旋性流與非旋性流對波浪影響之研究

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波斜行在穩定不均勻流上阻塞及反射現象

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摘要

當波浪行進方向與流向相反，且其羣速度在某一點與流速平衡時，波浪在此點將被流阻擋及反射爲不同的波長，此種現象已由 Shyu & Phillips (1990) 導出一具有明白形式的解，目前我們將 Shyu & Phillips 的方法進一步延伸，求當波向與流向具有一個角度時，波被流阻塞及反射現象的解，在此我們仍假設流爲二度空間及穩定，但目前的解配合以往的 ray solution 實際可決定一彎曲 caustic 上及其附近之水面位移變化.

誌謝

本篇報告所討論的內容最早由美國北卡州立大學的董啓超教授和本人在執行國科會的“海波及海流交互作用”計劃(本計劃另一共同主持人爲國立成功大學的歐善惠教授)時開始這項研究,在董教授訪問港研所的三個月期間,除完成“在逆流作用下的波譜”這篇論文外,且對波斜行在一穩定二度空間流上發生阻塞現象的問題導出一些關鍵性的結果,對於國科會的支持以及歐善惠教授的協助,我們特此誌謝.本研究的後續工作是在港研所八十一年度研(五)計劃的支持下完成,故對台灣省政府及省交通處我們也特表謝意.

目錄

一. 前言

當波浪行進在一不均勻流上,其振幅與波長將發生變化,此種變化最早由 Longuet-Higgins & Stewart (1960) 導出理論解, Bretherton & Garrett (1968) 且利用Whitham (1965)所提議的averaged Lagrangian method導出一般性的理論,但此兩種理論皆屬ray theory,故它們在某些點上無法適用,其中一種狀況是,當波浪逆行在流上,當其羣速度和流速相平衡時,以往的ray solution在此點將具有一個奇異點(singularity),表在此點上能量將變爲無窮大,但實際情況並非如此,爲解決此問題, Smith (1975)藉由Ludwig's (1966) 'ansatz'的應用導出能均勻適用於每一點上的解,此解顯示當波被流阻擋以後,實際將被反射爲不同波長的波. Smith的解除僅流限於非旋性流以外,可應用在非常廣泛的條件下,但此解所包含的一些參數不具有一明白的表式,其變化僅以滿足一些transport equations的形式出現,故它們在實際應用時將有所限制.

另一方面, Shyu & Phillips (1990)在研究表面張力阻塞現象時,亦導出另一種形式的解,此解具有一明白的形式,且它在忽略表面張力後,亦可應用在重力阻塞現象,利用此種形式的解, Tung & Shyu (1992)導出隨機波由靜水區行進到一逆流上,其波譜在各點(包括阻塞點)變化的表式,此表式具有一極簡單的形式,故極有利於實際的應用,但Shyu & Phillips (1990)的解仍僅限於波行進方向與流向平行時的狀況,故以下我們將採用類似的方法導出重力波斜行在一穩定二度空間流上發生阻塞現象的解,此解將可應用在較廣泛的情況下.

二. 基本方程式

在本節中我們將由原邊界值問題,考慮波浪緩慢變化性質,導出一描述振幅及波長皆作緩慢變化波列之水面起伏的方程式,此一方程式雖爲四階,但它的任意兩個independent solutions無法同時代表入射波和反射波,僅未來在第四節我們將它進一步分解並重新組合成一個二階常微分方程後,才可充份描述波浪被流反射現象.

若不考慮連結入射波和反射波的問題,且不考慮它們在阻塞點附近的變化,我們實際可由原邊界值問題建立任何階數的常微方,但目前已知,當流爲穩定時,離散關係式爲

$$n_0 = \left[g(k_x^2 + k_y^2)^{\frac{1}{2}}\right]^{\frac{1}{2}} + Uk_x = \text{constant}, \tag{2.1}$$

其中 n_0 表觀測頻率, g 表重力加速度, U 表當地的流速, k_x 和 k_y 表波數(wave-number)$\mathbf{k}$ 在 x 和 y 方向的分量,我們在此且定二度空間流的方向爲 x 軸方向. 由(2.1)可得

$$U^4k_x^4 - 4n_0U^3k_x^3 + (6n_0^2U^2 - g^2)k_x^2 - 4n_0^3Uk_x + (n_0^4 - g^2k_y^2) = 0. \tag{2.2}$$

由於(2.2)的左邊爲一個四次多項式,由Shyu & Phillips (1990)所述的方法可知,我們需由一個四階微方開始分解,才能得到一係數具有簡單形式且在阻塞點爲regular的二階微方.

爲導出控制水面位移 $\eta(x,y,t)$ 的四階微方,我們需利用以往ray solution的一些結果(不包括action conservation principle),後者雖無法應用到阻塞點,但經由這些中間步驟,包括未來分解及組合的過程,我們可建立一個二階微方,此微方在阻塞點上的奇異性質將互相抵消,因此可適用在阻塞點附近(包括阻塞點在內).

首先由以往的ray theory

$$\nabla \times \mathbf{k} = 0,$$

亦即

$$\frac{\partial k_y}{\partial x} - \frac{\partial k_x}{\partial y} = 0.$$

由於目前我們假設流在 y 方向無變化，故 $\partial/\partial y = 0$，因此可得在任何位置，任何時間，

$$k_y = \text{constant}.$$

由它及(2.1)以及以往的ray solution，我們可假設流速勢(velocity potential)

$$\phi = A(x)\exp\left[i\int k_x(x)\,dx + \int l(x,z)\,dz\right]\exp i(k_y y - n_0 t) \tag{2.3}$$

及

$$\eta = a(x)\exp\left[i\int k_x(x)\,dx\right]\exp i(k_y y - n_0 t), \tag{2.4}$$

其中 $A(x)$, $a(x)$ 和 $k_x(x)$ 在 x 方向作緩慢變化，$l(x,z)$ 在 x 和 z 方向皆作緩慢的變化．流速勢 ϕ 須滿足Laplace's equation，故我們可因此建立 k_x 和 l 的關係式如下．

將(2.3)代入Laplace's equation後可得

$$\left[\frac{\partial^2}{\partial x^2} - k_y^2 + \frac{\partial^2}{\partial z^2}\right]\phi = 0. \tag{2.5}$$

而由(2.3)，當式中對 z 積分下限訂在平均水面時，則在此位置

$$\frac{\partial^2\phi}{\partial x^2} = (-k_x^2 + i\frac{dk_x}{dx} + 2ik_x\frac{1}{A}\frac{dA}{dx})\phi, \tag{2.6}$$

$$\frac{\partial^2\phi}{\partial z^2} = (l^2 + \frac{\partial l}{\partial z})\phi. \tag{2.7}$$

(在(2.6)式以及以下的討論裡，我們將所有緩慢變化參數之高於一階的導函數皆忽略，此與Shyu & Phillips (1990)以及以往的ray theories所採用的近似相符合.)將(2.6)和(2.7)代入(2.5)得到

$$-k_x^2 + i\frac{dk_x}{dx} + 2ik_x\frac{1}{A}\frac{dA}{dx} - k_y^2 + l^2 + \frac{\partial l}{\partial z} = 0. \tag{2.8}$$

在(2.8)內我們尚未知 $\partial l/\partial z$ 和 dk_x/dx 之間的關係，為求得此關係式，我們先考慮波向與流向平行時的狀況，在此種狀況下，$k_y = 0$，故(2.5)可化簡為

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\phi=0.$$

由上式,若波的相位朝正x軸方向行進,則ϕ將滿足

$$\frac{\partial\phi}{\partial x}-i\frac{\partial\phi}{\partial z}=0 \tag{2.9}$$

(否則ϕ在負z軸方向將呈指數增加),此時(2.3)應改寫為

$$\phi=A(x)\exp\left[i\int k(x)\,dx+\int l(x,z)\,dz\right]e^{-in_0t}. \tag{2.10}$$

而(2.8)應改寫為

$$-k^2+i\frac{dk}{dx}+2ik\frac{1}{A}\frac{dA}{dx}+l^2+\frac{\partial l}{\partial z}=0. \tag{2.11}$$

將(2.10)代入(2.9),另外我們可獲得

$$ik+\frac{1}{A}\frac{dA}{dx}=il.$$

將上式左右兩邊平方,並忽略高階項可得

$$-k^2+2ik\frac{1}{A}\frac{dA}{dx}=-l^2. \tag{2.12}$$

由(2.11)和(2.12)立即顯示

$$\frac{\partial l}{\partial z}=-i\frac{dk}{dx}. \tag{2.13}$$

上式雖在2D的情況導出,但它經由座標轉換亦可應用在3D(因一個彎曲不大的波峯線對(2.13)的影響應僅為增加一修正項,但此修正項應為更高階的項,故可以忽略),故在目前的情況下

$$\frac{\partial l}{\partial z}=-i\frac{k_x^2}{k^2}\frac{dk_x}{dx}, \tag{2.14}$$

其中$k=(k_x^2+k_y^2)^{1/2}$表$\mathbf{k}$的量,將(2.14)代入(2.8)因此可得

$$l^2 = k^2 - 2ik_x\frac{1}{A}\frac{dA}{dx} - i(1-\frac{k_x^2}{k^2})\frac{dk_x}{dx},$$

或開根號,並採用相同的近似

$$l = k - i\frac{k_x}{k}\frac{A'}{A} - \frac{i}{2}\frac{k_x'}{k}(1-\frac{k_x^2}{k^2}). \tag{2.15}$$

(以下我們採用$A' = dA/dx, k_x' = dk_x/dx$等簡單的記號法來代表常微分.)

接下來我們利用上述結果將運動及動力兩邊界條件做各種的組合,以消去ϕ,並建立η的四階常微分方程,爲達到此目的,我們先由(2.3)和(2.15)獲得在$z=0$處

$$\frac{\partial\phi}{\partial x} = ik_x(1-\frac{i}{k_x}\frac{A'}{A})\phi,$$

$$\frac{\partial^2\phi}{\partial x^2} = -k_x^2(1-\frac{2i}{k_x}\frac{A'}{A} - i\frac{k_x'}{k_x^2})\phi,$$

$$\frac{\partial^3\phi}{\partial x^3} = -ik_x^3(1-\frac{3i}{k_x}\frac{A'}{A} - 3i\frac{k_x'}{k_x^2})\phi,$$

$$\frac{\partial^4\phi}{\partial x^4} = k_x^4(1-\frac{4i}{k_x}\frac{A'}{A} - 6i\frac{k_x'}{k_x^2})\phi,$$

$$\frac{\partial\phi}{\partial z} = k[1-i\frac{k_x}{k^2}\frac{A'}{A} - \frac{i}{2}\frac{k_x'}{k^2}(1-\frac{k_x^2}{k^2})]\phi,$$

$$\frac{\partial^2\phi}{\partial x\partial z} = ikk_x[1-i\frac{1}{k_x}\frac{A'}{A}(1+\frac{k_x^2}{k^2}) - \frac{i}{2}\frac{k_x'}{k^2}(3-\frac{k_x^2}{k^2})]\phi,$$

$$\frac{\partial^3\phi}{\partial x^2\partial z} = -kk_x^2[1-i\frac{1}{k_x}\frac{A'}{A}(2+\frac{k_x^2}{k^2}) - \frac{i}{2}\frac{k_x'}{k_x^2}(2+5\frac{k_x^2}{k^2} - \frac{k_x^4}{k^4})]\phi,$$

$$\frac{\partial^4\phi}{\partial x^3\partial z} = -ikk_x^3[1-i\frac{1}{k_x}\frac{A'}{A}(3+\frac{k_x^2}{k^2}) - \frac{i}{2}\frac{k_x'}{k_x^2}(6+7\frac{k_x^2}{k^2} - \frac{k_x^4}{k^4})]\phi.$$

由上述的結果我們進一步得到

$$\frac{\partial\phi}{\partial x} = i\frac{k_x}{k}[1 - ic_0\frac{A'}{A} + ic_1k_x']\frac{\partial\phi}{\partial z}, \tag{2.16a}$$

$$\frac{\partial^2\phi}{\partial x^2} = i\frac{k_x}{k}[1 - ic_0\frac{A'}{A} + ic_2k_x']\frac{\partial^2\phi}{\partial x\partial z}, \tag{2.16b}$$

$$\frac{\partial^3\phi}{\partial x^3} = i\frac{k_x}{k}[1 - ic_0\frac{A'}{A} + ic_3k_x']\frac{\partial^3\phi}{\partial x^2\partial z}, \tag{2.16c}$$

$$\frac{\partial^4\phi}{\partial x^4} = i\frac{k_x}{k}[1 - ic_0\frac{A'}{A} + ic_4k_x']\frac{\partial^4\phi}{\partial x^3\partial z}, \tag{2.16d}$$

其中

$$\left.\begin{aligned}
c_0 &= \frac{1}{k_x}(1-\frac{k_x^2}{k^2}),\\
c_1 &= \frac{1}{2k^2}(1-\frac{k_x^2}{k^2}),\\
c_2 &= -\frac{1}{k_x^2}(1-\frac{3}{2}\frac{k_x^2}{k^2}+\frac{1}{2}\frac{k_x^4}{k^4}),\\
c_3 &= -\frac{1}{k_x^2}(2-\frac{5}{2}\frac{k_x^2}{k^2}+\frac{1}{2}\frac{k_x^4}{k^4}),\\
c_4 &= -\frac{1}{k_x^2}(3-\frac{7}{2}\frac{k_x^2}{k^2}+\frac{1}{2}\frac{k_x^4}{k^4}),
\end{aligned}\right\} \tag{2.16e}$$

利用(2.16a–e)即可將運動及動力邊界條件加以組合.

