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台中港近岸波浪預報模式研究

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波被流反射現象之解析解及其應用於數值計算

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當行進於一大尺度流上的波之群速度在某一點和流速相平衡時，波將被阻塞於此點，並發生反射現象。在本研究中我們首先將 shyu and Phillips (1990)的理論延伸至深水波「斜行」於一穩定、二度空間及非旋性流上情況，在此情況下我們由拉普拉斯方程式及運動和動力邊界條件導出波被流反射現象之均勻漸近解及 WKBJ 解，這些解，除其中一些小項的表式外，其形式和 shyu and Phillips 的解相同。另外，我們應用離散關係式和波作用守恒方程式更証明，即使在一個由三度空間和不穩定之非旋性流所產生的彎曲和不穩定焦線附近，且波為中間水深波時，此兩種解的形式仍與 shyu 和 Phillips (1990)者相同，僅其小項此時需以數值方法求解，其解法我們經由一些數值試驗加以解說，其結果顯示，當僅由波作用守恒原理估計焦線附近的反射波將產生極嚴重的誤差放大現象時，目前的解法由於有一個明白形式的解可資應用，故絕大部份的誤差放大現象可加以避免。

Analytical solutions of the wave reflection phenomenon by currents and their application to numerical computations

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Abstract

Surface waves superimposed upon a larger-scale flow are blocked and reflected at the points where the group velocities balance the convection by the larger-scale flow. In this study, we first extend the theory of Shyu and Phillips (1990) to the situation when short deep-water gravity waves propagate *obliquely* upon a steady, two-dimensional, and irrotational current. In this case, the uniform asymptotic and the WKBJ solutions of the wave reflection phenomenon by the current at a straight caustic are derived from the Laplace equation and the kinematical and dynamic boundary conditions. These solutions, except the expressions for the minor terms, take the same forms as those derived by Shyu and Phillips. Furthermore, from considerations of the dispersion relation and the action conservation equation, we demonstrate that even for a curved and/or unsteady caustic induced by a three-dimensional and/or unsteady irrotational current, and for waves in an intermediate-depth region, the solutions in the vicinity of the caustic still take the same forms as those in Shyu & Phillips (1990), although the values of the minor terms can now be calculated only numerically. The algorithm for these calculations is illustrated through numerical tests, which indicate that while the error magnification phenomenon is very serious in the estimates of the reflected wave in the vicinity of the caustic from a consideration of the action conservation principle only, this phenomenon can for the most part be avoided by the present algorithm by taking advantage of the explicit forms of the present solutions.

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1. Introduction

The influence of long waves and currents on short waves has interested people for more than half a century (see, for example, Unna 1942; Johnson 1947), which recently received even more attention owing to the application of remote sensing techniques. An important feature of the wave-current interaction is that the wavelength and amplitude of the short waves both undergo changes during propagation. The rates of changes for infinitesimal short gravity waves riding on a long wave with a low amplitude have been calculated by Longuet-Higgins & Stewart (1960) using a rigorous perturbation approach, while a more general consideration was given by Bretherton & Garrett (1968) in terms of the action conservation principle. More recent studies in this respect include the detailed analyses of the modulations of linear short waves on a long wave of finite-amplitude (Phillips 1981; Longuet-Higgins 1987; Henyey *et al.* 1988), the prediction of the evolution of weakly nonlinear short waves on finite-amplitude long waves (Zhang & Melville 1990; Naciri & Mei 1992), and the experimental and numerical study of the interaction between a nonlinear wave and a current possessing an arbitrary distribution of vorticity (Thomas 1990).

On the other hand, the wave-current interactions can also lead to reflection of waves at the points where the group velocities balance the convection by the current. This process can cause waves to change even more drastically because the wavelength of the reflected wave is significantly different from that of the incident wave at each point. In spite of its importance, an asymptotic theory which is valid at the turning point and can take into account the reflection phenomenon was not available until the publication of Smith (1975) for gravity waves.

For gravity-capillary waves, a similar phenomenon but with opposite properties can also occur in the interaction between short wavelets and long waves or between short wavelets and currents, as suggested first by Phillips (1981). The uniformly valid solution of this capillary blockage phenomenon in a two-dimensional field has been given by Shyu & Phillips (1990), and more recently by Trulsen & Mei (1993) who even derived a solution near a triple turning point at which the two kinds of reflection points coalesce.

