I)=0.0
HZ(J,I)=0.0
W(J,I,1)=0.0
TAUX(J,I,1)=0.0
TAUY(J,I,1)=0.0
EPSLON(J,I,1)=A1*CNU
DO 100 K=1,KMAX
W(J,I,K+1)=0.0
TAUX(J,I,K+1)=0.0
TAUY(J,I,K+1)=0.0
EPSLON(J,I,K+1)=A1*CNU
PGRAX(J,I,K)=0.0
PGRAY(J,I,K)=0.0
RHO(J,I,K)=RHOT
DO 100 L=1,3
U(J,I,K,L)=0.0
V(J,I,K,L)=0.0
S(J,I,K,L)=SALINV
T(J,I,K,L)=TINV
E(J,I,K,L)=EINV
P(J,I,K,L)=PINV
100 CONTINUE
DO 200 L=1,3
SE(J,I,L)=0.0
200 CONTINUE
RETURN
END
SUBROUTINE DSVP
COMMON CP0,CLAM,CLAM0,CP1,CLAM1,RHOT
COMMON M,MM1,MM2,N,NM1,NM2,DX,DY,RDX,RDY,RDXS,RDYS
COMMON DT,DT2,DT4,CNU,L,L1,L2,L3,GRAV,KMAX,CKA,AHX,AHY,AMX,AMY,F
COMMON RHOM,RHOM2,ALPHO,TEMP,THETA,RHOA,WSPD,PHI,WINCH,DEPMIN
COMMON PI,KMAP,TD,TD2,TD4,KHT,CT,CKH,AHXT,AHYT

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COMMON RMM1,RMM2,CKP,AHXP,AHYP,PRETIM,PINV,DECAY,LMAX,CONVER
COMMON U(10,26,6,3),V(10,26,6,3),S(10,26,6,3),T(10,26,6,3)
COMMON W(10,26,7),TAUX(10,26,7),TAUY(10,26,7),EPSLON(10,26,7)
COMMON PGRAX(10,26,6),PGRAY(10,26,6),RHO(10,26,6),HK(10,26,6)
COMMON SE(10,26,3),KH(10,26),C(10,26),HS(7),YY(6),DEPTH(10,26)
COMMON RHOZ(10,26),HZ(10,26),NSE(10,26),ZL(6)
COMMON IB(100),JB(100),KB(100),UVB(100),AHUV(10,3,6)
COMMON UFLOS(100),UFLOT(100),UFLOP(100),PTURN(10,100)
COMMON STMP,LTURN(10,100),STURN(10,100),SET,TTURN(10,100)
COMMON SINI(10,100),TINI(10,100),SRETIM,TRETIM
COMMON ETURN(10,100),EINI(10,100),UFLOE(100),E(10,26,6,3)
COMMON ERETIM,EINV,TINV,SALINV,P(10,26,6,3),PINI(10,100)
DOUBLE PRECISION CP0,CLAM,CLAM0,CP1,CLAM1,RHOT
RIX=0.0
RIY=0.0
IF (WINCH.EQ.0.0) GO TO 50
TAUXS=0.0
TAUYS=0.0
1001 FORMAT (1X,'DSVP')
1002 CONTINUE
IF(L.LE.3) GO TO 50
WLAG=1.0
WL=L
WLAG=(WL-1984.)/128.
IF (WLAG.GE.1.0) WLAG=1.0
WSPS=WLAG*WSPD*51.447192
THETA=.001*(.8+.065*.514472*WSPD)
STRESS=THETA*RHOA*WSPS*WSPS
TAUXS=STRESS*SIN(PHI)
TAUYS=STRESS*COS(PHI)
50 CONTINUE
DO 75 I=1,M
DO 75 J=1,N
K1=KH(J,I)
IF (K1.EQ.0) GO TO 75
DO 70 K=1,K1
CP0=5890.+38*T(J,I,K,L1)-.375*T(J,I,K,L1)*T(J,I,K,L1)
CLAM=1779.5+11.25*T(J,I,K,L1)-.0745*T(J,I,K,L1)*T(J,I,K,L1)
CLAM0=3.8+0.01*T(J,I,K,L1)
CP1=CP0+3.0*S(J,I,K,L1)
CLAM1=CLAM-CLAM0*S(J,I,K,L1)
RHOT=CP1/(CLAM1+ALPH0*CP1)-RHOM
RHO(J,I,K)=RHOT
70 CONTINUE
75 CONTINUE
DO 250 I=2,MM1
IP1=I+1
IM1=I-1
DO 250 J=2,NM1
K=KH(J,I)
IF (K.EQ.0) GO TO 250
IF(K.EQ.1) GO TO 55
TEMP2=0.0
K3=1
K2=K
DO 51 KX=K3,K
YY(K2)=0.5*HK(J,I,K2)+TEMP2
TEMP2=TEMP2+HK(J,I,K2)
K2=K-KX
51 CONTINUE
55 IF(HK(J,I,1).LT.(.5*HS(1))) GO TO 56

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      YY(K)=0.5*HK(J,I,K)
      GO TO 59
56  YY(1)=.5*HK(J,I,1)
59  CONTINUE
      DO 60 KX=1,K
      ZL(KX)=0.4*YY(KX)*SQRT(1.-YY(KX)/DEPTH(J,I))
      IF(ZL(KX).GT..5*HK(J,I,K)) ZL(KX)=0.5*HK(J,I,K)
60  CONTINUE
      JP1=J+1
      JM1=J-1
      KR=KH(J,IP1)
      KT=KH(JP1,I)
      W0=0.0
      RHOB=RHOM+.5*(RHO(J,I,1)+RHO(J,IP1,1))
      TAUX(J,I,1)=TAUXS
      RHOB=RHOM+.5*(RHO(J,I,1)+RHO(JP1,I,1))
      TAUY(J,I,1)=TAUYS
      CB=.5*(C(J,I)+C(J,IP1))
      CB=CB*CB
      UB=U(J,I,K,L2)
      UU=UB*UB
      VB=.25*(V(J,I,K,L2)+V(J,IP1,K,L2)+V(JM1,I,K,L2)+V(JM1,IP1,K,L2))
      VV=VB*VB
      RHOB=RHOM+.5*(RHO(J,I,K)+RHO(J,IP1,K))
      TAUX(J,I,K+1)=GRAV*RHOB*UB*SQRT(UU+VV)/CB
      CB=.5*(C(J,I)+C(JP1,I))
      CB=CB*CB
      UB=.25*(U(J,I,K,L2)+U(J,IM1,K,L2)+U(JP1,I,K,L2)+U(JP1,IM1,K,L2))
      UU=UB*UB
      VB=V(J,I,K,L2)
      VV=VB*VB
      RHOB=RHOM+.5*(RHO(J,I,K)+RHO(JP1,I,K))
      TAUY(J,I,K+1)=GRAV*RHOB*VB*SQRT(UU+VV)/CB
      IF (KR.LT.1) GO TO 80
      RHOT=.5*(RHO(J,I,1)+RHO(J,IP1,1)+RHOM2)
      PGRAX(J,I,1)=GRAV*RDY*(RHOT*(SE(J,IP1,L1)-SE(J,I,L1))
      +(.5*(HK(J,I,1)+HK(J,IP1,1))+.5*(SE(J,I,L1)+SE(J,IP1,L1)))
      *
      *(RHO(J,IP1,1)-RHO(J,I,1)))
80  CONTINUE
      IF (KT.LT.1) GO TO 100
      RHOT=.5*(RHO(J,I,1)+RHO(JP1,I,1)+RHOM2)
      PGRAY(J,I,1)=GRAV*RDY*(RHOT*(SE(JP1,I,L1)-SE(J,I,L1))
      +(.5*(HK(J,I,1)+HK(JP1,I,1))+.5*(SE(J,I,L1)+SE(JP1,I,L1)))
      *
      *(RHO(JP1,I,1)-RHO(J,I,1)))
100 CONTINUE
      IF (K.NE.1) GO TO 150
      HC=SE(J,I,L1)+HK(J,I,1)
      HR=HC+SE(J,IP1,L1)+HK(J,IP1,1)
      HL=HC+SE(J,IM1,L1)+HK(J,IM1,1)
      HT=HC+SE(JP1,I,L1)+HK(JP1,I,1)
      HB=HC+SE(JM1,I,L1)+HK(JM1,I,1)
      W0=W0+.5*(RDY*(HR*U(J,I,K,L1)-HL*U(J,IM1,K,L1))
      +
      +RDY*(HT*V(J,I,K,L1)-HB*V(JM1,I,K,L1)))
      GO TO 200
150 CONTINUE
      HC=HK(J,I,K)
      HR=HC+HK(J,IP1,K)
      HL=HC+HK(J,IM1,K)
      HT=HC+HK(JP1,I,K)
      HB=HC+HK(JM1,I,K)
      W0=W0+.5*(RDY*(HR*U(J,I,K,L1)-HL*U(J,IM1,K,L1))