在目前的狀況,運動及動力邊界條件的近似可表爲

$$-in_0\eta + U\eta' + \eta U' = \frac{\partial\phi}{\partial z} \qquad at \qquad z=0, \tag{2.17}$$

$$-in_0\phi + g\eta + U\frac{\partial\phi}{\partial x} = 0 \qquad at \qquad z=0 \tag{2.18}$$

(參閱Shyu & Phillips (1990)). (η雖不僅包含x這項獨立變數,但在不慮混淆的情況下,我們以下仍用η', η'', η'''等代表$\partial\eta/\partial x, \partial^2\eta/\partial x^2, \partial^3\eta/\partial x^3$等.) 由(2.17)我們可得

$$(-in_0 + 2U')\eta' + U\eta'' = \frac{\partial^2\phi}{\partial x\partial z}, \tag{2.19a}$$

$$(-in_0 + 3U')\eta'' + U\eta''' = \frac{\partial^3\phi}{\partial x^2\partial z}, \tag{2.19b}$$

$$(-in_0 + 4U')\eta''' + U\eta^{IV} = \frac{\partial^4\phi}{\partial x^3\partial z}. \tag{2.19c}$$

由(2.18)可獲得

$$(-in_0 + U')\frac{\partial\phi}{\partial x} + g\eta' + U\frac{\partial^2\phi}{\partial x^2} = 0, \tag{2.20a}$$

$$(-in_0 + 2U')\frac{\partial^2\phi}{\partial x^2} + g\eta'' + U\frac{\partial^3\phi}{\partial x^3} = 0, \tag{2.20b}$$

$$(-in_0 + 3U')\frac{\partial^3\phi}{\partial x^3} + g\eta''' + U\frac{\partial^4\phi}{\partial x^4} = 0. \tag{2.20c}$$

將(2.16c,d)連同(2.19b,c)代入(2.20c),可得

$$U^2\eta^{IV}+\eta'''\Big[-i(2n_0U+\frac{gk}{k_x})+7UU'+c_0\frac{gk}{k_x}\frac{A'}{A}+(c_3-c_4)n_0Uk_x'-c_4\frac{gk}{k_x}k_x'\Big]$$
$$+\eta''\Big[-n_0^2-in_0^2(c_3-c_4)k_x'-6in_0U'\Big]=0. \tag{2.21}$$

再將(2.16b,c)連同(2.19a,b)代入(2.20b),得到

$$U^2\eta'''+\eta''\Big[-i(2n_0U+\frac{gk}{k_x})+5UU'+c_0\frac{gk}{k_x}\frac{A'}{A}+(c_2-c_3)n_0Uk_x'-c_3\frac{gk}{k_x}k_x'\Big]$$
$$+\eta'\Big[-n_0^2-in_0^2(c_2-c_3)k_x'-4in_0U'\Big]=0.$$

將上式分別乘以$-2in_0/U$和igk/U^2k_x可得

$$-2in_0U\eta'''-\eta''\Big[\frac{2n_0}{U}(2n_0U+\frac{gk}{k_x})+10in_0U'+c_0\frac{2in_0}{U}\frac{gk}{k_x}\frac{A'}{A}+2i(c_2-c_3)n_0^2k_x'$$
$$-c_3\frac{2in_0}{U}\frac{gk}{k_x}k_x'\Big]-\eta'\Big[-\frac{2in_0^3}{U}+(c_2-c_3)\frac{2n_0^3}{U}k_x'+8n_0^2\frac{U'}{U}\Big]=0 \tag{2.22}$$

和

$$i\frac{gk}{k_x}\eta'''+\eta''\Big[\frac{gk}{U^2k_x}(2n_0U+\frac{gk}{k_x})+5i\frac{gk}{k_x}\frac{U'}{U}+i\frac{c_0}{U^2}(\frac{gk}{k_x})^2\frac{A'}{A}+i(c_2-c_3)\frac{n_0}{U}\frac{gk}{k_x}k_x'$$
$$-i\frac{c_3}{U^2}(\frac{gk}{k_x})^2k_x'\Big]+\eta'\Big[-i\frac{n_0^2}{U^2}\frac{gk}{k_x}+(c_2-c_3)\frac{n_0^2}{U^2}\frac{gk}{k_x}k_x'+4\frac{n_0}{U^2}\frac{gk}{k_x}U'\Big]=0. \tag{2.23}$$

最後將(2.16a,b)連同(2.17)和(2.19a)代入(2.20a),其結果並乘以$-n_0^2/U$,得到

$$-n_0^2\eta''+\eta'\Big[i\frac{n_0^2}{U^2}(2n_0U+\frac{gk}{k_x})-3n_0^2\frac{U'}{U}-c_0\frac{n_0^2}{U^2}\frac{gk}{k_x}\frac{A'}{A}-(c_1-c_2)\frac{n_0^3}{U}k_x'$$
$$+c_2\frac{n_0^2}{U^2}\frac{gk}{k_x}k_x'\Big]+\eta\Big[\frac{n_0^4}{U^2}+i(c_1-c_2)\frac{n_0^4}{U^2}k_x'+2i\frac{n_0^3}{U^2}U'\Big]=0. \tag{2.24}$$

將(2.21)-(2.24)加在一起後可得

$$
\begin{aligned}
&U^2\eta^{IV}+\eta'''\Big[-4in_0U+7UU'+c_0\frac{gk}{k_x}\frac{A'}{A}+(c_3-c_4)n_0Uk_x'-c_4\frac{gk}{k_x}k_x'\Big]\\
&\quad+\eta''\Big\{-6n_0^2+(\frac{gk}{Uk_x})^2-(16in_0-5i\frac{gk}{Uk_x})U'+ic_0\frac{gk}{Uk_x}(\frac{gk}{Uk_x}-2n_0)\\
&\quad\cdot\frac{A'}{A}+\Big[-3i(c_2-c_3)n_0^2-ic_3(\frac{gk}{Uk_x})^2+i(c_2+c_3)n_0\frac{gk}{Uk_x}\Big]k_x'\Big\}\\
&\quad+\eta'\Big[\frac{4in_0^3}{U}-11n_0^2\frac{U'}{U}+4\frac{n_0}{U}\frac{gk}{Uk_x}U'-c_0\frac{n_0^2}{U}\frac{gk}{Uk_x}\frac{A'}{A}-3(c_2-c_3)\frac{n_0^3}{U}k_x'\\
&\quad+(2c_2-c_3)\frac{n_0^2}{U}\frac{gk}{Uk_x}k_x'\Big]+\eta\Big[\frac{n_0^4}{U^2}+i(c_1-c_2)\frac{n_0^4}{U^2}k_x'+2i\frac{n_0^3}{U^2}U'\Big]=0.
\end{aligned}
\tag{2.25}
$$

為獲得上式,我們且利用

$$c_1-c_2=c_2-c_3=c_3-c_4=\frac{k_y^2}{k^2k_x^2}. \tag{2.26}$$

以上的處理並非任意;這可由(2.25)式中的主要項和(2.2)一致看出,(2.25)式可進一步加以簡化,首先由(2.4)

$$\eta''=\big(2ik_x\frac{a'}{a}+ik_x'-k_x^2\big)\eta. \tag{2.27}$$

利用上式可將(2.25)中某些η''項轉換成η的項,但為達到真正簡化的目的,需將a'/a表為A'/A以及其它的項,此可由邊界條件達成.

若我們同時將(2.3)和(2.4)代入(2.17),可得

$$-in_0a+U(a'+ik_xa)+aU'=lA. \tag{2.28}$$

對x微分並應用(2.15),產生

$$-in_0a'+iUk_xa'+ik_xaU'+iUak_x'=kA'+Ak'.$$

在上式中所有高階項仍被忽略,另外由於

$$k'=\frac{k_x}{k}k_x',$$

上式可改寫爲

$$-in_0 a' + iUk_x a' + ik_x aU' + iUak_x' = kA' + A\frac{k_x}{k}k_x'. \tag{2.29}$$

另一方面,由對(2.1)微分我們亦可獲得

$$U' = -\Box\frac{k_x'}{k_x}, \tag{2.30}$$

其中

$$\Box = U + \frac{k_x}{2k^2}(gk)^{\frac{1}{2}}. \tag{2.31}$$

故由(2.29)

$$A'k + A\frac{k_x}{k}k_x' = -i(gk)^{\frac{1}{2}}\left(a' + a\frac{k_x k_x'}{2k^2}\right), \tag{2.32}$$

由(2.28)和(2.15)

$$A = -i\frac{a}{k}(gk)^{\frac{1}{2}}\left[1 + \frac{iU}{(gk)^{\frac{1}{2}}}\frac{a'}{a} + \frac{i}{(gk)^{\frac{1}{2}}}U'\right]. \tag{2.33}$$

用(2.33)除(2.32)並忽略高階項,可得

$$\frac{A'}{A}k + \frac{k_x}{k}k_x' = \frac{k}{a}\left(a' + a\frac{k_x k_x'}{2k^2}\right),$$

亦即

$$\frac{a'}{a} = \frac{A'}{A} + \frac{k_x k_x'}{2k^2}. \tag{2.34}$$

故(2.27)此時可改寫爲

$$\eta'' = \left[2ik_x\frac{A'}{A} + ik_x'\left(1 + \frac{k_x^2}{k^2}\right) - k_x^2\right]\eta. \tag{2.35}$$

利用(2.35)即可進行移項的工作.

在(2.25)中η''的係數包含

$$\left(\frac{gk}{Uk_x}\right)^2 = \frac{g^2}{U^2} + \left(\frac{gk_y}{Uk_x}\right)^2. \tag{2.36}$$

由(2.35)

$$\left(\frac{gk_y}{Uk_x}\right)^2\eta''=\left[2ik_x\left(\frac{gk_y}{Uk_x}\right)^2\frac{A'}{A}+ik_x'\left(1+\frac{k_x^2}{k^2}\right)\left(\frac{gk_y}{Uk_x}\right)^2-\left(\frac{gk_y}{U}\right)^2\right]\eta. \tag{2.37}$$

故目前在(2.25)中包含A'/A的項有

$$\eta'''\left[c_0\frac{gk}{k_x}\frac{A'}{A}\right]+\eta''\left[ic_0\frac{gk}{Uk_x}\left(\frac{gk}{Uk_x}-2n_0\right)\frac{A'}{A}\right]-\eta'\left[c_0\frac{n_0^2}{U}\frac{gk}{Uk_x}\frac{A'}{A}\right]+\eta\left[2ik_x\left(\frac{gk_y}{Uk_x}\right)^2\frac{A'}{A}\right]$$

$$=\eta\left\{-ik_x^3\left[c_0\frac{gk}{k_x}\frac{A'}{A}\right]-k_x^2\left[ic_0\frac{gk}{Uk_x}\left(\frac{gk}{Uk_x}-2n_0\right)\frac{A'}{A}\right]-ik_x\left[\frac{c_0}{U}\frac{gk}{Uk_x}\frac{A'}{A}(gk+2n_0Uk_x-U^2k_x^2)\right]\right.$$

$$\left.+\left[2ik_x\left(\frac{gk_y}{Uk_x}\right)^2\frac{A'}{A}\right]\right\}$$

$$=\eta\left\{-2ic_0k_x^2\left(\frac{gk}{Uk_x}\right)^2\frac{A'}{A}+2ik_x\left(\frac{gk_y}{Uk_x}\right)^2\frac{A'}{A}\right\}$$

$$=0,$$

(因(2.16e)).

接下來我們將(2.25)中所有含k_x'的係數亦移至η項,由於(2.26),這些係數的和,包括在(2.37)中含k_x'的係數,可寫成

$$ik_x^2k_x'\left\{(c_2-c_3)\left[-n_0Uk_x+3n_0^2-3\frac{n_0^3}{Uk_x}+\frac{n_0^2}{Uk_x}\frac{gk}{Uk_x}+\frac{n_0^4}{U^2k_x^2}\right]\right.$$

$$\left.+\frac{gk}{Uk_x}\left[c_4Uk_x+c_3\frac{gk}{Uk_x}-n_0(c_2+c_3)+c_2\frac{n_0^2}{Uk_x}\right]\right\}+i\left(1+\frac{k_x^2}{k^2}\right)\left(\frac{gk_y}{Uk_x}\right)^2k_x'$$

利用(2.16e)及離散關係式(2.1),上式可化簡爲

$$ik_x'\left[4\frac{k_y^2}{k^2}\frac{(gk)^{\frac{3}{2}}}{Uk_x}+2\left(1-\frac{k_x^2}{k^2}\right)\frac{g^2}{U^2}\right]. \tag{2.38}$$

同樣地,含U'的係數全部移至(2.25)的零階項後,可化簡爲

$$ik_x^2U'\left[-6\frac{gk}{Uk_x}+6\frac{(gk)^{\frac{3}{2}}}{U^2k_x^2}\right]. \tag{2.39}$$

利用(2.30)和(2.31), (2.38)和(2.39)並可組合成

$$iU'\left[-6\frac{gkk_x}{U}+2\frac{(gk)^{\frac{3}{2}}}{U^2}+4\frac{g^{\frac{3}{2}}k_x^2}{U^2k^{\frac{1}{2}}}\right].$$

故(2.25)最後可改寫爲

$$\begin{aligned}&U^2\eta^{IV}+\eta'''\left[-4in_0U\right]+\eta''\left[-6n_0^2+\frac{g^2}{U^2}\right]+\eta'\left[4i\frac{n_0^3}{U}\right]\\&+\eta\left[\frac{n_0^4}{U^2}-\frac{g^2k_y^2}{U^2}+iU'\left(-6\frac{gkk_x}{U}+2\frac{(gk)^{\frac{3}{2}}}{U^2}+4\frac{g^{\frac{3}{2}}k_x^2}{U^2k^{\frac{1}{2}}}\right)\right]=0.\end{aligned}\tag{2.40}$$

以上我們考慮波的緩變性質,利用Laplace's equation及邊界條件即導出(2.40)(離散關係式(2.1)實際亦可由邊界條件獲得),特別是A'/A的消除並未應用到action conservation principle,但在上式中仍包含k_x,此參數即使經由移項亦無法完全消去,故(2.40)在阻塞點仍爲singular.在下一節裡,我們將利用action conservation principle導出(2.40)中次要的項,此顯示(2.40)確爲正確,且wave action的確守恆.

三. 另一種導法

由離散關係式的展開式(2.2),我們已知所需之四階微方之主要項爲

$$U^2\eta^{IV}+\eta'''\Big[-4in_0U\Big]+\eta''\Big[-6n_0^2+\frac{g^2}{U^2}\Big]+\eta'\Big[4i\frac{n_0^3}{U}\Big]+\eta\Big[\frac{n_0^4}{U^2}-\frac{g^2k_y^2}{U^2}\Big]. \tag{3.1}$$

由(2.4)我們且已知

$$\eta'=(\frac{a'}{a}+ik_x)\eta,$$

$$\eta''=(2ik_x\frac{a'}{a}+ik_x'-k_x^2)\eta,$$

$$\eta'''=-(3k_x^2\frac{a'}{a}+3k_xk_x'+ik_x^3)\eta,$$

$$\eta^{IV}=(-4ik_x^3\frac{a'}{a}-6ik_x^2k_x'+k_x^4)\eta.$$

代入(3.1)可得

$$\eta\Big[U^2(-4ik_x^3\frac{a'}{a}-6ik_x^2k_x'+k_x^4)+4in_0U(3k_x^2\frac{a'}{a}+3k_xk_x'+ik_x^3)$$
$$+(-6n_0^2+\frac{g^2}{U^2})(2ik_x\frac{a'}{a}+ik_x'-k_x^2)+4i\frac{n_0^3}{U}(\frac{a'}{a}+ik_x)+\frac{n_0^4}{U^2}-\frac{g^2k_y^2}{U^2}\Big].$$

利用(2.2),上式中主要項皆如預期互相抵消,但以下我們將證明其次要項將無法抵消,表(3.1)需增加一些補充項方能加以平衡.

上式可化簡爲

$$\eta\Big[\big(-4iU^2k_x^3+12in_0Uk_x^2-12in_0^2k_x+2i\frac{g^2}{U^2}k_x+4i\frac{n_0^3}{U}\big)\frac{a'}{a}$$
$$+\big(-6iU^2k_x^2+12in_0Uk_x-6in_0^2+i\frac{g^2}{U^2}\big)k_x'\Big], \tag{3.2}$$

其中a'/a項可利用action conservation principle表爲k_x'或U',在目前的情況下,此principle可化簡爲

$$\frac{\partial}{\partial x}\Big[(U+C_{gx})\frac{E}{\sigma}\Big]=0,$$

其中$C_{gx} = k_x(gk)^{\frac{1}{2}}/2k^2$表波的羣速度在$x$方向的分量，$E = \rho g a^2/2$表波的能量密度，$\sigma = (gk)^{\frac{1}{2}}$表波的實際頻率(intrinsic frequency),故

$$\frac{\partial}{\partial x}\left[(U + \frac{1}{2}\frac{(gk)^{\frac{1}{2}}}{k^2}k_x)\frac{(gk)^{\frac{1}{2}}}{k}a^2\right] = 0.$$

在上式中已將常數項$\rho/2$消去,將上式展開並將(2.30)代入後,可得

$$\frac{a'}{a} = -\frac{U'}{2}\frac{[\,]}{[\,]^2}, \tag{3.3}$$

其中

$$[\,] = U(1 + \frac{1}{2}\frac{k_x^2}{k^2}) + \sigma\frac{k_x^3}{k^4}. \tag{3.4}$$

將(3.3)代入(3.2),且將(2.30)代入以取代k_x',再經由複雜的代數運算,(3.2)最後可化簡爲

$$\eta\left[iU'(6\frac{gkk_x}{U} - 2\frac{(gk)^{\frac{3}{2}}}{U^2} - 4\frac{g^{\frac{3}{2}}k_x^2}{U^2k^{\frac{1}{2}}})\right].$$

故與(2.40)相符合.

四. 連結入射波和反射波的方程式

由於(2.40)式在阻塞點仍爲singular,且式中k_x無法消去,表由上述的方法仍無法獲得一連結入射波和反射波的微分方程,但若我們繼續將(2.40)分解至一階微方,再分別用入射波和反射波的k_x代入以獲得兩個一階微方,再將兩者重新組合成一個二階微方後,上述的singularity即可互相抵消,因此表一連結入射波和反射波的方程式.要分解(2.40)式,我們採用Turrittin (1952)所建議的方法(亦參閱Shyu & Phillips (1990)).

4.1. 分解四階微方

首先我們將(2.40)表爲一矩陣方程式的形式

$$\overline{\eta}' = (B_0 + B_1)\overline{\eta}, \tag{4.1}$$

其中

$$\overline{\eta} = \begin{bmatrix} \eta \\ \eta' \\ \eta'' \\ \eta''' \end{bmatrix} \tag{4.2}$$

$$B_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ b_1 & b_2 & b_3 & b_4 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ b_5 & 0 & 0 & 0 \end{bmatrix}, \tag{4.3}$$

且矩陣中的元素

$$\left.\begin{aligned} &b_1 = \tfrac{g^2k_y^2}{U^4} - \tfrac{n_0^4}{U^4}, \quad b_2 = -4i\tfrac{n_0^3}{U^3}, \quad b_3 = 6\tfrac{n_0^2}{U^2} - \tfrac{g^2}{U^4}, \\ &b_4 = 4i\tfrac{n_0}{U}, \quad b_5 = iU'\big(6\tfrac{gkk_x}{U^3} - 2\tfrac{(gk)^{\frac{3}{2}}}{U^4} - 4\tfrac{g^{\frac{3}{2}}k_x^2}{U^4k^{\frac{1}{2}}}\big). \end{aligned}\right\} \tag{4.4}$$

應注意的是在(4.1)中的第二個係數矩陣B_1的量較小,我們特別將它與B_0分開,因此種安排對以下導公式極有幫助.

由(4.1),若B_0和B_1都能轉換成僅在對角線上的四方區(diagonal blocks)內的元素有值,則原四階系統即可被分解成兩個或兩個以上低階系統,爲達到此目的,我們首先令

$$\overline{\eta} = S\,\overline{y}, \tag{4.5}$$

其中S代表一有待決定之4×4轉換矩陣，$\overline{y}$取代$\overline{\eta}$爲新的相依變數，將(4.5)代入(4.1)後產生

$$S\,\overline{y}' + S'\,\overline{y} = (B_0 + B_1)S\,\overline{y}.$$

再將等號兩邊同時乘以S的逆矩陣S^{-1}，可得一新的微分方程

$$\overline{y}' = (C_0 + C_1)\,\overline{y}, \tag{4.6}$$

其中

$$C_0 = S^{-1}B_0S, \tag{4.7}$$

$$C_1 = S^{-1}B_1S - S^{-1}S'. \tag{4.8}$$

由於對緩慢變化的參數微分，S'的元素較小，故在(4.6)式中C_1仍僅含較小的量．另一方面(4.7)式代表一種對B_0所做的similarity transformation，對此種轉換我們已知，若S由B_0的四個eigenvectors組成，則轉換後的C_0即爲diagonal. 故由(4.3)我們先求B_0的四個eigenvalues λ_1, λ_2 λ_3和λ_4，它們須滿足

$$0 = \det(B_0 - \lambda I) = \begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ b_1 & b_2 & b_3 & b_4 - \lambda \end{vmatrix}$$

$$= -\lambda^4 + b_4\lambda^3 + b_3\lambda^2 + b_2\lambda + b_1, \tag{4.9}$$

其中I爲單位矩陣．上式代表一個四次多項式，它的解滿足下面幾個關係式

$$\left.\begin{aligned} &\lambda_1\lambda_2\lambda_3\lambda_4 = -b_1, \quad && \sum\lambda_i\lambda_j\lambda_k = b_2, \\ &\sum\lambda_i\lambda_j = -b_3, && \lambda_1+\lambda_2+\lambda_3+\lambda_4 = b_4. \end{aligned}\right\} \tag{4.10}$$

另外要指出的是，由比較(4.9)和(2.2)立即可發現，(4.9)和(2.2)的根具有以下關係

$$\lambda_j = ik_{xj}, \qquad j = 1, 2, 3, 4. \tag{4.11}$$

利用(4.9)我們可進一步求B_0的四個eigenvectors，分別爲

$$e_j = \begin{bmatrix} 1 \\ \lambda_j \\ \lambda_j^2 \\ \lambda_j^3 \end{bmatrix}, \qquad j = 1, 2, 3, 4.$$

故轉換矩陣應選擇為

$$S = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 \\ \lambda_1^3 & \lambda_2^3 & \lambda_3^3 & \lambda_4^3 \end{bmatrix}. \tag{4.12}$$

其逆矩陣為

$$S^{-1} = \frac{1}{\det S} \begin{bmatrix} -\lambda_2\lambda_3\lambda_4 L_1 & (\lambda_2\lambda_3 + \lambda_3\lambda_4 + \lambda_2\lambda_4)L_1 & -(\lambda_2 + \lambda_3 + \lambda_4)L_1 & L_1 \\ -\lambda_1\lambda_3\lambda_4 L_2 & (\lambda_1\lambda_3 + \lambda_3\lambda_4 + \lambda_1\lambda_4)L_2 & -(\lambda_1 + \lambda_3 + \lambda_4)L_2 & L_2 \\ -\lambda_1\lambda_2\lambda_4 L_3 & (\lambda_1\lambda_2 + \lambda_2 l\lambda_4 + \lambda_1\lambda_4)L_3 & -(\lambda_1 + \lambda_2 + \lambda_4)L_3 & L_3 \\ -\lambda_1\lambda_2\lambda_3 L_4 & (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1)L_4 & -(\lambda_1 + \lambda_2 + \lambda_3)L_4 & L_4 \end{bmatrix}. \tag{4.13}$$

其中

$$\left.\begin{aligned} L_1 &= (\lambda_3 - \lambda_2)(\lambda_4 - \lambda_3)(\lambda_2 - \lambda_4), \\ L_2 &= (\lambda_3 - \lambda_1)(\lambda_4 - \lambda_1)(\lambda_4 - \lambda_3), \\ L_3 &= (\lambda_2 - \lambda_1)(\lambda_4 - \lambda_1)(\lambda_2 - \lambda_4), \\ L_4 &= (\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2). \end{aligned}\right\} \tag{4.14}$$

且

$$\det S = -(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_4)(\lambda_4 - \lambda_3)(\lambda_4 - \lambda_1).$$

將(4.12)和(4.13)代入(4.7)可得

$$C_0 = S^{-1}B_0S = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix}. \tag{4.15}$$

上式明顯為一diagonal矩陣,但另一方面由(4.8)

$C_1 = S^{-1}B_1S - S^{-1}S'$

$$= \frac{1}{\det S} \begin{bmatrix} L_1(b_5 - N_1\lambda_1') & L_1[b_5 - (\lambda_4 - \lambda_2)(\lambda_3 - \lambda_2)\lambda_2'] & L_1[b_5 - (\lambda_4 - \lambda_3)(\lambda_2 - \lambda_3)\lambda_3'] & L_1[b_5 - (\lambda_4 - \lambda_3)(\lambda_4 - \lambda_2)\lambda_4'] \\ L_2[b_5 - (\lambda_4 - \lambda_1)(\lambda_3 - \lambda_1)\lambda_1'] & L_2(b_5 - N_2\lambda_2') & L_2[b_5 - (\lambda_4 - \lambda_3)(\lambda_1 - \lambda_3)\lambda_3'] & L_2[b_5 - (\lambda_4 - \lambda_3)(\lambda_4 - \lambda_1)\lambda_4'] \\ L_3[b_5 - (\lambda_4 - \lambda_1)(\lambda_2 - \lambda_1)\lambda_1'] & L_3[b_5 - (\lambda_4 - \lambda_2)(\lambda_1 - \lambda_2)\lambda_2'] & L_3(b_5 - N_3\lambda_3') & L_3[b_5 - (\lambda_4 - \lambda_1)(\lambda_4 - \lambda_2)\lambda_4'] \\ L_4[b_5 - (\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)\lambda_1'] & L_4[b_5 - (\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)\lambda_2'] & L_4[b_5 - (\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)\lambda_3'] & L_4(b_5 - N_4\lambda_4') \end{bmatrix} \tag{4.16}$$