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*      +RDY*(HT*V(J,I,K,L1)-HB*V(JM1,I,K,L1)))
200 CONTINUE
W(J,I,K)=-W0
KP1=K
K=K-1
IF (K.EQ.0) GO TO 250
HTX=HK(J,I,K)+HK(J,I,KP1)
HTY=HTX
HTS=HTX
IF (KR.GE.KP1) HTX=HTX+HK(J,I+1,KP1)
IF (KR.GE.K) HTX=HTX+HK(J,I+1,K)
IF (KT.GE.KP1) HTY=HTY+HK(J+1,I,KP1)
IF (KT.GE.K) HTY=HTY+HK(J+1,I,K)
IF (K.NE.1) GO TO 225
HTX=HTX+SE(J,I,L2)
HTY=HTY+SE(J,I,L2)
IF (KR.NE.0) HTX=HTX+SE(J,IP1,L2)
IF (KT.NE.0) HTY=HTY+SE(JP1,I,L2)
HTS=HTS+SE(J,I,L2)
225 CONTINUE
DRHOX=((RHO(J,I,K)+RHO(J,IP1,K))-(RHO(J,I,KP1)
*+RHO(J,IP1,KP1)))/HTX
DRHOY=((RHO(J,I,K)+RHO(JP1,I,K))-(RHO(J,I,KP1)
*+RHO(JP1,I,KP1)))/HTY
DUDZ=(U(J,I,K,L2)-U(J,I,KP1,L2))/(HTX*.5)
DVDZ=(V(J,I,K,L2)-V(J,I,K+1,L2))/(HTX*.5)
K1=KH(J,I)
IF (RIX.GT. 5.) RIX= 5.
IF (RIY.GT. 5.) RIY= 5.
IF (RIX.LT. -.5) RIX=-.5
IF (RIY.LT. -.5) RIY=-.5
HTX=.25*HTX
HTY=.25*HTY
UB=.5*(U(J,I,K,L1)+U(J,I,KP1,L1))
VB=.5*(V(J,I,K,L1)+V(J,I,KP1,L1))
WBX=(W(J,I,KP1)+W(J,IP1,KP1))* .5
WBY=(W(J,I,KP1)+W(JP1,I,KP1))* .5
DU=U(J,I,K,L2)-U(J,I,KP1,L2)
DV=V(J,I,K,L2)-V(J,I,KP1,L2)
UBK=.25*(U(J,I,K,L2)+U(J,IM1,K,L2)+U(JP1,I,K,L2)+U(JP1,IM1,K,L2))
UBKP=.25*(U(J,I,KP1,L2)+U(J,IM1,KP1,L2)
*      +U(JP1,I,KP1,L2)+U(JP1,IM1,KP1,L2))
VBK=.25*(V(J,I,K,L2)+V(J,IP1,K,L2)+V(JM1,I,K,L2)+V(JM1,IP1,K,L2))
VBKP=.25*(V(J,I,KP1,L2)+V(J,IP1,KP1,L2)
*      +V(JM1,I,KP1,L2)+V(JM1,IP1,KP1,L2))
DUB=UBK-UBKP
DVB=VBK-VBKP
RHOB=RHOM+.5*(RHO(J,I,K)+RHO(J,I,KP1))
CNUX=.5*(EPSLON(J,I,KP1)+EPSLON(J,IP1,KP1))
CNUY=.5*(EPSLON(J,I,KP1)+EPSLON(JP1,I,KP1))
RHOBXZ=RHOM+.25*(RHO(J,I,K)+RHO(J,I,KP1)
*      +RHO(J,IP1,K)+RHO(J,IP1,KP1))
RHOBYZ=RHOM+.25*(RHO(J,I,K)+RHO(J,I,KP1)
*      +RHO(JP1,I,K)+RHO(JP1,I,KP1))
TAUX(J,I,KP1)=CNUX*RHOBXZ
C *      *(SQRT(DU*DU+DVB*DVB))*RHOBXZ
TAUY(J,I,KP1)=CNUY*RHOBYZ
C *      *(SQRT(DUB*DUB+DV*DV))*RHOBYZ
GO TO 100
250 CONTINUE
DO 400 K=2,KMAX

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      KM1=K-1
      DO 300 I=1,M
      DO 300 J=1,N
      RHOZ(J,I)=RHO(J,I,K)+RHO(J,I,KM1)
      H=0.0
      IF (KH(J,I).GE.KM1) H=HK(J,I,KM1)
      IF (KH(J,I).GE.K) H=H+HK(J,I,K)
      IF (K.EQ.2) H=H+SE(J,I,L1)
      HZ(J,I)=H
300  CONTINUE
      DO 350 I=1,MM1
      IP1=I+1
      DO 350 J=1,NM1
      JP1=J+1
      K1=KH(J,I)
      IF (K1.LT.K) GO TO 350
      KR=KH(J,IP1)
      KT=KH(JP1,I)
      IF (KR.LT.K) GO TO 325
      PGRAX(J,I,K)=PGRAX(J,I,KM1)
      *      +.125*GRAV*(HZ(J,I)+HZ(J,IP1))*RDX*(RHOZ(J,IP1)-RHOZ(J,I))
325  CONTINUE
      IF (KT.LT.K) GO TO 350
      PGRAY(J,I,K)=PGRAY(J,I,KM1)
      *      +.125*GRAV*(HZ(J,I)+HZ(JP1,I))*RDY*(RHOZ(JP1,I)-RHOZ(J,I))
350  CONTINUE
400  CONTINUE
      RETURN
      END
      SUBROUTINE SESUV
      COMMON CPO,CLAM,CLAMO,CP1,CLAM1,RHOT
      COMMON M,MM1,MM2,N,NM1,NM2,DX,DY,RDX,RDY,RDXS,RDYS
      COMMON DT,DT2,DT4,CNU,L,L1,L2,L3,GRAV,KMAX,CKA,AHX,AHY,AMX,AMY,F
      COMMON RHOM,RHOM2,ALPHO,TEMP,THETA,RHOA,WSPD,PHI,WINCH,DEPMIN
      COMMON PI,KMAP,TD,TD2,TD4,KHT,CT,CKH,AHXT,AHYT
      COMMON RMM1,RMM2,CKP,AHXP,AHYP,PRETIM,PINV,DECAY,LMAX,CONVER
      COMMON U(10,26,6,3),V(10,26,6,3),S(10,26,6,3),T(10,26,6,3)
      COMMON W(10,26,7),TAUX(10,26,7),TAUY(10,26,7),EPSLON(10,26,7)
      COMMON PGRAX(10,26,6),PGRAY(10,26,6),RHO(10,26,6),HK(10,26,6)
      COMMON SE(10,26,3),KH(10,26),C(10,26),HS(7),YY(6),DEPTH(10,26)
      COMMON RHOZ(10,26),HZ(10,26),NSE(10,26),ZL(6)
      COMMON IB(100),JB(100),KB(100),UVB(100),AHUV(10,3,6)
      COMMON UFLOS(100),UFLOT(100),UFLOP(100),PTURN(10,100)
      COMMON STMP,LTURN(10,100),STURN(10,100),SET,TTURN(10,100)
      COMMON SINI(10,100),TINI(10,100),SRETIM,TRETIM
      COMMON ETURN(10,100),EINI(10,100),UFLOE(100),E(10,26,6,3)
      COMMON ERETIM,EINV,TINV,SALINV,P(10,26,6,3),PINI(10,100)
      DOUBLE PRECISION CPO,CLAM,CLAMO,CP1,CLAM1,RHOT
      DIMENSION XU(10),YU(10),XV(10),YV(10),XP(10),YP(10)
      DIMENSION XS(10),YS(10),XT(10),YT(10),XE(10),YE(10)
11  FORMAT(' SESUV: SE COMPUTED')
12  FORMAT(' SESUV: LEVEL ONE DEPTHS COMPUTED')
13  FORMAT(' SESUV: HEAT & SALINITY COMPUTED')
14  FORMAT(' SESUV: E-W VELOCITY COMPUTED')
15  FORMAT(' SESUV: N-S VELOCITY COMPUTED')
16  FORMAT(' SESUV: LEVELS SWITCHED')
17  FORMAT(' SESUV: INFLOW PARAMS SET')
18  FORMAT(' SESUV: OPEN BOUND. CONC. SET')
19  FORMAT(1X,'HEAT & SALINITY',2I10)
20  FORMAT(1X,'***** ESQRT1.LT.0.0 IN SESUV',4I5,1X,1PE12.5)
21  FORMAT(1X,'***** ESQRT2.LT.0.0 IN SESUV',4I5,1X,1PE12.5)

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DO 100 I=1,M
DO 100 J=1,N
SE(J,I,L1)=SE(J,I,L3)+DT2*W(J,I,1)
100 CONTINUE
LASTCK = 0
CHKOUT=4
IF (L .LE. LASTCK) WRITE (6,11)
CALL TIDAL
IAH=1
IAHP=2
IAHM=3
I=1
DO 110 K=1,KMAX
AHUV(1,IAHM,K)=0.0
AHUV(1,IAH,K)=0.0
AHUV(1,IAHP,K)=0.0
AHUV(N,IAHM,K)=0.0
AHUV(N,IAH,K)=0.0
AHUV(N,IAHP,K)=0.0
DO 110 J=2,NM1
AHUV(J,IAHM,K)=0.0
AHUV(J,IAH,K)=0.0
AHUV(J,IAHP,K)=.5*(DX+DY)*ABS(V(J,I+2,K,L3)-V(J,I+1,K,L3)
* -V(J-1,I+1,K,L3)+V(J-1,I,K,L3)-U(J+1,I+1,K,L3)
* +U(J,I+1,K,L3)+U(J,I,K,L3)-U(J-1,I,K,L3))
110 CONTINUE
DO 300 I=2,MM2
IM1=I-1
IP1=I+1
IP2=I+2
IT=IAHM
IAHM=IAH
IAH=IAHP
IAHP=IT
DO 300 J=2,NM2
DO 301 K=1,KMAX
AHUV(J+1,IAHP,K)=0.0
AHUV(J,IAHP,K)=.5*(DX+DY)*ABS(V(J,I+2,K,L3)-V(J,I+1,K,L3)
* -V(J-1,I+1,K,L3)+V(J-1,I,K,L3)-U(J+1,I+1,K,L3)
* +U(J,I+1,K,L3)+U(J,I,K,L3)-U(J-1,I,K,L3))
301 CONTINUE
K1=KH(J,I)
IF (K1.EQ.0) GO TO 300
IF(K1.EQ.1) GO TO 55
TEMP2=0.0
K3=1
K2=K1
DO 50 K=K3,K1
YY(K2)=0.5*HK(J,I,K2)+TEMP2
TEMP2=TEMP2+HK(J,I,K2)
K2=K1-K
50 CONTINUE
55 IF(HK(J,I,1).LT.(.5*HS(1))) GO TO 56
YY(K1)=0.5*HK(J,I,K1)
GO TO 59
56 YY(1)=.5*HK(J,I,1)
59 CONTINUE
DO 60 K=1,K1
ZL(K)=0.4*YY(K)*SQRT(1.-YY(K)/DEPTH(J,I))
IF(ZL(K).GT..5*HK(J,I,K)) ZL(K)=.5*HK(J,I,K)
60 CONTINUE

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```

K1U=K1
K1V=K1
BND1=1
BND2=1
BND3=1
BND4=1
BND5=1
DO 400 IJ=1,100
KBT=0
IBT=IB(IJ)
IF (IBT.EQ.0) GO TO 500
JBT=JB(IJ)
KBT=KB(IJ)
IF (IBT.NE.I) GO TO 370
IF (JBT.NE.J) GO TO 350
BND1=0
BND2=0
BND3=0
BND4=0
BND5=0
IF (KBT.EQ.1) BND4=1
IF (KBT.EQ.2) BND5=1
GO TO 500
350 CONTINUE
IF (IABS(KBT).NE.2) GO TO 360
IF (J.NE.JBT+1) GO TO 400
BND2=0
GO TO 500
360 CONTINUE
IF (IABS(KBT).NE.4) GO TO 400
IF (J.NE.JBT-1) GO TO 400
BND3=0
IF (KBT.LT.0) BND5=0
GO TO 500
370 CONTINUE
IF (J.NE.JBT) GO TO 400
IF (IABS(KBT).NE.1) GO TO 380
IF (I.NE.IBT+1) GO TO 400
BND3=0
GO TO 500
380 CONTINUE
IF (IABS(KBT).NE.3) GO TO 400
IF (I.NE.IBT-1) GO TO 400
BND2=0
IF (KBT.LT.0) BND4=0
GO TO 500
400 CONTINUE
500 CONTINUE
JM1=J-1
JP1=J+1
JP2=J+2
KR=KH(J,IP1)
KT=KH(JP1,I)
H=HK(J,I,1)
H2=H+H
K=1
KN1=1
KP1=2
KQ1=KP1
IF (K1.EQ.K) KQ1=K
HS1=SE(J,I,L1)+HK(J,I,1)