其中

$$\left.\begin{aligned}
N_1 &= \lambda_2\lambda_3+\lambda_3\lambda_4+\lambda_2\lambda_4-2\lambda_1(\lambda_2+\lambda_3+\lambda_4)+3\lambda_1^2,\\
N_2 &= \lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_3\lambda_4-2\lambda_2(\lambda_1+\lambda_3+\lambda_4)+3\lambda_2^2,\\
N_3 &= \lambda_1\lambda_2+\lambda_2\lambda_4+\lambda_1\lambda_4-2\lambda_3(\lambda_1+\lambda_2+\lambda_4)+3\lambda_3^2,\\
N_4 &= \lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1-2\lambda_4(\lambda_1+\lambda_2+\lambda_3)+3\lambda_4^2.
\end{aligned}\right\} \tag{4.17}$$

故C_1很明顯地不爲block diagonal,因此有必要進行另一次轉換,亦即令

$$\overline{y} = T\overline{\xi} \tag{4.18}$$

但這回我們將T漸近展開(asymptotic expansion)爲

$$T = T_0 + T_1 + \cdots, \tag{4.19}$$

其中T_1代表較小的量.將(4.18)和(4.19)代入(4.6)可得

$$(T_0+T_1+\cdots)\overline{\xi}' = [(C_0+C_1)(T_0+T_1+\cdots)-(T_0'+T_1'+\cdots)]\overline{\xi}.$$

若將其結果表爲

$$\overline{\xi}' = (D_0+D_1+\cdots)\overline{\xi}, \tag{4.20}$$

則經由展開和比較可得

$$T_0D_0 = C_0T_0, \tag{4.21}$$

$$T_0D_1+T_1D_0 = C_0T_1+C_1T_0-T_0', \tag{4.22}$$

其中(4.22)仍僅含次要的項.

由(4.21)可知,若定$T_0=I$,則

$$D_0 = C_0 = \begin{bmatrix} \lambda_1 & 0 & 0 & 0\\ 0 & \lambda_2 & 0 & 0\\ 0 & 0 & \lambda_3 & 0\\ 0 & 0 & 0 & \lambda_4 \end{bmatrix}. \tag{4.23}$$

表D_0不改變,仍爲一diagonal矩陣,此時(4.22)亦可簡化爲

$$D_1 = C_0T_1 - T_1C_0 + C_1. \tag{4.24}$$

故如今只要決定適當的T_1使D_1變爲block diagonal即可. 有關決定T_1的方法可參考Wasow (1985), 根據此法我們可獲得

$$T_1 \times \det S =$$

$$\begin{bmatrix} 0 & 0 & \frac{L_1}{\lambda_3-\lambda_1}[b_5-(\lambda_4-\lambda_3)(\lambda_2-\lambda_3)\lambda_3'] & \frac{L_1}{\lambda_4-\lambda_1}[b_5-(\lambda_4-\lambda_3)(\lambda_4-\lambda_2)\lambda_4'] \\ 0 & 0 & \frac{L_2}{\lambda_3-\lambda_2}[b_5-(\lambda_4-\lambda_3)(\lambda_1-\lambda_3)\lambda_3'] & \frac{L_2}{\lambda_4-\lambda_2}[b_5-(\lambda_4-\lambda_3)(\lambda_4-\lambda_1)\lambda_4'] \\ \frac{L_3}{\lambda_1-\lambda_3}[b_5-(\lambda_4-\lambda_1)(\lambda_2-\lambda_1)\lambda_1'] & \frac{L_3}{\lambda_2-\lambda_3}[b_5-(\lambda_4-\lambda_2)(\lambda_1-\lambda_2)\lambda_2'] & 0 & 0 \\ \frac{L_4}{\lambda_1-\lambda_4}[b_5-(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)\lambda_1'] & \frac{L_4}{\lambda_2-\lambda_4}[b_5-(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)\lambda_2'] & 0 & 0 \end{bmatrix}. \quad (4.25)$$

此時

$$D_1 = \frac{1}{\det S}\begin{bmatrix} L_1(b_5-N_1\lambda_1') & L_1[b_5-(\lambda_4-\lambda_2)(\lambda_3-\lambda_2)\lambda_2'] & 0 & 0 \\ L_2[b_5-(\lambda_4-\lambda_1)(\lambda_3-\lambda_1)\lambda_1'] & L_2(b_5-N_2\lambda_2') & 0 & 0 \\ 0 & 0 & L_3(b_5-N_3\lambda_3') & L_3[b_5-(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)\lambda_4'] \\ 0 & 0 & L_4[b_5-(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)\lambda_3'] & L_4(b_5-N_4\lambda_4') \end{bmatrix}. \quad (4.26)$$

因D_0和D_1兩者目前皆已轉換成diagonal blocks型態, 故此時我們可將(4.20)分成兩個不連結的系統, 首先令

$$\bar{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix}.$$

再令

$$\bar{\varsigma} = \begin{bmatrix} \xi_3 \\ \xi_4 \end{bmatrix}. \quad (4.27)$$

故將(4.23)和(4.26)代入(4.20)後, 可得

$$\bar{\varsigma}' = (\hat{D}_0 + \hat{D}_1)\bar{\varsigma}, \quad (4.28)$$

其中

$$\hat{D}_0 = \begin{bmatrix} \lambda_3 & 0 \\ 0 & \lambda_4 \end{bmatrix}, \qquad \hat{D}_1 = \frac{1}{\det S} \begin{bmatrix} L_3(b_5 - N_3\lambda_3') & \begin{matrix} L_3[b_5 - (\lambda_4 - \lambda_1) \\ (\lambda_4 - \lambda_2)\lambda_4'] \end{matrix} \\ \begin{matrix} L_4[b_5 - (\lambda_3 - \lambda_1) \\ (\lambda_3 - \lambda_2)\lambda_3'] \end{matrix} & L_4(b_5 - N_4\lambda_4') \end{bmatrix}. \tag{4.29}$$

由(4.28)式仍無法獲得原相依變數η的較低階微方,故必須將(4.28)式進一步分解爲一階系統才有效.

爲達到此目的,我們令

$$\bar{\varsigma} = V\bar{\zeta}, \tag{4.30}$$

其中

$$V = V_0 + V_1 + \cdots. \tag{4.31}$$

代入(4.28)可得

$$(V_0 + V_1 + \cdots)\bar{\zeta}' = [(\hat{D}_0 + \hat{D}_1)(V_0 + V_1 + \cdots) - (V_0' + V_1' + \cdots)]\bar{\zeta}.$$

若將其結果表爲

$$\bar{\zeta}' = (F_0 + F_1 + \cdots)\bar{\zeta}, \tag{4.32}$$

則經由展開和比較可得

$$V_0 F_0 = \hat{D}_0 V_0, \tag{4.33}$$

$$V_0 F_1 + V_1 F_0 = \hat{D}_0 V_1 + \hat{D}_1 V_0 - V_0'. \tag{4.34}$$

由於$\hat{D}_0$已爲diagonal形式,故由(4.33),若令$V_0 = I$,則

$$F_0 = \hat{D}_0 = \begin{bmatrix} \lambda_3 & 0 \\ 0 & \lambda_4 \end{bmatrix}, \tag{4.35}$$

且(4.34)可化簡爲

$$F_1 = \hat{D}_0 V_1 - V_1 \hat{D}_0 + \hat{D}_1. \tag{4.36}$$

故F_0仍爲diagonal, 且由(4.36)我們可看出,若

$$V_1 = \frac{1}{\det S}\begin{bmatrix} 0 & \frac{L_3}{\lambda_4-\lambda_3}[b_5-(\lambda_4-\lambda_1) \\ & (\lambda_4-\lambda_2)\lambda_4'] \\ \frac{L_4}{\lambda_3-\lambda_4}[b_5-(\lambda_3-\lambda_1) & 0 \\ (\lambda_3-\lambda_2)\lambda_3'] & \end{bmatrix},$$

則

$$F_1 = \frac{1}{\det S}\begin{bmatrix} L_3(b_5-N_3\lambda_3') & 0 \\ 0 & L_4(b_5-N_4\lambda_4') \end{bmatrix} \tag{4.37}$$

亦爲 diagonal. 令

$$\bar{\zeta} = \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix},$$

且將上述的 F_0 和 F_1 代入 (4.32), 我們於是可得到一階系統

$$\zeta_2' = \left[\lambda_4 + \frac{L_4}{\det S}(b_5 - N_4\lambda_4')\right]\zeta_2$$

或表爲

$$\left\{\frac{\partial}{\partial x} - \left[\lambda_4 + \frac{L_4}{\det S}(b_5 - N_4\lambda_4')\right]\right\}\zeta_2 = 0. \tag{4.38}$$

由 (4.38), 若將 ζ_2 逆轉換爲 η, 則上式將與原四階微方 (2.40) 相當, 但目前我們已將此四階微方 factorized, 故爲計算此四階微方之某一個獨立解, 我們僅需解

$$\zeta_2 = 0. \tag{4.39}$$

爲將 (4.39) 表爲 η 的方程式, 我們進行逆轉換, 首先由 (4.30) 和 (4.31)

$$\bar{\zeta} = (V_0 + V_1)^{-1}\bar{\varsigma}$$

經由直接計算可得

$$\zeta_2 = \xi_4 - (V_1)_{21}\xi_3 = 0, \tag{4.40}$$

其中 $(V_1)_{21}$ 表 V_1 之第二列第一行元素. 接下來由 (4.5), (4.18) 和 (4.19)

$$\bar{\eta} = S(T_0 + T_1)\bar{\xi},$$

故逆轉換

$$\bar{\xi} = (T_0 + T_1)^{-1}S^{-1}\bar{\eta}.$$

經由直接計算,並忽略高階項,產生

$$\begin{aligned}\xi_4 = &-(T_1)_{41}\left[(S^{-1})_{11}\eta + (S^{-1})_{12}\eta' + (S^{-1})_{13}\eta'' + (S^{-1})_{14}\eta'''\right]\\ &-(T_1)_{42}\left[(S^{-1})_{21}\eta + (S^{-1})_{22}\eta' + (S^{-1})_{23}\eta'' + (S^{-1})_{24}\eta'''\right]\\ &+\left[(S^{-1})_{41}\eta + (S^{-1})_{42}\eta' + (S^{-1})_{43}\eta'' + (S^{-1})_{44}\eta'''\right],\end{aligned} \tag{4.41}$$

和

$$\begin{aligned}\xi_3 = &-(T_1)_{31}\left[(S^{-1})_{11}\eta + (S^{-1})_{12}\eta' + (S^{-1})_{13}\eta'' + (S^{-1})_{14}\eta'''\right]\\ &-(T_1)_{32}\left[(S^{-1})_{21}\eta + (S^{-1})_{22}\eta' + (S^{-1})_{23}\eta'' + (S^{-1})_{24}\eta'''\right]\\ &+\left[(S^{-1})_{31}\eta + (S^{-1})_{32}\eta' + (S^{-1})_{33}\eta'' + (S^{-1})_{34}\eta'''\right],\end{aligned} \tag{4.42}$$

其中$(\cdot)_{ij}$表括號內矩陣之第i列第j行元素.將(4.41)和(4.42)代入(4.40),並將所有兩次要項的乘積皆忽略,可得

$$\begin{aligned}&\left[(S^{-1})_{41}\eta + (S^{-1})_{42}\eta' + (S^{-1})_{43}\eta'' + (S^{-1})_{44}\eta'''\right] - (T_1)_{41}\left[(S^{-1})_{11}\eta + (S^{-1})_{12}\eta'\right.\\ &\left.+ (S^{-1})_{13}\eta'' + (S^{-1})_{14}\eta'''\right] - (T_1)_{42}\left[(S^{-1})_{21}\eta + (S^{-1})_{22}\eta' + (S^{-1})_{23}\eta'' + (S^{-1})_{24}\eta'''\right]\\ &- (V_1)_{21}\left[(S^{-1})_{31}\eta + (S^{-1})_{32}\eta' + (S^{-1})_{33}\eta'' + (S^{-1})_{34}\eta'''\right] = 0,\end{aligned}$$

或表爲

$$\eta''' + E_2\eta'' + E_1\eta' + E_0\eta = 0, \tag{4.43}$$

其中

$$\begin{aligned}E_2 = &-(\lambda_1+\lambda_2+\lambda_3) - (T_1)_{41}\frac{(S^{-1})_{13}}{(S^{-1})_{44}} - (T_1)_{42}\frac{(S^{-1})_{23}}{(S^{-1})_{44}} - (V_1)_{21}\frac{(S^{-1})_{33}}{(S^{-1})_{44}}\\ &+\Psi\frac{(S^{-1})_{43}}{(S^{-1})_{44}},\end{aligned} \tag{4.44a}$$

$$\begin{aligned}E_1 = &(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1) - (T_1)_{41}\frac{(S^{-1})_{12}}{(S^{-1})_{44}} - (T_1)_{42}\frac{(S^{-1})_{22}}{(S^{-1})_{44}} - (V_1)_{21}\frac{(S^{-1})_{32}}{(S^{-1})_{44}}\\ &+\Psi\frac{(S^{-1})_{42}}{(S^{-1})_{44}},\end{aligned} \tag{4.44b}$$

$$E_0 = -\lambda_1\lambda_2\lambda_3 - (T_1)_{41}\frac{(S^{-1})_{11}}{(S^{-1})_{44}} - (T_1)_{42}\frac{(S^{-1})_{21}}{(S^{-1})_{44}} - (V_1)_{21}\frac{(S^{-1})_{31}}{(S^{-1})_{44}} + \Psi\frac{(S^{-1})_{41}}{(S^{-1})_{44}}, \tag{4.44c}$$