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```

HBU1=SE(J,I,L1)+SE(J,I+1,L1)+HK(J,I,1)+HK(J,I+1,1)
HBV1=SE(J,I,L1)+SE(J+1,I,L1)+HK(J,I,1)+HK(J+1,I,1)
HSR=SE(J,I+1,L3)+HK(J,I+1,1)
HC=SE(J,I,L3)+HK(J,I,1)
HSL=SE(J,I-1,L3)+HK(J,I-1,1)
HST=SE(J+1,I,L3)+HK(J+1,I,1)
HSB=SE(J-1,I,L3)+HK(J-1,I,1)
HBR=SE(J,I+1,L2)+SE(J,I+2,L2)+HK(J,I+1,1)+HK(J,I+2,1)
HB=SE(J,I,L2)+SE(J,I+1,L2)+HK(J,I,1)+HK(J,I+1,1)
HBL=SE(J,I-1,L2)+SE(J,I,L2)+HK(J,I-1,1)+HK(J,I,1)
HR=SE(J,I,L2)+SE(J+1,I,L2)+HK(J,I,1)+HK(J+1,I,1)
HBR=SE(J,I+1,L2)+SE(J+1,I+1,L2)+HK(J,I+1,1)+HK(J+1,I+1,1)
HBLB=SE(J-1,I,L2)+SE(J,I,L2)+HK(J-1,I,1)+HK(J,I,1)
HBRB=SE(J,I+1,L2)+SE(J-1,I+1,L2)+HK(J,I+1,1)+HK(J-1,I+1,1)
HBTR=SE(J+1,I,L2)+SE(J+1,I+1,L2)+HK(J+1,I,1)+HK(J+1,I+1,1)
HBT=SE(J+1,I,L2)+SE(J+1,I,L2)+HK(J+1,I-1,1)+HK(J+1,I,1)
HBLT=SE(J,I-1,L2)+SE(J+1,I-1,L2)+HK(J,I-1,1)+HK(J+1,I-1,1)
HBR3=SE(J,I+1,L3)+SE(J,I+2,L3)+HK(J,I+1,1)+HK(J,I+2,1)
HB3=SE(J,I,L3)+SE(J,I+1,L3)+HK(J,I,1)+HK(J,I+1,1)
HBL3=SE(J,I-1,L3)+SE(J,I,L3)+HK(J,I-1,1)+HK(J,I,1)
HBTR3=SE(J+1,I,L3)+SE(J+1,I+1,L3)+HK(J+1,I,1)+HK(J+1,I+1,1)
HBB3=SE(J-1,I,L3)+SE(J-1,I+1,L3)+HK(J-1,I,1)+HK(J-1,I+1,1)
HBR3=SE(J,I+1,L3)+SE(J+1,I+1,L3)+HK(J,I+1,1)+HK(J+1,I+1,1)
HBLT3=SE(J,I,L3)+SE(J+1,I,L3)+HK(J,I,1)+HK(J+1,I,1)
HBLT3=SE(J,I-1,L3)+SE(J+1,I-1,L3)+HK(J,I-1,1)+HK(J+1,I-1,1)
HBT3=SE(J+1,I,L3)+SE(J+1,I,L3)+HK(J+1,I,1)+HK(J+1,I,1)
HBLB3=SE(J-1,I,L3)+SE(J,I,L3)+HK(J-1,I,1)+HK(J,I,1)
HC2=SE(J,I,L2)+HK(J,I,1)
CU=0.0
CV=0.0
UUP=0.0
VVP=0.0
XU(1)=0.0
XV(1)=0.0
YU(1)=0.0
YV(1)=0.0
CS=0.0
CTT=0.0
CE=0.0
CP=0.0
SSP=0.0
TTP=0.0
EEP=0.0
PPP=0.0
XS(1)=0.0
XT(1)=0.0
XE(1)=0.0
XP(1)=0.0
YS(1)=0.0
YT(1)=0.0
YE(1)=0.0
YP(1)=0.0
Q1=0.0
Q3=0.0
Q5=0.0
Q7=0.0
HQ=.5*(H+HK(J,I,KQ1))
IF (L .LE. LASTCK) WRITE (6,12)
DUVDZ=0.0
200 CONTINUE

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      AHUV(J+1,IAHP,K)=.5*(DX+DY)*ABS(V(J+1,I+2,K,L3)-V(J+1,I+1,K,L3)
*      -V(J,I+1,K,L3)+V(J,I,K,L3)-U(J+2,I+1,K,L3)
*      +U(J+1,I+1,K,L3)+U(J+1,I,K,L3)-U(J,I,K,L3))
      IF (HQ.EQ.0.0) GO TO 1000
      IF (HS1.EQ.0.0) GO TO 1000
      IF (HBU1.EQ.0.0) GO TO 1000
      IF (HBV1.EQ.0.0) GO TO 1000
      IF (RHO(J,I,K)+RHO(J,IP1,K)+RHOM2.EQ.0.0) GO TO 1000
      IF (RHO(J,I,K)+RHO(JP1,I,K)+RHOM2.EQ.0.0) GO TO 1000
      GO TO 1010
1000 WRITE (6,1001) I,J,K,L,HQ,HS1,HBU1,HBV1
      WRITE (6,1002) RHOM2,RHO(J,I,K),RHO(J,IP1,K),RHO(JP1,I,K)
1001 FORMAT (1H0,'SESUV D.C.',4I10,4(1X,1PE12.5))
1002 FORMAT (10X,'RHO',4(1X,1PE12.5))
1010 CONTINUE
      HTM=HK(J,I,K)+HK(J,I,KN1)
      HTX=HK(J,I,K)+HK(J,I,KQ1)
      HTY=HTX
      HTS=HTX
      HTD=HTX
      IF (KR.GE.KQ1) HTX=HTX+HK(J,I+1,KQ1)
      IF (KR.GE.K) HTX=HTX+HK(J,I+1,K)
      IF (KT.GE.KQ1) HTY=HTY+HK(J+1,I,KQ1)
      IF (KT.GE.K) HTY=HTY+HK(J+1,I,K)
      IF (K.NE.1) GO TO 2250
      HTX=HTX+SE(J,I,L2)
      HTY=HTY+SE(J,I,L2)
      IF (KR.NE.0) HTX=HTX+SE(J,IP1,L2)
      IF (KT.NE.0) HTY=HTY+SE(JP1,I,L2)
      HTS=HTS+SE(J,I,L2)
      HTM=HTM+SE(J,I,L2)
2250 CONTINUE
      IF (BND1.EQ.0.0) GO TO 212
      A1=.68
      A2=.1
      A3=1.0
      A4=1.0
      IF (L.LE.LASTCK) WRITE (6,19) I,J
      RHOB=RHOM+.5*(RHO(J,I,K)+RHO(J,I,KQ1))
      DRHO=(RHO(J,I,K)-RHO(J,I,KQ1))/(0.5*HTS)
      ZLB=.5*(ZL(K)+ZL(KQ1))
      EBZ=.5*(E(J,I,K,L3)+E(J,I,KQ1,L3))
      RICH=-(GRAV*DRHO*ZLB*ZLB)/(RHOB*EBZ+.000001)
      IF(RICH.GT. 5.) RICH= 5.
      IF(RICH.LT. -5.) RICH=-5.
      ZEXP=EXP(-RMM2*RICH)
      ZEXP=1.0
      DUVDZ1=DUVDZ
      UBAR=U(J,I,K,L2)+U(J,I-1,K,L2)
      VBAR=V(J,I,K,L2)+V(J-1,I,K,L2)
      DUVDZ2= ((UBAR-(U(J,I,KQ1,L2)+U(J,I-1,KQ1,L2)))*2
*      +(VBAR-(V(J,I,KQ1,L2)+V(J-1,I,KQ1,L2)))*2)/(HTS*HTS)
      IF(DUVDZ2.LT..00000001) DUVDZ2=.00000001
      DUVDZ=SQRT(DUVDZ2)
      EN=E(J,I,K,L2)
      IF (EN.GE.0.0) GO TO 2256
      WRITE (6,2255) EN,I,J,K,L
2255 FORMAT (1X,'NEGATIVE ENERGY = ',1PE10.3,' I,J,K,L= ',4I5)
      E(J,I,K,L2)=EINV
      EN=EINV
2256 CONTINUE

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ESQRT1=E(J,I,K,L3)
IF (ESQRT1.LT.0.0) WRITE (6,20) I,J,K,L,ESQRT1
IF (ESQRT1.LT.0.0) ESQRT1=0.0
ESQRT1=SQRT(ESQRT1)
ESQRT2=E(J,I,KQ1,L3)
IF (ESQRT2.LT.0.0) WRITE (6,21) I,J,K,L,ESQRT2
IF (ESQRT2.LT.0.0) ESQRT2=0.0
ESQRT2=SQRT(ESQRT2)
IF(K1.NE.1) GOTO 2352
GENN1=.0
VEL=.25*(UBAR*UBAR+VBAR*VBAR)
GEN=A3*GRAV*VEL*SQRT(VEL)/(C(J,I)*C(J,I))
DISP=((H+SE(J,I,L2))*A2*ESQRT1)/ZL(K)
DENGRA=0.0
GOTO 2360
2352 CONTINUE
IF(K.NE.K1) GOTO 2354
HESQ=HK(J,I,K) *ESQRT1
GENN1=GENN2
VEL=.25*(UBAR*UBAR+VBAR*VBAR)
GENN2=A3*GRAV*VEL*SQRT(VEL)/(C(J,I)*C(J,I))
GEN=.5*GENN1+GENN2
DISP=(HK(J,I,K1) *A2*ESQRT1)/ZL(K)
DENGRA=A3*ZL(K)*ESQRT1*(GRAV/(RHOM+RHO(J,I,K)))*
* ((RHO(J,I,K-1)-RHO(J,I,K)))/(.5*HTS)
GOTO 2360
2354 CONTINUE
IF(K.NE.1) GOTO 2356
GENN1=0.
EPSLON(J,I,2)=.5*(ZL(K)*ESQRT1
*+ZL(KQ1)*ESQRT2)
HESQ=(H+SE(J,I,L2))*ESQRT1
GENN2=A3*ZEXP*EPSLON(J,I,2)*.5*HTS*DUVDZ*DUVDZ
DISP=(HESQ*A2)/ZL(K)
GEN=.5*GENN2
DENGRA=A3*ZL(K)*ESQRT1*(GRAV/(RHOM+RHO(J,I,K)))*
* ((RHO(J,I,K)-RHO(J,I,K+1)))/(.5*HTS)
GOTO 2360
2356 CONTINUE
EPSLON(J,I,K+1)=.5*(ZL(K)*ESQRT1
*+ZL(KQ1)*ESQRT2)
HESQ= HK(J,I,K) *ESQRT1
GENN1=GENN2
GENN2=A3*ZEXP*EPSLON(J,I,K+1)*.5*HTS*DUVDZ*DUVDZ
GEN=.5*(GENN1+GENN2)
DISP=(HESQ*A2)/ZL(K)
DENGRA=A3*ZL(K)*ESQRT1*(GRAV/(RHOM+RHO(J,I,K)))*
* ((RHO(J,I,K-1)-RHO(J,I,K+1)))/HK(J,I,K)
2360 CONTINUE
IF (L.LE.LASTCK) WRITE (6,19) I,J
CKA=A4*EPSLON(J,I,KQ1)
CKH=A4*EPSLON(J,I,KQ1)
CKE=A1*EPSLON(J,I,KQ1)
CKP=A4*EPSLON(J,I,KQ1)
Q2A=CKA*ZEXP/HTD
Q2=-0.5*W(J,I,KP1)*(S(J,I,K,L2)+S(J,I,KQ1,L2))
Q4A=CKH*ZEXP/HTD
Q4=-0.5*W(J,I,KP1)*(T(J,I,K,L2)+T(J,I,KQ1,L2))
Q6A=CKE *ZEXP/HTD
Q6=-0.5*W(J,I,KP1)*(E(J,I,K,L2)+E(J,I,KQ1,L2))
Q8A=CKP*ZEXP/HTD

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Q8=-0.5*W(J,I,KP1)*(P(J,I,K,L2)+P(J,I,KQ1,L2))
IF (L.LE.LASTCK) WRITE (6,19) I,J
AHXRB=.5*(AHUV(J,IAHP,K)+AHUV(J,IAH,K))
AHXTB=.5*(AHUV(J+1,IAHP,K)+AHUV(J+1,IAH,K))
AHXBB=.5*(AHUV(J-1,IAHP,K)+AHUV(J-1,IAH,K))
AHXLB=.5*(AHUV(J,IAH,K)+AHUV(J,IAHM,K))
AHYTB=.5*(AHUV(J+1,IAH,K)+AHUV(J,IAH,K))
AHYRB=.5*(AHUV(J+1,IAHP,K)+AHUV(J,IAHP,K))
AHYLB=.5*(AHUV(J+1,IAHM,K)+AHUV(J,IAHM,K))
AHYBB=.5*(AHUV(J,IAH,K)+AHUV(J-1,IAH,K))
IF (KH(JP1,I).GE.K) GO TO 901
AHYST=0.0
AHYTT=0.0
AHYPT=0.0
GO TO 902
901 CONTINUE
AHYST=AHYTB+AHY
AHYTT=AHYTB+AHYT
AHYPT=AHYTB+AHYP
902 CONTINUE
IF (KH(JM1,I).GE.K) GO TO 903
AHYSB=0.0
AHYTB=0.0
AHYPB=0.0
GO TO 904
903 CONTINUE
AHYSB=AHYBB+AHY
AHYTB=AHYBB+AHYT
AHYPB=AHYBB+AHYP
904 CONTINUE
IF (KH(J,IP1).GE.K) GO TO 905
AHXSR=0.0
AHXTR=0.0
AHXPR=0.0
GO TO 906
905 CONTINUE
AHXSR=AHXRB+AHX
AHXTR=AHXRB+AHXT
AHXPR=AHXRB+AHXP
906 CONTINUE
IF (KH(J,IM1).GE.K) GO TO 907
AHXSL=0.0
AHXTL=0.0
AHXPL=0.0
GO TO 908
907 CONTINUE
AHXSL=AHXLB+AHX
AHXTL=AHXLB+AHXT
AHXPL=AHXLB+AHXP
908 CONTINUE
T1=RDY*(HB*U(J,I,K,L2)*(S(J,I,K,L2)+S(J,IP1,K,L2))
* -HBL*U(J,IM1,K,L2)*(S(J,IM1,K,L2)+S(J,I,K,L2)))
T2=RDY*(HR*V(J,I,K,L2)*(S(J,I,K,L2)+S(JP1,I,K,L2))
* -HBLB*V(JM1,I,K,L2)*(S(JM1,I,K,L2)+S(J,I,K,L2)))
HSK=(HC+HC)*S(J,I,K,L3)
SJI=S(J,I,K,L3)
T3=(HB3*(S(J,IP1,K,L3)-SJI)*AHXSR
* -HBL3*(SJI-S(J,IM1,K,L3))*AHXSL)*RDYS*.5
T4=(HBLT3*(S(JP1,I,K,L3)-SJI)*AHYST
* -HBLB3*(SJI-S(JM1,I,K,L3))*AHYSB)*RDYS*.5
DHS=-.25*(T1+T2)+Q1-Q2+T3+T4

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T1=RDY*(HB*U(J,I,K,L2)*(T(J,I,K,L2)+T(J,IP1,K,L2))
*   -HBL*U(J,IM1,K,L2)*(T(J,IM1,K,L2)+T(J,I,K,L2)))
T2=RDY*(HR*V(J,I,K,L2)*(T(J,I,K,L2)+T(JP1,I,K,L2))
*   -HBLB*V(JM1,I,K,L2)*(T(JM1,I,K,L2)+T(J,I,K,L2)))
HTK=(HC+HC)*T(J,I,K,L3)
TJI=T(J,I,K,L3)
T3=(HB3*(T(J,IP1,K,L3)-TJI)*AHXTR
*   -HBL3*(TJI-T(J,IM1,K,L3))*AHXTL)*RDXS*.5
T4=(HBLT3*(T(JP1,I,K,L3)-TJI)*AHYTT
*   -HBLB3*(TJI-T(JM1,I,K,L3))*AHYTB)*RDYS*.5
DHT=-.25*(T1+T2)+Q3-Q4+T3+T4
T1=RDY*(HB*U(J,I,K,L2)*(E(J,I,K,L2)+E(J,IP1,K,L2))
*   -HBL*U(J,IM1,K,L2)*(E(J,IM1,K,L2)+E(J,I,K,L2)))
T2=RDY*(HR*V(J,I,K,L2)*(E(J,I,K,L2)+E(JP1,I,K,L2))
*   -HBLB*V(JM1,I,K,L2)*(E(JM1,I,K,L2)+E(J,I,K,L2)))
HEK=(HC+HC)*E(J,I,K,L3)
EJI=E(J,I,K,L3)
T3=(HB3*(E(J,IP1,K,L3)-EJI)*AHXTR
*   -HBL3*(EJI-E(J,IM1,K,L3))*AHXTL)*RDXS*.5
T4=(HBLT3*(E(JP1,I,K,L3)-EJI)*AHYTT
*   -HBLB3*(EJI-E(JM1,I,K,L3))*AHYTB)*RDYS*.5
DHE=-.25*(T1+T2)+Q5-Q6+T3+T4+GEN-DISP*(E(J,I,K,L3)+E(J,I,K,L2))* .5
*   +DENGRA
T1=RDY*(HB*U(J,I,K,L2)*(P(J,I,K,L2)+P(J,IP1,K,L2))
*   -HBL*U(J,IM1,K,L2)*(P(J,IM1,K,L2)+P(J,I,K,L2)))
T2=RDY*(HR*V(J,I,K,L2)*(P(J,I,K,L2)+P(JP1,I,K,L2))
*   -HBLB*V(JM1,I,K,L2)*(P(JM1,I,K,L2)+P(J,I,K,L2)))
HPK=(HC+HC)*P(J,I,K,L3)
PJI=P(J,I,K,L3)
T3=(HB3*(P(J,IP1,K,L3)-PJI)*AHXPR
*   -HBL3*(PJI-P(J,IM1,K,L3))*AHXPL)*RDXS*.5
T4=(HBLT3*(P(JP1,I,K,L3)-PJI)*AHYPT
*   -HBLB3*(PJI-P(JM1,I,K,L3))*AHYPB)*RDYS*.5
DHP=-.25*(T1+T2)+Q7-Q8+T3+T4-DECAY*P(J,I,K,L3)*HC2
GO TO 211
210 CONTINUE
DHS=Q1-Q2
DHT=Q3-Q4
DHE=Q5-Q6
DHEI=Q5-Q6+GEN-DISP*E(J,I,K,L3)
DHP=Q7-Q8
211 CONTINUE
IF (K.GE.K1) GO TO 2421
AS=CS
CS=Q2A
SSM=SSP
SSP=S(J,I,K,L3)-S(J,I,KQ1,L3)
BS=HS1/DT2+AS+CS
DS=DHS+HC*S(J,I,K,L3)/DT2-CS*SSP+AS*SSM
DEN=BS-AS*XS(K)
XS(KP1)=CS/DEN
YS(KP1)=(DS+AS*YS(K))/DEN
GO TO 2441
2421 CONTINUE
IF (K1.LE.1) GO TO 2461
BS=HK(J,I,K)/DT2
DS=DHS+BS*S(J,I,K,L3)+CS*(S(J,I,K-1,L3)-S(J,I,K,L3))
BS=BS+CS
IF (CS.EQ.0.0) S(J,I,K,L1)=(HC*S(J,I,K,L3)+DT2*DHS)/HS1
IF (CS.NE.0.0) S(J,I,K,L1)=(DS+CS*YS(K))/(BS-CS*XS(K))
IF (ABS(S(J,I,K,L1)).LT.1.0E-20) S(J,I,K,L1)=0.0