且

$$\Psi = (T_1)_{41}\frac{(S^{-1})_{14}}{(S^{-1})_{44}} + (T_1)_{42}\frac{(S^{-1})_{24}}{(S^{-1})_{44}} + (V_1)_{21}\frac{(S^{-1})_{34}}{(S^{-1})_{44}}. \tag{4.45}$$

爲獲得(4.44a–c),我們且應用(4.13),故(4.43)最後可表爲

$$\eta''' - (d_3 + d_6)\eta'' - (d_2 + d_5)\eta' - (d_1 + d_4)\eta = 0, \tag{4.46}$$

其中

$$d_1 = \lambda_1\lambda_2\lambda_3, \quad d_2 = -(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1), \quad d_3 = \lambda_1 + \lambda_2 + \lambda_3, \tag{4.47}$$

$$\left.\begin{aligned} d_4 &= (T_1)_{41}\frac{(S^{-1})_{11}}{(S^{-1})_{44}} + (T_1)_{42}\frac{(S^{-1})_{21}}{(S^{-1})_{44}} + (V_1)_{21}\frac{(S^{-1})_{31}}{(S^{-1})_{44}} - \Psi\frac{(S^{-1})_{41}}{(S^{-1})_{44}}, \\ d_5 &= (T_1)_{41}\frac{(S^{-1})_{12}}{(S^{-1})_{44}} + (T_1)_{42}\frac{(S^{-1})_{22}}{(S^{-1})_{44}} + (V_1)_{21}\frac{(S^{-1})_{32}}{(S^{-1})_{44}} - \Psi\frac{(S^{-1})_{42}}{(S^{-1})_{44}}, \\ d_6 &= (T_1)_{41}\frac{(S^{-1})_{13}}{(S^{-1})_{44}} + (T_1)_{42}\frac{(S^{-1})_{23}}{(S^{-1})_{44}} + (V_1)_{21}\frac{(S^{-1})_{33}}{(S^{-1})_{44}} - \Psi\frac{(S^{-1})_{43}}{(S^{-1})_{44}}. \end{aligned}\right\} \tag{4.48}$$

4.2. 分解三階微方

獲得三階微方(4.46)後,接下來需將它分解爲二階微方,再分解爲一階(用以上的方法無法直接獲得一階微方),使用和Shyu & Phillips (1990)完全相同的步驟可得

$$\eta'' + [-(\lambda_1 + \lambda_2) + \hat{Q}]\eta' + [\lambda_1\lambda_2 + \hat{P}]\eta = 0, \tag{4.49}$$

其中

$$\hat{Q} = \frac{1}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}\big[d_4 + \lambda_3 d_5 + (\lambda_3\lambda_1 + \lambda_3\lambda_2 - \lambda_1\lambda_2)d_6 - (\lambda_3 - \lambda_2)\lambda_1' - (\lambda_3 - \lambda_1)\lambda_2'\big], \tag{4.50a}$$

$$\hat{P} = \frac{1}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}\big[(\lambda_3 - \lambda_2 - \lambda_1)d_4 - \lambda_1\lambda_2 d_5 - \lambda_1\lambda_2\lambda_3 d_6 + \lambda_2(\lambda_3 - \lambda_2)\lambda_1' + \lambda_1(\lambda_3 - \lambda_1)\lambda_2'\big]. \tag{4.50b}$$

代入(4.48), (4.25), (4.13)和(4.14),並化簡後,可得

$$\hat{Q}=\frac{1}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\left\{\frac{b_5}{(\lambda_2-\lambda_4)(\lambda_4-\lambda_1)}(\lambda_3+\lambda_4-\lambda_2-\lambda_1)-\frac{\lambda_3-\lambda_2}{\lambda_4-\lambda_1}\right.$$
$$\left.\cdot(\lambda_3+\lambda_4-2\lambda_1)\lambda_1'-\frac{\lambda_3-\lambda_1}{\lambda_4-\lambda_2}(\lambda_3+\lambda_4-2\lambda_2)\lambda_2'\right\}, \tag{4.51a}$$

$$\hat{P}=\frac{1}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\left\{\frac{b_5}{(\lambda_2-\lambda_4)(\lambda_4-\lambda_1)}[\lambda_3\lambda_4-(\lambda_1+\lambda_2)(\lambda_3+\lambda_4)+\lambda_1\lambda_2\right.$$
$$\left.+(\lambda_1^2+\lambda_2^2)]+\lambda_2\frac{\lambda_3-\lambda_2}{\lambda_4-\lambda_1}(\lambda_3+\lambda_4-2\lambda_1)\lambda_1'+\lambda_1\frac{\lambda_3-\lambda_1}{\lambda_4-\lambda_2}(\lambda_3+\lambda_4-2\lambda_2)\lambda_2'\right\} \tag{4.51b}$$

在(4.49)內的$\hat{P}$和$\hat{Q}$,特別是含b_5的項,在阻塞點仍爲singular,故此二階微方仍無法適用於阻塞點附近,且無法連結入射波和反射波,唯有將(4.49)再分解成一階,並重新組合成另一個二階微方後才能達到此目的.

4.3. 分解二階微方

我們採用的方法仍與§4.1和§4.2內所採用者類似,由於在前面所使用的矩陣僅出現在中間步驟裡,故在不慮混淆的情況下,我們仍將(4.49)改寫爲

$$\overline{\eta}'=(B_0+B_1)\overline{\eta}, \tag{4.52}$$

其中

$$\overline{\eta}=\begin{bmatrix}\eta\\ \eta'\end{bmatrix},\quad B_0=\begin{bmatrix}0 & 1\\ -\lambda_1\lambda_2 & \lambda_1+\lambda_2\end{bmatrix},\quad B_1=\begin{bmatrix}0 & 0\\ -\hat{P} & -\hat{Q}\end{bmatrix}. \tag{4.53}$$

由(4.51a,b)我們可知B_1仍僅含較高階的項.

爲分解(4.52),我們如以往需將B_0和B_1轉換爲diagonal,故令

$$\overline{\eta}=S\overline{y}, \tag{4.54}$$

因此

$$\overline{y}'=(C_0+C_1)\overline{y}, \tag{4.55}$$

其中

$$C_0=S^{-1}B_0S, \tag{4.56}$$

$$C_1=S^{-1}B_1S-S^{-1}S'. \tag{4.57}$$

由(4.56),若S的每一行皆由B_0的一個eigenvector組成,則C_0將爲diagonal,且其元素即爲B_0的兩個eigenvalues,後者我們可很容易證明即爲λ_1和λ_2,因此

$$S = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}, \tag{4.58}$$

且

$$C_0 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}. \tag{4.59}$$

將(4.58)和(4.53)代入(4.57)可得

$$C_1 = \frac{1}{\lambda_2 - \lambda_1} \begin{bmatrix} \hat{P} + \lambda_1 \hat{Q} + \lambda_1' & \hat{P} + \lambda_2 \hat{Q} + \lambda_2' \\ -\hat{P} - \lambda_1 \hat{Q} - \lambda_1' & -\hat{P} - \lambda_2 \hat{Q} - \lambda_2' \end{bmatrix}. \tag{4.60}$$

由於C_1尚未能轉換爲diagonal,故需從事另一次轉換,這回令

$$\overline{y} = T\overline{\xi}, \tag{4.61}$$

且

$$T = T_0 + T_1 + \cdots . \tag{4.62}$$

代表一漸近展開.將(4.61)和(4.62)代入(4.55)可得

$$(T_0 + T_1 + \cdots)\overline{\xi}' = [(C_0 + C_1)(T_0 + T_1 + \cdots) - (T_0' + T_1' + \cdots)]\overline{\xi}.$$

若將其結果表爲

$$\overline{\xi}' = (D_0 + D_1 + \cdots)\overline{\xi}, \tag{4.63}$$

則經由展開和比較得到

$$T_0 D_0 = C_0 T_0,$$

$$T_0 D_1 + T_1 D_0 = C_0 T_1 + C_1 T_0 - T_0'.$$

若定$T_0 = I$,則

$$D_0 = C_0 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \tag{4.64}$$

且

$$D_1 = C_0T_1 - T_1C_0 + C_1. \tag{4.65}$$

故 D_0 仍爲 diagonal, 且由 (4.65) 若我們選

$$T_1 = \begin{bmatrix} 0 & T_{11} \\ T_{12} & 0 \end{bmatrix},$$

其中

$$\left.\begin{aligned} T_{11} &= \frac{1}{(\lambda_2-\lambda_1)^2}(\hat{P}+\lambda_2\hat{Q}+\lambda_2'), \\ T_{12} &= \frac{1}{(\lambda_2-\lambda_1)^2}(\hat{P}+\lambda_1\hat{Q}+\lambda_1'), \end{aligned}\right\}$$

則

$$D_1 = \frac{1}{\lambda_2-\lambda_1}\begin{bmatrix} \hat{P}+\lambda_1\hat{Q}+\lambda_1' & 0 \\ 0 & -\hat{P}-\lambda_2\hat{Q}-\lambda_2' \end{bmatrix} \tag{4.66}$$

亦爲 diagonal.

將 (4.64) 和 (4.66) 代入 (4.63), 且令

$$\bar{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix},$$

可得兩個不連結的方程式, 其中一個爲

$$\xi_2' = \left\{\lambda_2 + \frac{1}{\lambda_2-\lambda_1}(-\hat{P}-\lambda_2\hat{Q}-\lambda_2')\right\}\xi_2,$$

或表爲

$$\left[\frac{\partial}{\partial x} - (\lambda_2+\hat{E})\right]\xi_2 = 0, \tag{4.67}$$

其中

$$\hat{E} = \frac{1}{\lambda_2-\lambda_1}(-\hat{P}-\lambda_2\hat{Q}-\lambda_2'). \tag{4.68}$$

接下來我們將 $\bar{\xi}$ 轉換爲 $\bar{\eta}$, 由 (4.54), (4.61) 和 (4.62)

$$\bar{\xi} = (T_0+T_1)^{-1}S^{-1}\bar{\eta}.$$

故經由直接計算可得

$$\xi_2 = \frac{1}{\lambda_2-\lambda_1}\frac{1}{\det(T_0+T_1)}\left[-\lambda_1\eta+\eta'+R_1(\lambda_2\eta-\eta')\right]$$
$$= \frac{1}{\lambda_2-\lambda_1}\frac{1}{\det(T_0+T_1)}\left[\eta'(1-R_1)-\eta(\lambda_1-\lambda_2R_1)\right],$$

其中

$$R_1 = \frac{1}{(\lambda_2-\lambda_1)^2}(-\hat{P}-\lambda_1\hat{Q}-\lambda_1'). \tag{4.69}$$

代入(4.67)後得到

$$\left[\frac{\partial}{\partial x}-(\lambda_2+\hat{E})\right]\frac{1}{\lambda_2-\lambda_1}\frac{1}{\det(T_0+T_1)}\left[\eta'(1-R_1)-\eta(\lambda_1-\lambda_2R_1)\right]=0.$$

上式若忽略高階項相當於原來的二階微方(4.49);但它目前已被factorized,故爲考慮單獨一個k_1分量,我們僅需考慮一階微方

$$\eta'(1-R_1)-\eta(\lambda_1-\lambda_2R_1)=0.$$

或在相同的準確度情況下

$$\eta'-\eta[\lambda_1+(\lambda_1-\lambda_2)R_1]=0. \tag{4.70}$$

由於R_1包含$\hat{P}$和$\hat{Q}$,故由(4.51a,b),上式實際包含b_5此一參數,故(4.4)中的b_5此時須以$k_{x1}=\lambda_1/i$代入才能符合原邊界值問題(此時$\hat{P}$和$\hat{Q}$我們以$\hat{P}_1$和$\hat{Q}_1$來代表,b_5用b_5^1來代表),但(4.70)仍無法適用於阻塞點附近.

在上面導公式的過程裡,我們並未定λ_1是入射波或反射波,故上式可同樣適用於入射波和反射波,若以下我們定λ_1代表入射波,λ_2代表反射波,則代表反射波的一階微方可表爲

$$\eta'-\eta[\lambda_2+(\lambda_2-\lambda_1)R_2]=0, \tag{4.71}$$

其中

$$R_2 = \frac{1}{(\lambda_2-\lambda_1)^2}(-\hat{P}_2-\lambda_2\hat{Q}_2-\lambda_2'), \tag{4.72}$$

且 $\hat{P}_2$ 和 $\hat{Q}_2$ 乃爲(4.51a,b)中的 $\hat{P}$ 和 $\hat{Q}$,將其中的 b_5 以 k_{x2} 代入(表爲 b_5^2)所獲得的結果.