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DO 2431 KKK=2,K1
KK=K1-KKK+2
S(J,I,KK-1,L1)=XS(KK)*S(J,I,KK,L1)+YS(KK)
IF (ABS(S(J,I,KK-1,L1)).LT.1.0E-20) S(J,I,KK-1,L1)=0.0
2431 CONTINUE
GO TO 2441
2461 CONTINUE
S(J,I,K,L1)=(HC*S(J,I,K,L3)+DT2*DHS)/HS1
IF (ABS(S(J,I,K,L1)).LT.1.0E-20) S(J,I,K,L1)=0.0
2441 CONTINUE
IF (K.GE.K1) GO TO 2422
AT=CTT
CTT=Q4A
TTM=TTP
TTP=T(J,I,K,L3)-T(J,I,KQ1,L3)
BT=HS1/DT2+AT+CTT
DTT=DHT+HC*T(J,I,K,L3)/DT2-CTT*TTP+AT*TTM
DEN=BT-AT*XT(K)
XT(KP1)=CTT/DEN
YT(KP1)=(DTT+AT*YT(K))/DEN
GO TO 2442
2422 CONTINUE
IF (K1.LE.1) GO TO 2462
BT=HK(J,I,K)/DT2
DTT=DHT+BT*T(J,I,K,L3)+CTT*(T(J,I,K-1,L3)-T(J,I,K,L3))
BT=BT+CTT
IF (CTT.EQ.0.0) T(J,I,K,L1)=(HC*T(J,I,K,L3)+DT2*DHT)/HS1
IF (CTT.NE.0.0) T(J,I,K,L1)=(DTT+CTT*YT(K))/(BT-CTT*XT(K))
IF (ABS(T(J,I,K,L1)).LT.1.0E-20) T(J,I,K,L1)=0.0
DO 2432 KKK=2,K1
KK=K1-KKK+2
T(J,I,KK-1,L1)=XT(KK)*T(J,I,KK,L1)+YT(KK)
IF (ABS(T(J,I,KK-1,L1)).LT.1.0E-20) T(J,I,KK-1,L1)=0.0
2432 CONTINUE
GO TO 2442
2462 CONTINUE
T(J,I,K,L1)=(HC*T(J,I,K,L3)+DT2*DHT)/HS1
IF (ABS(T(J,I,K,L1)).LT.1.0E-20) T(J,I,K,L1)=0.0
2442 CONTINUE
IF (K.GE.K1) GO TO 2424
AP=CP
CP=Q8A
PPM=PPP
PPP=P(J,I,K,L3)-P(J,I,KQ1,L3)
BP=HS1/DT2+AP+CP
DP=DHP+HC*P(J,I,K,L3)/DT2-CP*PPP+AP*PPM
DEN=BP-AP*XP(K)
XP(KP1)=CP/DEN
YP(KP1)=(DP+AP*YP(K))/DEN
GO TO 2444
2424 CONTINUE
IF (K1.LE.1) GO TO 2464
BP=HK(J,I,K)/DT2
DP=DHP+BP*P(J,I,K,L3)+CP*(P(J,I,K-1,L3)-P(J,I,K,L3))
BP=BP+CP
IF (CP.EQ.0.0) P(J,I,K,L1)=(HC*P(J,I,K,L3)+DT2*DHP)/HS1
IF (CP.NE.0.0) P(J,I,K,L1)=(DP+CP*YP(K))/(BP-CP*XP(K))
IF (ABS(P(J,I,K,L1)).LT.1.0E-20) P(J,I,K,L1)=0.0
DO 2434 KKK=2,K1
KK=K1-KKK+2
P(J,I,KK-1,L1)=XP(KK)*P(J,I,KK,L1)+YP(KK)

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      IF (ABS(P(J,I,KK-1,L1)).LT.1.0E-20) P(J,I,KK-1,L1)=0.0
2434 CONTINUE
      GO TO 2444
2464 CONTINUE
      P(J,I,K,L1)=(HC*P(J,I,K,L3)+DT2*DHP)/HS1
      IF (ABS(P(J,I,K,L1)).LT.1.0E-20) P(J,I,K,L1)=0.0
2444 CONTINUE
      IF (K.GE.K1) GO TO 2423
      AE=CE
      CE=Q6A
      EEM=E*P
      EEP=E(J,I,K,L3)-E(J,I,KQ1,L3)
      BE=HS1/DT2+AE+CE
      DE=DHE+HC*E(J,I,K,L3)/DT2-CE*EEP+AE*EEM
      DEN=BE-AE*XE(K)
      XE(KP1)=CE/DEN
      YE(KP1)=(DE+AE*YE(K))/DEN
      GO TO 2443
2423 CONTINUE
      IF (K1.LE.1) GO TO 2463
      BE=HK(J,I,K)/DT2
      DE=DHE+BE*E(J,I,K,L3)+CE*(E(J,I,K-1,L3)-E(J,I,K,L3))
      BE=BE+CE
      IF (CE.EQ.0.0) E(J,I,K,L1)=(HC*E(J,I,K,L3)+DT2*DHE)/HS1
      IF (CE.NE.0.0) E(J,I,K,L1)=(DE+CE*YE(K))/(BE-CE*XE(K))
      IF (ABS(E(J,I,K,L1)).LT.1.0E-20) E(J,I,K,L1)=0.001
      DO 2433 KKK=2,K1
      KK=K1-KKK+2
      E(J,I,KK-1,L1)=XE(KK)*E(J,I,KK,L1)+YE(KK)
      IF (ABS(E(J,I,KK-1,L1)).LT.1.0E-20) E(J,I,KK-1,L1)=0.001
2433 CONTINUE
      GO TO 2443
2463 CONTINUE
      E(J,I,K,L1)=(HC*E(J,I,K,L3)+DT2*DHE)/HS1
      IF (ABS(E(J,I,K,L1)).LT.1.0E-20) E(J,I,K,L1)=0.001
2443 CONTINUE
212 CONTINUE
      IF (L.LE.LASTCK) WRITE (6,13)
      IF (KR.EQ.0) GO TO 225
      IF (BND4.EQ.0.0) GOTO 225
      GO TO 215
2351 CONTINUE
      IF (BND2.EQ.0.0) GO TO 215
      IF (K.LE.KR) GO TO 215
      IF (KR.LE.1) GO TO 215
      DO 214 KK=K,K1
      XU(KK)=0.0
      YU(KK)=0.0
214 CONTINUE
      GO TO 217
215 CONTINUE
      RHOBX=(RHO(J,I,K)+RHO(J,IP1,K))+RHOM2
      UBP=U(J,I,K,L2)
      IF (K.LT.K1) UBP=UBP+U(J,I,KQ1,L2)
      WBXP=W(J,I,KP1)+W(J,IP1,KP1)
      T6=HB*PGRAX(J,I,K)/RHOBX
      IF (BND2.EQ.0.0) GO TO 220
      T1=(U(J,I,K,L2)+U(J,IP1,K,L2))
      * *(HBR*U(J,IP1,K,L2)+HB*U(J,I,K,L2))
      T2=(U(J,IM1,K,L2)+U(J,I,K,L2))
      * *(HB*U(J,I,K,L2)+HBL*U(J,IM1,K,L2))

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T3=(U(J,I,K,L2)+U(JP1,I,K,L2))
* *(HR*V(J,I,K,L2)+HBRT*V(J,IP1,K,L2))
T4=(U(J,I,K,L2)+U(JM1,I,K,L2))
* *(HBLB*V(JM1,I,K,L2)+HBRB*V(JM1,IP1,K,L2))
T5=F*HB*(V(J,I,K,L2)+V(J,IP1,K,L2)+V(JM1,I,K,L2)+V(JM1,IP1,K,L2))
T7=RDXS*(HSR*AHUV(J,IAHP,K)*(U(J,I+1,K,L3)-U(J,I,K,L3))
* -HC*AHUV(J,IAH,K)*(U(J,I,K,L3)-U(J,I-1,K,L3)))
T8=.25*RDYS*((HBTR3*AHXTB+HB3*AHXRB)*(U(J+1,I,K,L3)-U(J,I,K,L3))
* -(HB3*AHXRB+HBB3*AHXBB)*(U(J,I,K,L3)-U(J-1,I,K,L3)))
T9=0.0
IF (K.EQ.1) T9=-(TAUX(J,I,K)+TAUX(J,I,K))/RHOBX
IF (K.EQ.1) GO TO 216
UBM=U(J,I,K,L2)+U(J,I,K-1,L2)
WBXM=W(J,I,K)+W(J,IP1,K)
T9=-.25*(UBP*WBXP-UBM*WBXM)
216 CONTINUE
DHU=.125*(T5-RDX*(T1-T2)-RDY*(T3-T4))-T6+(T7+T8)-T9
GO TO 221
220 CONTINUE
T5=F*HB*(V(J,I+1,K,L2)+V(J-1,I+1,K,L2))
IF (BND1.NE.0.0) T5=F*HB*(V(J,I,K,L2)+V(J-1,I,K,L2))
DHU=-T6+T5*.25
221 CONTINUE
IF (KR.LE.1) GO TO 226
IF (K.GE.K1) GO TO 222
AU=CU
CU=2.*TAUX(J,I,KP1)/(HTD*RHOBX)
UUM=UUP
UUP=U(J,I,K,L3)-U(J,I,KQ1,L3)
BU=HBU1/DT4+AU+CU
DU=DHU+HB3*U(J,I,K,L3)/DT4-CU*UUP+AU*UUM
DEN=BU-AU*XU(K)
XU(KP1)=CU/DEN
YU(KP1)=(DU+AU*YU(K))/DEN
IF (K.LT.KR) GO TO 224
K1U=KR
BND4=0.0
IF (KR.LE.1) GO TO 226
GO TO 410
222 CONTINUE
IF (K1.LE.1) GO TO 226
CB=.5*(C(J,I)+C(J,IP1))
UB=U(J,I,K,L2)
VB=.25*(V(J,I,K,L2)+V(J,IP1,K,L2)+V(JM1,I,K,L2)+V(JM1,IP1,K,L2))
CB=CB*CB
UB=UB*UB
VB=VB*VB
RU=.5*GRAV*SQRT(UB+VB)/CB
BU=HK(J,I,K)/DT2
DU=DHU+(BU-RU)*U(J,I,K,L3)
* +CU*(U(J,I,K-1,L3)-U(J,I,K,L3))
BU=BU+RU+CU
410 CONTINUE
IF (CU.NE.0.0) GO TO 222
RHOBXZ=RHOM2+.5*(RHO(J,I,K)+RHO(J,I,KQ1)
* +RHO(J,IP1,K)+RHO(J,IP1,KQ1))
TAUZ=TAUX(J,I,KP1)*4.0*(U(J,I,K,L3)-U(J,I,KQ1,L3))/HTX
DHU=DHU-(TAUZ+TAUZ-.25*UBP*WBXP*RHOBXZ)/RHOBX
U(J,I,K,L1)=(HB3*U(J,I,K,L3)+DT4*DHU)/HBU1
IF (ABS(U(J,I,K,L1)).LT.1.0E-20) U(J,I,K,L1)=0.0
GO TO 217

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2222 CONTINUE
      U(J,I,K,L1)=(DU+CU*YU(K))/(BU-CU*XU(K))
      IF (ABS(U(J,I,K,L1)).LT.1.0E-20) U(J,I,K,L1)=0.0
217 CONTINUE
      DO 223 KKK=2,K1U
      KK=K1U-KKK+2
      U(J,I,KK-1,L1)=XU(KK)*U(J,I,KK,L1)+YU(KK)
      IF (ABS(U(J,I,KK-1,L1)).LT.1.0E-20) U(J,I,KK-1,L1)=0.0
223 CONTINUE
      GO TO 224
226 CONTINUE
      RHOBXZ=RHOM2+RHO(J,I,K)+RHO(J,I+1,K)
      TAUZ=TAUX(J,I,KP1)
      DHU=DHU-(TAUZ+TAUZ-.25*UBP*WBXP*RHOBXZ)/RHOBX
      U(J,I,K,L1)=(HB3*U(J,I,K,L3)+DT4*DHU)/HBU1
      IF (ABS(U(J,I,K,L1)).LT.1.0E-20) U(J,I,K,L1)=0.0
224 CONTINUE
      IF (L.LE.LASTCK) WRITE (6,14)
225 CONTINUE
      IF (KT.EQ.0) GO TO 250
      IF (BND5.EQ.0.0) GO TO 250
      GO TO 227
2353 CONTINUE
      IF (BND3.EQ.0.0) GO TO 227
      IF (K.LE.KT) GO TO 227
      IF (KT.LE.1) GO TO 227
      DO 234 KK=K,K1
      XV(KK)=0.0
      YV(KK)=0.0
234 CONTINUE
      GO TO 235
227 CONTINUE
      RHOBY=(RHO(J,I,K)+RHO(JP1,I,K))+RHOM2
      VBP=V(J,I,K,L2)
      IF (K.LT.K1) VBP=VBP+V(J,I,KQ1,L2)
      WBYP=W(J,I,KP1)+W(JP1,I,KP1)
      T6=HR*PGRAY(J,I,K)/RHOBY
      IF (BND3.EQ.0.0) GO TO 230
      T1=(V(J,I,K,L2)+V(JP1,I,K,L2))
      * *(HBT*V(JP1,I,K,L2)+HR*V(J,I,K,L2))
      T2=(V(JM1,I,K,L2)+V(J,I,K,L2))
      * *(HR*V(J,I,K,L2)+HBLB*V(JM1,I,K,L2))
      T3=(V(J,I,K,L2)+V(J,IP1,K,L2))
      * *(HBTR*U(JP1,I,K,L2)+HB*U(J,I,K,L2))
      T4=(V(J,IM1,K,L2)+V(J,I,K,L2))
      * *(HBT*U(JP1,IM1,K,L2)+HBL*U(J,IM1,K,L2))
      T5=F*HR*(U(J,I,K,L2)+U(J,IM1,K,L2)+U(JP1,I,K,L2)+U(JP1,IM1,K,L2))
      T7=RDYS*(HST*AHUV(J+1,IAH,K)*(V(J+1,I,K,L3)-V(J,I,K,L3))
      * -HC*AHUV(J,IAH,K)*(V(J,I,K,L3)-V(J-1,I,K,L3)))
      T8=.25*RDXS*((HBRT3*AHYRB+HBLT3*AHYTB)*(V(J,I+1,K,L3)-V(J,I,K,L3))
      * -(HBLT3*AHYTB+HBLT3*AHYLB)*(V(J,I,K,L3)-V(J,I-1,K,L3)))
      T9=0.0
      IF (K.EQ.1) T9=-(TAUY(J,I,K)+TAUY(J,I,K))/RHOBY
      IF (K.EQ.1) GO TO 228
      VBM=V(J,I,K,L2)+V(J,I,K-1,L2)
      WBYM=W(J,I,K)+W(JP1,I,K)
      T9=-.25*(VBP*WBYP-VBM*WBYM)
228 CONTINUE
      DHV=.125*(-T5-RDY*(T1-T2)-RDX*(T3-T4))-T6+(T7+T8)-T9
      GO TO 231
230 CONTINUE

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T5=F*HR*(U(J+1,I,K,L2)+U(J+1,I-1,K,L2))
IF (BND1.NE.0.0) T5=F*HR*(U(J,I,K,L2)+U(J,I-1,K,L2))
DHV=-T6-T5*.25
231 CONTINUE
IF (KT.LE.1) GO TO 246
IF (K.GE.K1) GO TO 242
AV=CV
CV=2.*TAUY(J,I,KP1)/(HTD*RHOBY)
VVM=VVP
VVP=V(J,I,K,L3)-V(J,I,KQ1,L3)
BV=HBV1/DT4+AV+CV
DV=DHV+HBLT3*V(J,I,K,L3)/DT4-CV*VVP+AV*VVM
DEN=BV-AV*XV(K)
XV(KP1)=CV/DEN
YV(KP1)=(DV+AV*YV(K))/DEN
IF (K.LT.KT) GO TO 244
K1V=KT
BND5=0.0
IF (KT.LE.1) GO TO 246
GO TO 420
242 CONTINUE
IF (K1.LE.1) GO TO 246
CB=.5*(C(J,I)+C(JP1,I))
VB=V(J,I,K,L2)
UB=.25*(U(J,I,K,L2)+U(J,IM1,K,L2)+U(JP1,I,K,L2)+U(JP1,IM1,K,L2))
CB=CB*CB
VB=VB*VB
UB=UB*UB
RV=.5*GRAV*SQRT(UB+VB)/CB
BV=HK(J,I,K)/DT2
DV=DHV+(BV-RV)*V(J,I,K,L3)
*      +CV*(V(J,I,K-1,L3)-V(J,I,K,L3))
BV=BV+RV+CV
420 CONTINUE
IF (CV.NE.0.0) GO TO 2240
RHOBYZ=RHO2+.5*(RHO(J,I,K)+RHO(J,I,KQ1)
*      +RHO(JP1,I,K)+RHO(JP1,I,KQ1))
TAUZ=TAUY(J,I,KP1)*4.0*(V(J,I,K,L3)-V(J,I,KQ1,L3))/HTY
DHV=DHV-(TAUZ+TAUZ-.25*VBP*WBYP*RHOBYZ)/RHOBY
V(J,I,K,L1)=(HBLT3*V(J,I,K,L3)+DT4*DHV)/HBV1
IF (ABS(V(J,I,K,L1)).LT.1.0E-20) V(J,I,K,L1)=0.0
GO TO 235
2240 CONTINUE
V(J,I,K,L1)=(DV+CV*YV(K))/(BV-CV*XV(K))
IF (ABS(V(J,I,K,L1)).LT.1.0E-20) V(J,I,K,L1)=0.0
235 CONTINUE
DO 243 KKK=2,K1V
KK=K1V-KKK+2
V(J,I,KK-1,L1)=XV(KK)*V(J,I,KK,L1)+YV(KK)
IF (ABS(V(J,I,KK-1,L1)).LT.1.0E-20) V(J,I,KK-1,L1)=0.0
243 CONTINUE
GO TO 244
246 CONTINUE
RHOBYZ=RHO2+RHO(J,I,K)+RHO(J+1,I,K)
TAUZ=TAUY(J,I,KP1)
DHV=DHV-(TAUZ+TAUZ-.25*VBP*WBYP*RHOBYZ)/RHOBY
V(J,I,K,L1)=(HBLT3*V(J,I,K,L3)+DT4*DHV)/HBV1
IF (ABS(V(J,I,K,L1)).LT.1.0E-20) V(J,I,K,L1)=0.0
244 CONTINUE
IF (L.LE.LASTCK) WRITE (6,15)
250 CONTINUE

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      K=K+1
      IF (K.GT.K1) GO TO 280
      KP1=K+1
      KQ1=KP1
      IF (K1.EQ.K) KQ1=K
      H=HK(J,I,K)
      H2=H+H
      HS1=H
      HBU1=H2
      HBV1=H2
      HSR=H
      HC=H
      HSL=H
      HST=H
      HSB=H
      HBR=H2
      HB=H2
      HBL=H2
      HC2=H
      HR=H2
      HBRT=H2
      HBLB=H2
      HBRB=H2
      HBTR=H2
      HBTL=H2
      HBT=H2
      HBLLT=H2
      HBR3=H2
      HB3=H2
      HBL3=H2
      HBTR3=H2
      HBB3=H2
      HBRT3=H2
      HBLT3=H2
      HBLLT3=H2
      HBT3=H2
      HBLB3=H2
      Q1=Q2
      Q3=Q4
      Q5=Q6
      Q7=Q8
      HQ=.5*(H+HK(J,I,KP1))
      IF (L.LE.LASTCK) WRITE (6,16)
      GO TO 200
280  CONTINUE
      IF (K1.LE.KR) GO TO 282
      KK=KR+1
      DO 281 K=KK,K1
      U(J,I,K,L1)=0.0
281  CONTINUE
282  CONTINUE
      IF (K1.LE.KT) GO TO 300
      KK=KT+1
      DO 283 K=KK,K1
      V(J,I,K,L1)=0.0
283  CONTINUE
300  CONTINUE
      CALL SGSLOS
      RETURN
      END
      SUBROUTINE SGSLOS

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COMMON CPO,CLAM,CLAMO,CP1,CLAM1,RHOT
COMMON M,MM1,MM2,N,NM1,NM2,DX,DY,RDX,RDY,RDXS,RDYS
COMMON DT,DT2,DT4,CNU,L,L1,L2,L3,GRAV,KMAX,CKA,AHX,AHY,AMX,AMY,F
COMMON RHOM,RHOM2,ALPHO,TEMP,THETA,RHOA,WSPD,PHI,WINCH,DEPMIN
COMMON PI,KMAP,TD,TD2,TD4,KHT,CT,CKH,AHXT,AHYT
COMMON RMM1,RMM2,CKP,AHXP,AHYP,PRETIM,PINV,DECAY,LMAX,CONVER
COMMON U(10,26,6,3),V(10,26,6,3),S(10,26,6,3),T(10,26,6,3)
COMMON W(10,26,7),TAUX(10,26,7),TAUY(10,26,7),EPSLON(10,26,7)
COMMON PGRAX(10,26,6),PGRAY(10,26,6),RHO(10,26,6),HK(10,26,6)
COMMON SE(10,26,3),KH(10,26),C(10,26),HS(7),YY(6),DEPTH(10,26)
COMMON RHOZ(10,26),HZ(10,26),NSE(10,26),ZL(6)
COMMON IB(100),JB(100),KB(100),UVB(100),AHUV(10,3,6)
COMMON UFLOS(100),UFLOT(100),UFLOP(100),PTURN(10,100)
COMMON STMP,LTURN(10,100),STURN(10,100),SET,TTURN(10,100)
COMMON SINI(10,100),TINI(10,100),SRETIM,TRETIM
COMMON ETURN(10,100),EINI(10,100),UFLOE(100),E(10,26,6,3)
COMMON ERETIM,EINV,TINV,SALINV,P(10,26,6,3),PINI(10,100)
DOUBLE PRECISION CPO,CLAM,CLAMO,CP1,CLAM1,RHOT
WSPS=WSPD*51.447192
WSALE=1.0
XL=L
IF(L.LT.120) WSALE=XL/120.
DO 3100 I=2,MM2
DO 3100 J=2,NM2
IF (HK(J,I,1).LE.0.0) GO TO 3100
A=21.8*SQRT(HK(J,I,1))
IF (A.GT.WSPS) A=WSPS
B=WSALE*(.0000000028*A*A*A/A/HK(J,I,1))
IF(B.EQ.0.0) GO TO 3100
IF (B.GT.E(J,I,1,L1)) E(J,I,1,L1)=B
3100 CONTINUE
PIH=.5*PI
DO 310 IJ=1,100
IF (IB(IJ).EQ.0) GO TO 320
IBT=IB(IJ)
JBT=JB(IJ)
KI1=0
KI2=0
KI3=0
KI4=0
KI5=0
KI6=0
KBT=IABS(KB(IJ))
IF (MOD(KBT,2).EQ.1) KI1=1
IF (MOD(KBT,2).EQ.0) KI2=1
IF (KBT.LT.3) GO TO 100
KI1=-KI1
KI2=-KI2
KI3=KI1
KI4=KI2
KI5=KI1
KI6=KI2
100 CONTINUE
KI5=KI5+KI1
KI6=KI6+KI2
K1=KH(JBT,IBT)
DO 210 K=1,K1
IF (KI1.EQ.0) GO TO 200
IF (KH(JBT+1,IBT).LT.K) GO TO 210
V(JBT,IBT,K,L1)=V(JBT,IBT+KI1,K,L1)
GO TO 210