以上所獲得的(4.70)和(4.71)可被重新組合成一個新的二階微方,此微方將可應用在阻塞點附近.

4.4. 組合一階微方爲二階

由(4.70)和(4.71)最直接的一種組合方式爲

$$\left\{\frac{\partial}{\partial x}-[\lambda_2+(\lambda_2-\lambda_1)R_2]\right\}\left\{\frac{\partial}{\partial x}-[\lambda_1+(\lambda_1-\lambda_2)R_1]\right\}\eta=0. \tag{4.73}$$

上式明顯地適用於 λ_1 分量,但由於 λ_1 本身亦爲 x 的函數,故上式將無法適用於 λ_2 分量,若將上式展開可得

$$\eta''-[\lambda_1+\lambda_2+(\lambda_1-\lambda_2)(R_1-R_2)]\eta'+[\lambda_1\lambda_2+(\lambda_1-\lambda_2)(\lambda_2R_1-\lambda_1R_2)-\lambda_1']\eta=0.$$

故很明顯地 λ_1 和 λ_2 在上式中並不對稱,但若我們將上式和

$$\frac{\lambda_1'-\lambda_2'}{\lambda_2-\lambda_1}\eta'-\lambda_1\frac{\lambda_1'-\lambda_2'}{\lambda_2-\lambda_1}\eta=0$$

相加,則可獲得

$$\begin{aligned}\eta''&-\left[\lambda_1+\lambda_2+(\lambda_1-\lambda_2)(R_1-R_2)-\frac{\lambda_1'-\lambda_2'}{\lambda_2-\lambda_1}\right]\eta'\\&+\left[\lambda_1\lambda_2+(\lambda_1-\lambda_2)(\lambda_2R_1-\lambda_1R_2)-\frac{\lambda_2\lambda_1'-\lambda_1\lambda_2'}{\lambda_2-\lambda_1}\right]\eta=0.\end{aligned} \tag{4.74}$$

上式中 λ_1 和 λ_2 出現的形式完全對稱,故(4.74)若能適用於 λ_1 分量,應亦能適用於 λ_2 分量,實際上,若我們由(4.73)開始,將 λ_2 和 λ_1 互換,以及將 R_2 和 R_1 互換,經由相同的步驟可得和(4.74)完全相同的結果.以下我們將證明此一結果不僅適用於入射波和反射波兩者,且能適用在包括阻塞點在內的每一點上.

五. 證明(4.74)在阻塞點爲regular

由四次多項式(2.2)根的解可知λ_1和λ_2具有以下形式:

$$\lambda_1 = M - N, \qquad \lambda_2 = M + N, \tag{5.1}$$

其中N具有$\sqrt{\psi}$的形式, M和ψ皆爲entire functions, 且ψ在阻塞點爲零,故

$$NN' = \frac{1}{2}\psi \tag{5.2}$$

在阻塞點爲analytic, 且N^2, N^4, N^6等皆爲analytic, 因此我們以下將證明(4.74)內的N皆以此種方式存在,且各係數的分母不爲零.

首先由(5.1)

$$\lambda_1 + \lambda_2 = 2M, \qquad \lambda_1\lambda_2 = M^2 - N^2$$

故(4.74)中的主要項皆爲analytic; 以下僅需對次要項加以分析.

由(4.69)和(4.72),

$$-(\lambda_1-\lambda_2)(R_1-R_2) + \frac{\lambda_1'-\lambda_2'}{\lambda_2-\lambda_1} = \frac{1}{\lambda_1-\lambda_2}\left[(\hat{P}_1-\hat{P}_2)+(\lambda_1\hat{Q}_1-\lambda_2\hat{Q}_2)\right], \tag{5.3}$$

$$(\lambda_1-\lambda_2)(\lambda_2R_1-\lambda_1R_2) - \frac{\lambda_2\lambda_1'-\lambda_1\lambda_2'}{\lambda_2-\lambda_1} = \frac{1}{\lambda_1-\lambda_2}\left[-(\lambda_2\hat{P}_1-\lambda_1\hat{P}_2)-\lambda_1\lambda_2(\hat{Q}_1-\hat{Q}_2)\right]. \tag{5.4}$$

由於$\hat{P}_1$和$\hat{P}_2$以及$\hat{Q}_1$和$\hat{Q}_2$的差別僅在於(4.51a,b)中的b_5分別使用k_{x1}和k_{x2}代入(此兩種b_5分別用b_5^1和b_5^2來代表),故

$$\hat{P}_1-\hat{P}_2 = \frac{1}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_2-\lambda_4)(\lambda_4-\lambda_1)}\Big[\lambda_3\lambda_4-(\lambda_1+\lambda_2)(\lambda_3+\lambda_4)$$
$$+\lambda_1\lambda_2+(\lambda_1^2+\lambda_2^2)\Big](b_5^1-b_5^2), \tag{5.5a}$$

$$\hat{Q}_1-\hat{Q}_2 = \frac{1}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_2-\lambda_4)(\lambda_4-\lambda_1)}(\lambda_3+\lambda_4-\lambda_2-\lambda_1)(b_5^1-b_5^2), \tag{5.5b}$$

$$\lambda_2\hat{P}_1-\lambda_1\hat{P}_2 = \frac{1}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_2-\lambda_4)(\lambda_4-\lambda_1)}\Big\{\Big[\lambda_3\lambda_4-(\lambda_1+\lambda_2)(\lambda_3+\lambda_4)$$
$$+\lambda_1\lambda_2+(\lambda_1^2+\lambda_2^2)\Big](\lambda_2b_5^1-\lambda_1b_5^2)+(\lambda_1-\lambda_2)\Big[(\lambda_3+\lambda_4-2\lambda_1)$$
$$(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)\lambda_2\lambda_1'+(\lambda_3+\lambda_4-2\lambda_2)(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)\lambda_1\lambda_2'\Big]\Big\}, \tag{5.5c}$$

$$\lambda_1\hat{Q}_1-\lambda_2\hat{Q}_2=\frac{1}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_2-\lambda_4)(\lambda_4-\lambda_1)}\Big\{(\lambda_3+\lambda_4-\lambda_2-\lambda_1)(\lambda_1b_5^1-\lambda_2b_5^2)$$
$$+(\lambda_1-\lambda_2)\Big[(\lambda_3+\lambda_4-2\lambda_1)(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)\lambda_1'+(\lambda_3+\lambda_4-2\lambda_2)$$
$$\cdot(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)\lambda_2'\Big]\Big\}. \tag{5.5d}$$

再由(4.4)和(2.1)

$$b_5^j=U'\Big[\frac{12}{U}n_0^2\lambda_j-8U\lambda_j^3+18in_0\lambda_j^2-\frac{2i}{U^2}n_0^3$$
$$-\frac{1}{k_y^2-\lambda_j^2}(-\frac{4i}{U^2}n_0^3\lambda_j^2+12in_0\lambda_j^4+12\frac{n_0^2}{U}\lambda_j^3-4U\lambda_j^5)\Big],\qquad j=1,2$$

故

$$b_5^1-b_5^2=U'(\lambda_1-\lambda_2)\Big\{\frac{12}{U}n_0^2-8U(\lambda_1^2+\lambda_2^2)-8U\lambda_1\lambda_2+18in_0(\lambda_1+\lambda_2)$$
$$-\frac{1}{(k_y^2-\lambda_1^2)(k_y^2-\lambda_2^2)}\Big[k_y^2(-\frac{4i}{U^2}n_0^3)(\lambda_1+\lambda_2)+12in_0k_y^2(\lambda_1+\lambda_2)(\lambda_1^2+\lambda_2^2)$$
$$+12k_y^2\frac{n_0^2}{U}(\lambda_1^2+\lambda_2^2)+12k_y^2\frac{n_0^2}{U}\lambda_1\lambda_2-4Uk_y^2(\lambda_1^4+\lambda_2^4)-4Uk_y^2\lambda_1\lambda_2(\lambda_1^2+\lambda_2^2)$$
$$-4Uk_y^2\lambda_1^2\lambda_2^2-12in_0\lambda_1^2\lambda_2^2(\lambda_1+\lambda_2)-12\frac{n_0^2}{U}\lambda_1^2\lambda_2^2+4U\lambda_1^2\lambda_2^2(\lambda_1^2+\lambda_2^2)$$
$$+4U\lambda_1^3\lambda_2^3\Big]\Big\}, \tag{5.6a}$$

$$\lambda_2b_5^1-\lambda_1b_5^2=U'(\lambda_1-\lambda_2)\Big\{-8U\lambda_1\lambda_2(\lambda_1+\lambda_2)+18in_0\lambda_1\lambda_2+\frac{2i}{U^2}n_0^3$$
$$-\frac{1}{(k_y^2-\lambda_1^2)(k_y^2-\lambda_2^2)}\Big[k_y^2(-\frac{4i}{U^2}n_0^3)\lambda_1\lambda_2+12in_0k_y^2\lambda_1\lambda_2(\lambda_1^2+\lambda_2^2)$$
$$+12in_0k_y^2\lambda_1^2\lambda_2^2+12k_y^2\frac{n_0^2}{U}\lambda_1\lambda_2(\lambda_1+\lambda_2)-4Uk_y^2\lambda_1\lambda_2(\lambda_1+\lambda_2)(\lambda_1^2+\lambda_2^2)$$
$$-\frac{4i}{U^2}n_0^3\lambda_1^2\lambda_2^2-12in_0\lambda_1^3\lambda_2^3+4U\lambda_1^3\lambda_2^3(\lambda_1+\lambda_2)\Big]\Big\}, \tag{5.6b}$$

$$
\begin{aligned}
\lambda_1 b_5^1 - \lambda_2 b_5^2 =& U'(\lambda_1-\lambda_2)\Big\{\frac{12}{U}n_0^2(\lambda_1+\lambda_2) - 8U(\lambda_1+\lambda_2)(\lambda_1^2+\lambda_2^2) + 18in_0(\lambda_1^2+\lambda_2^2) \\
&+ 18in_0\lambda_1\lambda_2 - \frac{2i}{U^2}n_0^3 - \frac{1}{(k_y^2-\lambda_1^2)(k_y^2-\lambda_2^2)}\Big[k_y^2(-\frac{4i}{U^2}n_0^3)(\lambda_1^2+\lambda_2^2+\lambda_1\lambda_2) \\
&+ 12in_0k_y^2(\lambda_1^4+\lambda_2^4) + 12in_0k_y^2\lambda_1\lambda_2(\lambda_1^2+\lambda_2^2+\lambda_1\lambda_2) + 12k_y^2\frac{n_0^2}{U}(\lambda_1+\lambda_2)(\lambda_1^2+\lambda_2^2) \\
&- 4Uk_y^2(\lambda_1^3+\lambda_2^3)(\lambda_1^2+\lambda_2^2+\lambda_1\lambda_2) + \frac{4i}{U^2}n_0^3\lambda_1^2\lambda_2^2 - 12in_0\lambda_1^2\lambda_2^2(\lambda_1^2+\lambda_2^2+\lambda_1\lambda_2) \\
&- 12\frac{n_0^2}{U}\lambda_1^2\lambda_2^2(\lambda_1+\lambda_2) + 4U\lambda_1^2\lambda_2^2(\lambda_1+\lambda_2)(\lambda_1^2+\lambda_2^2)\Big]\Big\}, \qquad (5.6c)
\end{aligned}
$$

因此(5.3)和(5.4)的分母$(\lambda_1-\lambda_2)$可抵消掉，且在(5.6a–c)大括號內各項亦爲解析函數，另一方面在(5.5a–d)中剩餘的各項也很明顯地爲解析（因λ_1'和λ_2'的項具有對稱性質，故將(5.1)代入後，所有N（包括N')的奇數乘冪將互相抵消，僅餘NN', N^2, N^4, N^3N'等)，故(4.74)即使在阻塞點亦爲regular，因此它能均勻適用於每一點.

六. 反射現象的解

由於(4.74)的兩獨立解分別代表入射波和反射波,且它能適用在阻塞點附近(包括阻塞點在內),故由(4.74)我們可連結入射波和反射波,並完全獲得反射現象的解. 目前我們仍採用Shyu & Phillips (1990)的解法,此方法是根據Smith (1975)的結果所採用的一種極簡明的導法.

首先我們將(4.74)改寫為

$$\eta'' + [-i(k_{x1} + k_{x2}) + Q]\eta' + [-k_{x1}k_{x2} + P]\eta = 0, \tag{6.1}$$

其中

$$\left.\begin{aligned} Q &= -(\lambda_1 - \lambda_2)(R_1 - R_2) + \frac{\lambda_1' - \lambda_2'}{\lambda_2 - \lambda_1}, \\ P &= (\lambda_1 - \lambda_2)(\lambda_2 R_1 - \lambda_1 R_2) - \frac{\lambda_2\lambda_1' - \lambda_1\lambda_2'}{\lambda_2 - \lambda_1}. \end{aligned}\right\} \tag{6.2}$$

接下來令

$$\eta = v(x)e^{i(k_y y - n_0 t)} \exp\left\{-\frac{1}{2}\int[-i(k_{x1} + k_{x2}) + Q]\,dx\right\}. \tag{6.3}$$

代入(6.1)可得

$$v'' + v(H + G) = 0, \tag{6.4}$$

其中

$$H = \frac{1}{4}(k_{x1} - k_{x2})^2, \tag{6.5}$$

$$G = P + \frac{i}{2}(k_{x1} + k_{x2})Q + \frac{i}{2}(k_{x1}' + k_{x2}'). \tag{6.6}$$

上式在阻塞點仍為regular, 且由Smith (1975)的結果,我們可期待

$$v \approx A_0 A_i(-r) - C_0 A_i'(-r), \tag{6.7}$$

其中

$$\left.\begin{aligned} &\tfrac{2}{3}r^{\frac{3}{2}} = -\textstyle\int_0^x H^{\frac{1}{2}}\,dx, \\ &A_0 = \left(\tfrac{r}{H}\right)^{\frac{1}{4}} \cos\left(-\textstyle\int_0^x \tfrac{1}{2}G/H^{\frac{1}{2}}\,dx\right), \quad C_0 = r^{-\frac{1}{4}}H^{-\frac{1}{4}} \sin\left(-\textstyle\int_0^x \tfrac{1}{2}G/H^{\frac{1}{2}}\,dx\right). \end{aligned}\right\} \tag{6.8}$$

爲明確起見,在此我們定阻塞點的位置爲$x=0$,故在$x>0$區域波浪將無法行進至內部,上式中$Ai(-r)$和$Ai'(-r)\equiv\{dAi(x)/dx\}_{x=-r}$代表Airy function和它的導函數,前者滿足

$$\frac{d^2Ai(-r)}{dr^2}+rAi(-r)=0\,. \tag{6.9}$$

若將(6.7)和(6.8)代入(6.4)可得

$$v''+v(H+G)=A_0''Ai(-r)-C_0''Ai'(-r). \tag{6.10}$$

由於A_0和C_0皆爲緩慢變化的參數,故(6.10)和(6.4)的差異仍和以前一樣可以忽略,且由於A_0, C_0,和Ai在阻塞點爲analytical,因此(6.7)和(6.8)連同(6.3)代表一能均勻適用於每一點的漸近近似解.

由(6.3), (6.7)和(6.8)我們尚無法將入射波和反射波清楚地加以區分,但由於$Ai(-r)$和$Ai'(-r)$在r爲正且較大時,可近似表爲

$$Ai(-r)=\pi^{-\frac{1}{2}}r^{-\frac{1}{4}}\sin\left(\frac{2}{3}r^{\frac{3}{2}}+\frac{1}{4}\pi\right),$$

$$Ai'(-r)=-\pi^{-\frac{1}{2}}r^{\frac{1}{4}}\cos\left(\frac{2}{3}r^{\frac{3}{2}}+\frac{1}{4}\pi\right).$$

故由(6.3), (6.7)和(6.8),亦可得在$x<0$區域

$$\begin{aligned}\eta\approx{}&H^{-\frac{1}{4}}\exp\left[\int_0^x\frac{1}{2}(-Q-iG/H^{\frac{1}{2}})\,dx\right]\exp i\left[\int_0^x k_1\,dx+k_yy-n_0t-\frac{1}{4}\pi\right]\\&+H^{-\frac{1}{4}}\exp\left[\int_0^x\frac{1}{2}(-Q+iG/H^{\frac{1}{2}})\,dx\right]\exp i\left[\int_0^x k_2\,dx+k_yy-n_0t+\frac{1}{4}\pi\right].\end{aligned} \tag{6.11}$$

上式即爲一般的WKBJ solution,由它可清楚地顯示入射波和反射波的相對振幅與相位,但此解無法適用在阻塞點附近.由(6.11)我們可証明入射波和反射波的振幅變化滿足action conservation principle.

七. 証明滿足action conservation principle

在這一節裡我們將証明入射波和反射波的振幅變化除各自能滿足action conservation principle外,兩者的action fluxes亦相等,表action在阻塞點亦守恆.

7.1. 單獨一個波的action flux

由(3.3),(3.4)和(2.31),入射波或反射波若滿足action conservation principle,須

$$\frac{a'}{a} = -\frac{U'}{2}\frac{[\square]}{\square^2}, \tag{3.3}$$

其中

$$[\square] = U\left(1 + \frac{1}{2}\frac{k_x^2}{k^2}\right) + \sigma\frac{k_x^3}{k^4}, \tag{3.4}$$

$$\square = U + \frac{k_x}{2k^2}(gk)^{\frac{1}{2}}. \tag{2.31}$$

我們先考慮入射波,由(6.11),入射波的振幅為

$$a = H^{-\frac{1}{4}} \exp\left[\int_0^x \frac{1}{2}(-Q - iG/H^{\frac{1}{2}})\,dx\right]. \tag{7.1}$$

故

$$a^2 = \frac{2}{k_{x2} - k_{x1}} \exp\left[\int_0^x (-Q - iG/H^{\frac{1}{2}})\,dx\right], \tag{7.1}$$

因此

$$\frac{a'}{a} = \frac{1}{2}(-Q - iG/H^{\frac{1}{2}}) - \frac{1}{2}\frac{k'_{x2} - k'_{x1}}{k_{x2} - k_{x1}}.$$

將(6.2), (6.5)和(6.6)代入後可得

$$\frac{a'}{a} = (\lambda_1 - \lambda_2)R_1.$$

應用(4.69)和(4.51a,b),上式可展開為

$$\frac{a'}{a} = \frac{1}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_1)}(b_5 + b_6), \tag{7.2}$$

其中

$$b_6 = \left[(\lambda_3 + \lambda_4 - 2\lambda_1)(\lambda_1 - \lambda_2) - (\lambda_3 - \lambda_1)(\lambda_4 - \lambda_1)\right]\lambda_1'.$$

由(4.10), (4.4)和(2.30)

$$\begin{aligned}
b_6 =& \lambda_1' \left[(4i\frac{n_0}{U} - 3\lambda_1 - \lambda_2)(\lambda_1 - \lambda_2) - (\frac{-b_1}{\lambda_1\lambda_2} - 4i\frac{n_0}{U}\lambda_1 + 2\lambda_1^2 + \lambda_1\lambda_2)\right] \\
=& \frac{\lambda_1'}{\lambda_1\lambda_2}\left[8i\frac{n_0}{U}\lambda_1^2\lambda_2 - 4i\frac{n_0}{U}\lambda_1\lambda_2^2 + \lambda_1\lambda_2^3 + \lambda_1^2\lambda_2^2 - 5\lambda_1^3\lambda_2 + \frac{g^2k_y^2}{U^4} - \frac{n_0^4}{U^4}\right] \\
=& -\frac{U'}{\lambda_2\square_1}\left[8i\frac{n_0}{U}\lambda_1^2\lambda_2 - 4i\frac{n_0}{U}\lambda_1\lambda_2^2 + \lambda_1\lambda_2^3 + \lambda_1^2\lambda_2^2 - 5\lambda_1^3\lambda_2 + \frac{g^2k_y^2}{U^4} - \frac{n_0^4}{U^4}\right], \qquad (7.3)
\end{aligned}$$

且

$$\begin{aligned}
(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_1) &= \lambda_1\lambda_3\lambda_4 + \lambda_1^3 - \lambda_1^2\lambda_4 - \lambda_1^2\lambda_3 - \lambda_2\lambda_3\lambda_4 - \lambda_1^2\lambda_2 + \lambda_1\lambda_2(\lambda_4 + \lambda_3) \\
&= -4i\frac{n_0^3}{U^3} + 2\frac{b_1}{\lambda_1} - \lambda_1^2\left(4i\frac{n_0}{U} - \lambda_1\right) + \lambda_1^3 \\
&= -4i\frac{n_0^3}{U^3} + \frac{2}{\lambda_1}\left(\frac{g^2k_y^2}{U^4} - \frac{n_0^4}{U^4}\right) + 2\lambda_1^3 - 4i\frac{n_0}{U}\lambda_1^2 \\
&= \frac{2i}{U^4k_{x1}}\left(-2n_0^3Uk_{x1} - g^2k_y^2 + n_0^4 - U^4k_{x1}^4 + 2n_0U^3k_{x1}^3\right) \\
&= \frac{2i}{U^4k_{x1}}\left[(n_0 - Uk_{x1})^2(n_0^2 - U^2k_{x1}^2) - g^2k_y^2\right] \\
&= \frac{2i}{U^4k_{x1}}\left[\frac{g^2(k_y^2 + k_{x1}^2)(n_0 + Uk_{x1})}{n_0 - Uk_{x1}} - g^2k_y^2\right] \\
&= \frac{2i}{U^4k_{x1}}\frac{g^2}{n_0 - Uk_{x1}}\left[2Uk_{x1}k_y^2 + n_0k_{x1}^2 + Uk_{x1}^3\right] \\
&= \frac{2i}{U^4}\frac{g^2}{n_0 - Uk_{x1}}\left[2Uk_y^2 + n_0k_{x1} + Uk_{x1}^2\right]
\end{aligned}$$

由於

$$\begin{aligned}
\square_1 = U + \frac{(gk_1)^{\frac{1}{2}}}{2k_1^2}k_{x1} &= U + \frac{n_0 - UK_{x1}}{2k_1^2}k_{x1} \\
&= \frac{1}{2k_1^2}\left[2U(k_y^2 + k_{x1}^2) + n_0k_{x1} - Uk_{x1}^2\right] \\
&= \frac{1}{2k_1^2}\left[2Uk_y^2 + n_0k_{x1} + Uk_{x1}^2\right],
\end{aligned}$$

故上式最後可表爲

$$(\lambda_1-\lambda_2)(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)=\frac{4ik_1^2}{U^4}\frac{g^2}{n_0-Uk_{x1}}\square_1. \tag{7.4}$$

比較(3.3), (7.2)和(7.4)顯示,若

$$-\frac{1}{2}[\square]_1U'=\frac{U^4(n_0-Uk_{x1})}{4ik_1^2g^2}\square_1(b_5+b_6),$$

則(6.11)可滿足action conservation principle,上式或可改寫爲

$$\frac{4ik_1^2g^2}{U^4}[\square]_1U'+2\square_1(n_0-Uk_{x1})b_5=-2\square_1(n_0-Uk_{x1})b_6, \tag{7.5}$$

故以下我們將証明(7.5)左右兩邊相等.

由(4.4)和(2.1), b_5 可展開爲

$$\begin{aligned}b_5=iU'\Big(&-4\frac{n_0^3}{U^4}\frac{k_{x1}^2}{k_1^2}-2\frac{n_0^3}{U^4}+12\frac{n_0^2}{U^3}\frac{k_{x1}^3}{k_1^2}+12\frac{n_0^2}{U^3}k_{x1}\\&-12\frac{n_0}{U^2}\frac{k_{x1}^4}{k_1^2}-18\frac{n_0}{U^2}k_{x1}^2+4\frac{1}{U}\frac{k_{x1}^5}{k_1^2}+8\frac{1}{U}k_{x1}^3\Big).\end{aligned} \tag{7.6}$$

故將(7.6)和(7.3)代入並應用(2.1)和(2.2),可得(7.5)等號左右兩邊 U' 的係數分別爲

$$\left.\begin{aligned}\text{L.H.}=&12i\frac{n_0^3}{U^2}k_{x1}-36i\frac{n_0^2}{U}k_{x1}^2+36in_0k_{x1}^3-12iUk_{x1}^4-2i\frac{n_0}{U^4}k_{x1}g^2\\&+2i\frac{1}{U^3}k_{x1}^2g^2,\\\text{R.H.}=&8i\frac{n_0^2}{U}k_{x1}k_{x2}-10in_0k_{x1}^2k_{x2}-16i\frac{n_0^2}{U}k_{x1}^2+26in_0k_{x1}^3+2iUk_{x1}^2k_{x2}^2\\&+2iUk_{x1}^3k_{x2}-10iUk_{x1}^4-2in_0k_{x1}k_{x2}^2+2i\frac{n_0}{U^4}\frac{1}{k_{x2}}(n_0^4-g^2k_y^2)\\&-2i\frac{1}{U^3}\frac{k_{x1}}{k_{x2}}(n_0^4-g^2k_y^2).\end{aligned}\right\} \tag{7.7}$$

另一方面,因

$$g^2k_{x1}k_{x2} = \begin{cases} \frac{k_{x1}}{k_{x2}}(g^2k_{x2}^2); \\ \\ \frac{k_{x2}}{k_{x1}}(g^2k_{x1}^2). \end{cases}$$

故由(2.2)

$$\begin{aligned} &\frac{k_{x1}}{k_{x2}}(n_0^4 - 4n_0^3Uk_{x2} + 6n_0^2U^2k_{k2}^2 - 4n_0U^3k_{x2}^3 + U^4k_{x2}^4 - g^2k_y^2) \\ =&\frac{k_{x2}}{k_{x1}}(n_0^4 - 4n_0^3Uk_{x1} + 6n_0^2U^2k_{k1}^2 - 4n_0U^3k_{x1}^3 + U^4k_{x1}^4 - g^2k_y^2). \end{aligned}$$

化簡後可得

$$n_0^4 - g^2k_y^2 = U^4k_{x1}^2k_{x2}^2 + \frac{k_{x1}k_{x2}}{k_{x1}+k_{x2}}(4n_0^3U - 4n_0U^3k_{x1}k_{x2}). \tag{7.8}$$