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200 CONTINUE
  IF (KH(JBT,IBT+1).LT.K) GO TO 210
  U(JBT,IBT,K,L1)=U(JBT+KI2,IBT,K,L1)
210 CONTINUE
  IF (KB(IJ).GT.0) GO TO 305
  S(JBT,IBT,1,L1)=-S(JBT+KI2,IBT+KI1,1,L1)+UFLOS(IJ)+UFLOS(IJ)
  T(JBT,IBT,1,L1)=-T(JBT+KI2,IBT+KI1,1,L1)+UFLOT(IJ)+UFLOT(IJ)
  E(JBT,IBT,1,L1)=-E(JBT+KI2,IBT+KI1,1,L1)+UFLOE(IJ)+UFLOE(IJ)
  P(JBT,IBT,1,L1)=-P(JBT+KI2,IBT+KI1,1,L1)+UFLOP(IJ)+UFLOP(IJ)
  SE(JBT,IBT,L1)=-SE(JBT+KI2,IBT+KI1,L1)
  IF (KI2.EQ.0) U(JBT+KI4,IBT+KI3,1,L1)=UVB(IJ)
  IF (KI1.EQ.0) V(JBT+KI4,IBT+KI3,1,L1)=UVB(IJ)
  GO TO 310
305 CONTINUE
  K1=KH(JBT,IBT)
  K2=KH(JBT+KI2,IBT+KI1)
  IF (K1.GT.K2) K1=K2
  IF (K1.EQ.0) GO TO 310
  DO 345 K=1,K1
  IF (KI2.EQ.0) GO TO 307
  IF (KI2.GT.0) GO TO 306
  IF (V(JBT+KI4,IBT+KI3,K,L1).GT.0.0) GO TO 335
  IF (V(JBT+KI6,IBT+KI5,K,L1).LT.0.0) GO TO 330
  GO TO 325
306 CONTINUE
  IF (V(JBT+KI4,IBT+KI3,K,L1).LT.0.0) GO TO 335
  IF (V(JBT+KI6,IBT+KI5,K,L1).GT.0.0) GO TO 330
  GO TO 325
307 CONTINUE
  IF (KI1.LT.0) GO TO 308
  IF (U(JBT+KI4,IBT+KI3,K,L1).LT.0.0) GO TO 335
  IF (U(JBT+KI6,IBT+KI5,K,L1).GT.0.0) GO TO 330
  GO TO 325
308 CONTINUE
  IF (U(JBT+KI4,IBT+KI3,K,L1).GT.0.0) GO TO 335
  IF (U(JBT+KI6,IBT+KI5,K,L1).LT.0.0) GO TO 330
325 CONTINUE
  LTURN(K,IJ)=L-1
  STURN(K,IJ)=S(JBT+KI2,IBT+KI1,K,L2)
  TTURN(K,IJ)=T(JBT+KI2,IBT+KI1,K,L2)
  ETURN(K,IJ)=E(JBT+KI2,IBT+KI1,K,L2)
  PTURN(K,IJ)=P(JBT+KI2,IBT+KI1,K,L2)
330 CONTINUE
  IF (LTURN(K,IJ).LT.0) GO TO 325
  SFRAC = (DT*(L-LTURN(K,IJ)))/(SRETIM*3600.)
  TFRAC = (DT*(L-LTURN(K,IJ)))/(TRETIM*3600.)
  EFRAC = (DT*(L-LTURN(K,IJ)))/(ERETIM*3600.)
  PFRAC = (DT*(L-LTURN(K,IJ)))/(PRETIM*3600.)
  IF (SFRAC .GT. 1.0) SFRAC = 1.0
  IF (TFRAC .GT. 1.0) TFRAC = 1.0
  IF (EFRAC .GT. 1.0) EFRAC = 1.0
  IF (PFRAC .GT. 1.0) PFRAC = 1.0
  S(JBT,IBT,K,L1)=STURN(K,IJ)+(SINI(K,IJ)-STURN(K,IJ))
  * *SIN (PIH*SFRAC)
  T(JBT,IBT,K,L1)=TTURN(K,IJ)+(TINI(K,IJ)-TTURN(K,IJ))
  * *SIN (PIH*TFRAC)
  E(JBT,IBT,K,L1)=ETURN(K,IJ)+(EINI(K,IJ)-ETURN(K,IJ))
  * *SIN (PIH*EFRAC)
  P(JBT,IBT,K,L1)=PTURN(K,IJ)+(PINI(K,IJ)-PTURN(K,IJ))
  * *SIN (PIH*PFRAC)
  GO TO 345

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```

335 CONTINUE
    LTURN(K,IJ)=-1
    IF (KI2.EQ.0) GO TO 340
    RDRD=RDY
    UVXY=DT*V(JBT+KI4,IBT+KI3,K,L2)
    GO TO 342
340 CONTINUE
    RDRD=RDX
    UVXY=DT*U(JBT+KI4,IBT+KI3,K,L2)
342 CONTINUE
    SGRAD=(S(JBT+KI2,IBT+KI1,K,L2)-S(JBT,IBT,K,L2))*RDRD
    TGRAD=(T(JBT+KI2,IBT+KI1,K,L2)-T(JBT,IBT,K,L2))*RDRD
    EGRAD=(E(JBT+KI2,IBT+KI1,K,L2)-E(JBT,IBT,K,L2))*RDRD
    PGRAD=(P(JBT+KI2,IBT+KI1,K,L2)-P(JBT,IBT,K,L2))*RDRD
    S(JBT,IBT,K,L1)=S(JBT,IBT,K,L2)+SGRAD*UVXY
    T(JBT,IBT,K,L1)=T(JBT,IBT,K,L2)+TGRAD*UVXY
    E(JBT,IBT,K,L1)=E(JBT,IBT,K,L2)+EGRAD*UVXY
    P(JBT,IBT,K,L1)=P(JBT,IBT,K,L2)+PGRAD*UVXY
345 CONTINUE
310 CONTINUE
320 CONTINUE
    RETURN
    END
    SUBROUTINE PRNT1 (X,BX,M,N,L1,L)
    DIMENSION X(N,M,3)
    1 FORMAT (1H1,10X,A4,5X,'TIME=',I6)
    2 FORMAT (I4,1X,21F6.1)
    3 FORMAT (3X,21I6)
    I2=0
100 CONTINUE
    I1=I2+1
    I2=I2+21
    IF (I1.GT.M) RETURN
    IF (I2.GT.M) I2=M
    WRITE (6,1) BX,L
    WRITE (6,3) (I,I=I1,I2)
    DO 200 J1=1,N
    J=N+1-J1
200 WRITE (6,2) J,(X(J,I,L1),I=I1,I2)
    GO TO 100
    END
    SUBROUTINE PRNT2 (X,BX,M,N,K,L1,K1,L)
    DIMENSION X(N,M,K,3)
    1 FORMAT (1H1,10X,A4,5X,'TIME=',I6,5X,'LEVEL',I3)
    2 FORMAT (I4,1X,21F6.1)
    3 FORMAT (3X,21I6)
    I2=0
100 CONTINUE
    I1=I2+1
    I2=I2+21
    IF (I1.GT.M) RETURN
    IF (I2.GT.M) I2=M
    WRITE (6,1) BX,L,K1
    WRITE (6,3) (I,I=I1,I2)
    DO 200 J1=1,N
    J=N+1-J1
200 WRITE (6,2) J,(X(J,I,K1,L1),I=I1,I2)
    GO TO 100
    END
    SUBROUTINE PRNT3 (X,BX,M,N,K,K1,L)
    DIMENSION X(N,M,K)

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1 FORMAT (1H1,10X,A4,5X,'TIME=',I6,5X,'LEVEL',I3)
2 FORMAT (I4,1X,21F6.0)
3 FORMAT (3X,21I6)
  I2=0
100 CONTINUE
  I1=I2+1
  I2=I2+21
  IF (I1.GT.M) RETURN
  IF (I2.GT.M) I2=M
  WRITE (6,1) BX,L,K1
  WRITE (6,3) (I,I=I1,I2)
  DO 200 J1=1,N
    J=N+1-J1
200 WRITE (6,2) J,(X(J,I,K1),I=I1,I2)
  GO TO 100
  END
  SUBROUTINE PRNT4 (X,BX,M,N,K1,L)
  DIMENSION X(N,M)
1 FORMAT (1H0,10X,A4,5X,'TIME=',I6,5X,'LEVEL',I3)
2 FORMAT (I4,1X,9(1X,1PE12.5))
3 FORMAT (3X,9I13)
4 FORMAT (1H1)
  WRITE (6,4)
  I2=0
100 CONTINUE
  I1=I2+1
  I2=I2+9
  IF (I1.GT.M) RETURN
  IF (I2.GT.M) I2=M
  WRITE (6,1) BX,L,K1
  WRITE (6,3) (I,I=I1,I2)
  DO 200 J1=1,N
    J=N+1-J1
200 WRITE (6,2) J,(X(J,I),I=I1,I2)
  GO TO 100
  END
  SUBROUTINE PRNT5 (X,BX,M,N,L1,L)
  DIMENSION X(N,M,3)
1 FORMAT (1H0,10X,A4,5X,'TIME=',I6)
2 FORMAT (I4,10(1X,1PE11.4))
3 FORMAT (6X,10(I3,9X))
4 FORMAT (1H1)
  WRITE (6,4)
  I2=0
100 CONTINUE
  I1=I2+1
  I2=I2+10
  IF (I1.GT.M) RETURN
  IF (I2.GT.M) I2=M
  WRITE (6,1) BX,L
  WRITE (6,3) (I,I=I1,I2)
  DO 200 J1=1,N
    J=N+1-J1
200 WRITE (6,2) J,(X(J,I,L1),I=I1,I2)
  GO TO 100
  END
  SUBROUTINE PRNT6 (X,BX,M,N,K,L1,K1,L)
  DIMENSION X(N,M,K,3)
1 FORMAT (1H0,10X,A4,5X,'TIME=',I6,5X,'LEVEL',I3)
2 FORMAT (I4,10(1X,1PE11.4))
3 FORMAT (6X,10(I3,9X))

```