將(7.8)代入(7.7)後可得

$$\begin{aligned} \text{R.H.} =&8i\frac{n_0^2}{U}k_{x1}k_{x2} - 8in_0k_{x1}^2k_{x2} - 16i\frac{n_0^2}{U}k_{x1}^2 + 26in_0k_{x1}^3 + 2iUk_{x1}^2k_{x2}^2 - 10iUk_{x1}^4 \\ &- 2in_0k_{x1}k_{x2}^2 + \frac{1}{k_{x1}+k_{x2}}\left(2i\frac{n_0}{U^4}k_{x1} - 2i\frac{1}{U^3}k_{x1}^2\right)(4n_0^3U - 4n_0U^3k_{x1}k_{x2}). \end{aligned}$$

或

$$\begin{aligned} \text{R.H.} \times (k_{x1}+k_{x2})^2 =& -8i\frac{n_0^2}{U}k_{x1}^2k_{x2}^2 + 16in_0k_{x1}^3k_{x2}^2 - 12in_0k_{x1}^2k_{x2}^3 - 16i\frac{n_0^2}{U}k_{x1}^4 \\ &- 32i\frac{n_0^2}{U}k_{x1}^3k_{x2} + 26in_0k_{x1}^5 + 52in_0k_{x1}^4k_{x2} - 8iUk_{x1}^4k_{x2}^2 \\ &+ 2iUk_{x1}^2k_{x2}^4 + 4iUk_{x1}^3k_{x2}^3 - 10iUk_{x1}^6 - 20iUk_{x1}^5k_{x2} \\ &- 2in_0k_{x1}k_{x2}^4 - 8i\frac{n_0^3}{U^2}k_{x1}^2(k_{x1}+k_{x2}). \end{aligned} \tag{7.9}$$

同理

$$
\begin{aligned}
\text{L.H.} \times (k_{x1}+k_{x2}) =& 12i\frac{n_0^3}{U^2}k_{x1}(k_{x1}+k_{x2}) - 36i\frac{n_0^2}{U}k_{x1}^2(k_{x1}+k_{x2}) + 36in_0k_{x1}^3(k_{x1}+k_{x2}) \\
& - 12iUk_{x1}^4(k_{x1}+k_{x2}) - 2i\frac{n_0}{U^4}\Big(n_0^4 - 4n_0^3Uk_{x1} + 6n_0^2U^2k_{x1}^2 - 4n_0U^3k_{x1}^3 \\
& + U^4k_{x1}^4 - g^2k_y^2\Big) - 2i\frac{n_0}{U^4}\frac{k_{x1}}{k_{x2}}\Big(n_0^4 - 4n_0^3Uk_{x2} + 6n_0^2U^2k_{x2}^2 - 4n_0U^3k_{x2}^3 \\
& + U^4k_{x2}^4 - g^2k_y^2\Big) + 2i\frac{1}{U^3}\Big(n_0^4 - 4n_0^3Uk_{x1} + 6n_0^2U^2k_{x1}^2 - 4n_0U^3k_{x1}^3 \\
& + U^4k_{x1}^4 - g^2k_y^2\Big)(k_{x1}+k_{x2}) \\
=& -16i\frac{n_0^2}{U}k_{x1}^3 - 24i\frac{n_0^2}{U}k_{x1}^2k_{x2} + 26in_0k_{x1}^4 + 28in_0k_{x1}^3k_{x2} - 10iUk_{x1}^5 \\
& - 10iUk_{x1}^4k_{x2} + 16i\frac{n_0^4}{U^3}k_{x1} + 8i\frac{n_0^2}{U}k_{x1}k_{x2}^2 - 2in_0k_{x1}k_{x2}^3 - 8i\frac{n_0^3}{U^2}k_{x1}^2 \\
& - 8i\frac{n_0^3}{U^2}k_{x1}k_{x2} - 2i\frac{n_0}{U^4}\left[U^4k_{x1}^2k_{x2}^2 + \frac{k_{x1}k_{x2}}{k_{x1}+k_{x2}}(4n_0^3U - 4n_0U^3k_{x1}k_{x2})\right] \\
& - 2i\frac{n_0}{U^4}\frac{k_{x1}}{k_{x2}}\left[U^4k_{x1}^2k_{x2}^2 + \frac{k_{x1}k_{x2}}{k_{x1}+k_{x2}}(4n_0^3U - 4n_0U^3k_{x1}k_{x2})\right] \\
& + \frac{2i}{U^3}\left[U^4k_{x1}^2k_{x2}^2 + \frac{k_{x1}k_{x2}}{k_{x1}+k_{x2}}(4n_0^3U - 4n_0U^3k_{x1}k_{x2})\right](k_{x1}+k_{x2}).
\end{aligned}
$$

上式再乘以 $k_{x1}+k_{x2}$ 可得和(7.9)完全相同的結果,故証明入射波的振幅變化確滿足action conservation principle. 同理我們亦可証明(6.11)中的反射波的解亦滿足action conservation principle.

7.2. 比較入射波和反射波的action fluxes

前面的(3.3)式實際代表入射波和反射波的action fluxes在 x 方向爲定值(在 y 方向亦爲定值),接下來我們將証明此兩fluxes實際相等.

由於我們已証明單獨一個波的action flux在各處皆相同,故要計算此flux,我們可選擇在一較方便的地點進行,而由(6.11)或(7.1),在阻塞點上action fluxes

$$
(U + C_{gx})\frac{E}{\sigma} = \begin{cases} \left(U + \frac{1}{2}\frac{\sigma_1}{k_1^2}k_{x1}\right)\frac{\sigma_1}{k_1}H^{-\frac{1}{2}} & (k_1 \text{ component}); \\ \left(U + \frac{1}{2}\frac{\sigma_2}{k_2^2}k_{x2}\right)\frac{\sigma_2}{k_2}H^{-\frac{1}{2}} & (k_2 \text{ component}). \end{cases} \tag{7.10}
$$

此乃爲最簡單者,故以下我們將計算(7.10)等號右邊的量.

我們仍由計算k_1 component開始,將(6.5)代入後可得

$$(E + C_{gx1})\frac{E_1}{\sigma_1} = \frac{\sigma_1}{k_1^3}(Uk_1^2 + \frac{1}{2}\sigma_1 k_{x1})\frac{2}{k_{x2} - k_{x1}}. \tag{7.11}$$

在阻塞點上,$k_{x2} = k_{x1}$,故上式中的分母爲零,此與WKBJ solution的振幅在阻塞點爲無窮大此一事實相吻合,但因在同一點上,$U + C_{gx1} = 0$,故上式仍具有一有限的極限值,此一極限值可由L'Hôpital's rule獲得.

令

$$f(x) = Uk_1^2 + \frac{1}{2}\sigma_1 k_{x1}$$

$$= U(k_{x1}^2 + k_y^2) + \frac{1}{2}(n_0 - Uk_{x1})k_{x1}, \tag{7.12}$$

$$g(x) = k_{x2} - k_{x1}. \tag{7.13}$$

故由L'Hôpital's rule

$$\lim_{x\to 0^-}\frac{f(x)}{g(x)} = \lim_{x\to 0^-}\frac{f'(x)}{g'(x)}. \tag{7.14}$$

利用(2.30), (2.31)和(2.1)可以獲得

$$\frac{f'(x)}{g'(x)} = \frac{\mathrm{Nu}}{\mathrm{De}},$$

其中

$$\begin{aligned}\mathrm{Nu} =& \frac{1}{2}n_0Uk_y^4k_{x2} - \frac{1}{4}n_0Uk_y^2k_{x1}^3 - \frac{1}{8}n_0Uk_{x1}^3k_{x2}^2 - \frac{1}{8}n_0^2k_{x1}^3k_{x2} - \frac{3}{4}U^2k_y^2k_{x1}^4 - \frac{3}{8}U^2k_{x1}^4k_{x2}^2\\ &+ U^2k_y^6 + \frac{1}{2}U^2k_y^4k_{x2}^2 - \frac{3}{8}n_0Uk_{x1}^4k_{x2},\\ \mathrm{De} =& UK_y^4(k_{x1} - k_{x2}) - UK_y^2(k_{x2}^3 - k_{x1}^3) - \frac{1}{2}n_0k_{x1}k_{x2}(k_{x2}^2 - k_{x1}^2) - \frac{1}{2}Uk_y^2k_{x1}k_{x2}(k_{x1} - k_{x2})\\ &- \frac{1}{2}Uk_{x1}^2k_{x2}^2(k_{x2} - k_{x1}).\end{aligned}$$

再將(5.1)代入可得

$$
\begin{aligned}
\mathrm{Nu} =&\Big(\frac{1}{2}n_0Uk_y^4M-\frac{1}{4}n_0Uk_y^2M^3-\frac{1}{2}n_0UM^5-\frac{1}{8}n_0^2M^4-\frac{3}{4}U^2k_y^2M^4-\frac{3}{8}U^2M^6+U^2k_y^6\\
&+\frac{1}{2}U^2k_y^4M^2\Big)+N\Big(\frac{1}{2}n_0Uk_y^4+\frac{3}{4}n_0Uk_y^2M^2+\frac{5}{4}n_0UM^4+\frac{1}{4}n_0^2M^3+3U^2k_y^2M^3\\
&+\frac{3}{4}U^2M^5+U^2k_y^4M\Big)+N^2(\cdots)+\cdots, \qquad (7.15a)\\
\mathrm{De} =&N(-2UK_y^4-5UK_y^2M^2-2n_0M^3-UM^4)-N^3(3Uk_y^2-2n_0M-2UM^2)\\
&-UN^5. \qquad (7.15b)
\end{aligned}
$$

由於在阻塞點

$$U+C_g=0,$$

故在此點上

$$UM^2+n_0M+2Uk_y^2=0,$$

或

$$U=\frac{-n_0M}{M^2+2k_y^2},$$

因此(7.15a,b)可化簡爲

$$\mathrm{Nu}=-\frac{1}{4}N\frac{n_0^2M}{(M^2+2k_y^2)^2}(M^2-2k_y^2)^2(M^2+k_y^2)+N^2(\cdots)+\cdots,$$

$$\mathrm{De}=-N\frac{n_0M}{M^2+2k_y^2}(M^2-2k_y^2)(M^2+k_y^2)-N^3(7Uk_y^2)-UN^5.$$

故由(7.14)我們可獲得

$$\lim_{x\to 0-}\frac{f(x)}{g(x)}=\frac{1}{4}n_0\frac{M^2-2k_y^2}{M^2+2k_y^2}. \qquad (7.16)$$

上式明顯地爲一有限值.

上式似乎顯示,若在阻塞點上$M^2=2k_y^2$,則入射波的action flux將爲零,但此並非事實,因爲當$M^2=2k_y^2$時,在(7.15a)和(7.15b)內N的係數兩者皆爲零,不

僅如此,且(7.15a)內N^2的係數亦爲零,故兩者皆僅餘N^3以及更高次的項,因此(7.15a)和(7.15b)的商仍具有一不爲零且爲有限的極限值(實際等於$3n_0/28$).

另一方面,由於反射波的action flux

$$(E+C_{gx2})\frac{E_2}{\sigma_2}=\frac{\sigma_2}{k_2^3}(Uk_2^2+\frac{1}{2}\sigma_2k_{x2})\frac{2}{k_{x2}-k_{x1}}.$$

故它與入射波的差別僅在於(7.15a)內所含$N, N^3, N^5, \ldots$的項符號應相反,但(7.15b)內的符號不變,因此証明入射波和反射波的action fluxes在x方向具有相同的量(雖方向相反).

八. 結論

當波斜行在一穩定二度空間流上，應用較複雜的處理，仍獲得可均勻適用於每一點有關水面位移變化的二階常微分方程，由此微分方程我們直接求得同樣可均勻適用於每一點的漸近近似解，此解在遠離阻塞點區域(實際僅需離開不到1/2波長的距離)被進一步展開爲WKBJ解，由後者我們証明目前的解可滿足action conservation principle，且入射波的action flux和反射波量相等.

以上的結果清楚地顯示波被流阻擋後，確被反射爲不同的波長，且它明確地決定入射波和反射波的相對振幅與相位，此外它也提供了在阻塞點附近水面位移的詳細變化. 另外要強調的是，目前的結果甚至對波浪場和流場爲完全三度空間的狀況也有幫助；因在此種狀況下，當應用在阻塞點附近，(6.7)中屬於迅速變化的項應不變(若x軸仍與當地流向平行)，而其餘作緩慢變化的項在離阻塞點不到一個波長的距離內變化也很小，故當我們利用以往的ray solution求得入射波和反射波的解至離阻塞點相當近的範圍內後，應用目前的(6.11)可求得當地的$-\int_0^x Q/2\,dx$和$-\int_0^x G/2H^{\frac{1}{2}}\,dx$，甚或它們兩者在當地的變化率，亦即當地的$-Q/2$和$-G/2H^{\frac{1}{2}}$(此時有關$Q$和$G$的表式須做一些修改，但目前我們實際不需考慮它們的詳細內容)，將其結果代入(6.3)，(6.7)和(6.8)後即可獲得一彎曲caustic上及其附近水面位移的變化.

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