```

4 FORMAT (1H1)
WRITE (6,4)
I2=0
100 CONTINUE
I1=I2+1
I2=I2+10
IF (I1.GT.M) RETURN
IF (I2.GT.M) I2=M
WRITE (6,1) BX,L,K1
WRITE (6,3) (I,I=I1,I2)
DO 200 J1=1,N
J=N+1-J1
200 WRITE (6,2) J,(X(J,I,K1,L1),I=I1,I2)
GO TO 100
END
SUBROUTINE RESAL
COMMON CPO,CLAM,CLAMO,CP1,CLAM1,RHOT
COMMON M,MM1,MM2,N,NM1,NM2,DX,DY,RDX,RDY,RDXS,RDYS
COMMON DT,DT2,DT4,CNU,L,L1,L2,L3,GRAV,KMAX,CKA,AHX,AHY,AMX,AMY,F
COMMON RHOM,RHOM2,ALPHO,TEMP,THETA,RHOA,WSPD,PHI,WINCH,DEPMIN
COMMON PI,KMAP,TD,TD2,TD4,KHT,CT,CKH,AHXT,AHYT
COMMON RMM1,RMM2,CKP,AHXP,AHYP,PRETIM,PINV,DECAY,LMAX,CONVER
COMMON U(10,26,6,3),V(10,26,6,3),S(10,26,6,3),T(10,26,6,3)
COMMON W(10,26,7),TAUX(10,26,7),TAUY(10,26,7),EPSLON(10,26,7)
COMMON PGRAX(10,26,6),PGRAY(10,26,6),RHO(10,26,6),HK(10,26,6)
COMMON SE(10,26,3),KH(10,26),C(10,26),HS(7),YY(6),DEPTH(10,26)
COMMON RHOZ(10,26),HZ(10,26),NSE(10,26),ZL(6)
COMMON IB(100),JB(100),KB(100),UVB(100),AHUV(10,3,6)
COMMON UFLOS(100),UFLOT(100),UFLOP(100),PTURN(10,100)
COMMON STMP,LTURN(10,100),STURN(10,100),SET,TTURN(10,100)
COMMON SINI(10,100),TINI(10,100),SRETIM,TRETIM
COMMON ETURN(10,100),EINI(10,100),UFLOE(100),E(10,26,6,3)
COMMON ERETIM,EINV,TINV,SALINV,P(10,26,6,3),PINI(10,100)
DOUBLE PRECISION CPO,CLAM,CLAMO,CP1,CLAM1,RHOT
1 FORMAT (9F8.2)
2 FORMAT (12F6.0)
9 FORMAT (20F4.0/20F4.0)
31 FORMAT (1H0,'SALINITY AT LEVEL',I3,' IS BEING READ')
RETURN
50 CONTINUE
J1=12
MM2=M-2
NM2=N-2
READ (5,2) (S(J1,I,1,1),I=2,MM2)
DO 152 K=1,KMAX
KM1=K-1
ZKM1=KM1
DO 151 J=2,NM2
DJ=IABS(J-J1)
DO 150 I=2,MM2
IF (K.GT.KH(J,I)) GO TO 150
IF (K.EQ.1) GO TO 120
IF (S(J,I,KM1,1).EQ.0.0) GO TO 150
120 CONTINUE
S(J,I,K,1)=S(J1,I,1,1)+.076*ZKM1-.05*DJ
IF (S(J,I,K,1).LT.0.0) S(J,I,K,1)=0.0
S(J,I,K,2)=S(J,I,K,1)
S(J,I,K,3)=S(J,I,K,1)
150 CONTINUE
151 CONTINUE
152 CONTINUE

```

```

RETURN
END
SUBROUTINE TIDAL
COMMON CPO,CLAM,CLAMO,CP1,CLAM1,RHOT
COMMON M,MM1,MM2,N,NM1,NM2,DX,DY,RDX,RDY,RDXS,RDYS
COMMON DT,DT2,DT4,CNU,L,L1,L2,L3,GRAV,KMAX,CKA,AHX,AHY,AMX,AMY,F
COMMON RHOM,RHOM2,ALPHO,TEMP,THETA,RHOA,WSPD,PHI,WINCH,DEPMIN
COMMON PI,KMAP,TD,TD2,TD4,KHT,CT,CKH,AHXT,AHYT
COMMON RMM1,RMM2,CKP,AHXP,AHYP,PRETIM,PINV,DECAY,LMAX,CONVER
COMMON U(10,26,6,3),V(10,26,6,3),S(10,26,6,3),T(10,26,6,3)
COMMON W(10,26,7),TAUX(10,26,7),TAUY(10,26,7),EPSLON(10,26,7)
COMMON PGRAX(10,26,6),PGRAY(10,26,6),RHO(10,26,6),HK(10,26,6)
COMMON SE(10,26,3),KH(10,26),C(10,26),HS(7),YY(6),DEPTH(10,26)
COMMON RHOZ(10,26),HZ(10,26),NSE(10,26),ZL(6)
COMMON IB(100),JB(100),KB(100),UVB(100),AHUV(10,3,6)
COMMON UFLOS(100),UFLOT(100),UFLOP(100),PTURN(10,100)
COMMON STMP,LTURN(10,100),STURN(10,100),SET,TTURN(10,100)
COMMON SINI(10,100),TINI(10,100),SRETIM,TRETIM
COMMON ETURN(10,100),EINI(10,100),UFLOE(100),E(10,26,6,3)
COMMON ERETIM,EINV,TINV,SALINV,P(10,26,6,3),PINI(10,100)
DOUBLE PRECISION CPO,CLAM,CLAMO,CP1,CLAM1,RHOT
DATA IFRST/0/
100 CONTINUE
SCALE=1.0
XL=L
TX=0.0017453*XL*DT
IF (IFRST.NE.0) GO TO 150
IFRST=1
FM2=0.080515
FS2=0.0833333
FN2=0.0789889
FK2=0.0835421
FK1=0.0417885
FO1=0.0387295
FP1=0.0415454
A1=-3.485
B1=10.0
A2=2.26
B2=5.6
A3=0.0
B3=0.0
A4=0.0
B4=0.0
A5=-2.44
B5=-3.48
A6=-1.174
B6=-1.954
A7=0.0
B7=0.0
150 CONTINUE
SETIDE =A1*COS(FM2*TX)+B1*SIN(FM2*TX)
* +A2*COS(FS2*TX)+B2*SIN(FS2*TX)
* +A3*COS(FN2*TX)+B3*SIN(FN2*TX)
* +A4*COS(FK2*TX)+B4*SIN(FK2*TX)
* +A5*COS(FK1*TX)+B5*SIN(FK1*TX)
* +A6*COS(FO1*TX)+B6*SIN(FO1*TX)
* +A7*COS(FP1*TX)+B7*SIN(FP1*TX)
IF (I.EQ.I) GO TO 400
200 CONTINUE
IF (IFRST.NE.0) GO TO 300
IFRST=1

```

```

RADEG=PI/180.0
FM2=.0805114006
FS2=.0833333333
FK1=.0417807462
FO1=.0387306544
FN2=.0789992487
FP1=.0415525871
FK2=.0835614924
FQ1=.08372185025
FMF=.003050091944
A0=.0
A1=5.5
A2=2.
A3=2.1
A4=2.
A5=.8
A6=.9
A7=.4
A8=.8
A9=2.3
P1=RADEG*308.1
P2=RADEG*342.5
P3=RADEG*160.
P4=RADEG*203.
P5=RADEG*287.9
P6=RADEG*144.
P7=RADEG*345.5
P8=RADEG*181.1
P9=RADEG*231.2
300 CONTINUE
SETIDE=.5*A0
*      +A1*COS(FM2*TX-P1)
*      +A2*COS(FS2*TX-P2)
*      +A3*COS(FK1*TX-P3)
*      +A4*COS(FO1*TX-P4)
*      +A5*COS(FN2*TX-P5)
*      +A6*COS(FP1*TX-P6)
*      +A7*COS(FK2*TX-P7)
*      +A8*COS(FQ1*TX-P8)
*      +A9*COS(FMF*TX-P9)
400 CONTINUE
DO 110 IJ=1,100
IF (IB(IJ).EQ.0) GO TO 120
IF (KB(IJ).LT.0) GO TO 110
IBT=IB(IJ)
JBT=JB(IJ)
SE(JBT,IBT,L1)=SETIDE
110 CONTINUE
120 CONTINUE
RETURN
END

```

8.9 輸入資料範例

TEST CASE

400	26	10	6	45.0	200000.0	200000.0
0	9999990	9999990				
00.0	0.0	0.0	0.0	0.0	15.0	15.0
15.0	0.0	0.0	5.0	1.0	1.0	1.0

1.0	.001226	15.0	.0026	105.3	0.0	
100	100	100	10000			
0	1000000	9999990	6			
650.0	30.0	-4.6	.4	.8	.4	
15.0	1000000.0	1000000.0	1.0	0.0	.000005	100.00
200.0	200.0	200.0	200.0	200.0	200.0	

```

00121212121212121212121212121212121212121212121212121212 0 0 0
0121212121212121212121212121212121212121212121212121212 0 0 0
0121212121212121212121212121212121212121212121212121212 0 0 0
0121212121212121212121212121212121212121212121212121212 2 0 0
0121212121212121212121212121212121212121212121212121212 0 0 0
0121212121212121212121212121212121212121212121212121212 0 0 0
0121212121212121212121212121212121212121212121212121212 0 0 0
0121212121212121212121212121212121212121212121212121212 0 0 0

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2	5	BASIN OPENING
12	5	MIDDLE OF BASIN
23	5	END OF BASIN

2	5	6	BASIN OPENING
12	5	6	MIDDLE OF BASIN
23	5	6	END OF BASIN

2	2	1					
2	3	1					
2	4	1					
2	5	1					
2	6	1					
2	7	1					
2	8	1					
24	5	-3	-5.0	00.0	00.0	1.0	10.0
0.0	0.0		105.3				
.4	.8						