Though the effect of surface tension is disregarded, Smith's (1975) theory is valid for a more general case than the one considered by Shyu & Phillips (1990). According to Smith, the uniform solution for gravity waves superimposed upon a slowly varying irrotational current, that can be three-dimensional and unsteady, can be expressed as

$$u = \{A\,\mathrm{Ai}(\rho) + iC\,\mathrm{Ai}'(\rho)\}\exp(is) \tag{1.1}$$

with

$$\left.\begin{aligned} \rho = -\left[\tfrac{3}{4}(\chi^{(1)} - \chi^{(2)})\right]^{\frac{2}{3}}, \quad s = \tfrac{1}{2}(\chi^{(1)} + \chi^{(2)}), \\ A = \pi^{\frac{1}{2}}(-\rho)^{\frac{1}{4}}(a^{(1)} + a^{(2)}), \quad C = \pi^{\frac{1}{2}}(-\rho)^{-\frac{1}{4}}(a^{(1)} - a^{(2)}), \end{aligned}\right\} \tag{1.2}$$

where u denotes any instantaneous property of the waves, $\mathrm{Ai}(\rho)$ and $\mathrm{Ai}'(\rho)$ represent respectively the Airy function and its derivative, and $a^{(1)}, a^{(2)}, \chi^{(1)}$ and $\chi^{(2)}$ correspond to the local amplitudes and phases of the incident and reflected waves satisfying various transport equations that apply to a ray solution. The above unified formulae were also summarized by Peregrine & Smith (1979).

In (1.2), the requirement that C remains finite and analytic at the turning point implies that $a^{(1)}$ and $a^{(2)}$ have equal singularities there. This, together with the action conservation equation 'enables us to conclude that the flux of wave action normal to the caustic carried by the incident and by the reflected waves are equal and opposite' (Smith 1975). Thus the amplitude of the reflected wave relative to that of the incident wave can be determined in theory. However, in the immediate vicinity of the caustic, the amplitude of the incident wave itself cannot be solved accurately from the action conservation equation owing to the singularities at the caustic, and in the region at a certain distance from the caustic, the difference between the amplitudes of the incident and reflected waves becomes significant, but as shown in the following, when the action conservation principle is applied, this difference is featured of an even smaller quantity divided by a small one so that an error magnification phenomenon may seriously influence the estimates of this quantity.

On the other hand, the approach adopted by Shyu & Phillips (1990) can directly lead to explicit solutions for ρ, s, A and C even without using the action conservation principle (though the satisfaction of the principle by these solutions can easily be proved). This method first reduces the free-surface boundary conditions to a third-order ordinary differential equation (or directly to a second-order one if the surface tension term is neglected), which was then factored and reduced further to a uniformly valid second-order ordinary differential equation. The uniform asymptotic solution of this equation, essentially devoid of any singularities in the expressions, can readily be obtained by using a treatment suggested by the results of Smith (1975).

Shyu & Phillips' (1990) analysis was still restricted to the case in which the current and the waves vary in only one horizontal direction. In this paper, we will show that this technique can be extended to the case when short waves *obliquely* propagate upon the same current. This task is not trivial, but more importantly, the analysis and the results will lead us to hope that even for a curved and/or unsteady caustic induced by a three-dimensional and/or unsteady irrotational flow, and for waves in an intermediate-depth region, the solutions in the vicinity of the caustic, except the expressions for the minor terms, should take the same forms as those in Shyu & Phillips (1990) and those in the case when waves obliquely propagate upon a two-dimensional current (in which the caustic is still straight). This suggestion will be verified by considerations of the dispersion relation and the action conservation equation which themselves have been demonstrated by Smith (1975) to be valid in the vicinity of the caustic in a general situation.

In these solutions, the minor terms are responsible for the difference between the amplitudes of the incident and reflected waves in the vicinity of the caustic, but in the case of a curved and/or unsteady caustic and in the case when the waves are in an intermediate-depth region, these minor terms can be calculated only numerically. Therefore it is of practical importance to develop a numerical algorithm to estimate these minor terms in a general case. This will be achieved through simulation tests in which the strategies to avoid the error magnification phenomenon by taking advantage of the explicit forms of the present solutions will be developed.

2. Formulation of the problem in terms of ordinary differential equations

For a two-dimensional wave-current field including the effects of surface tension, Shyu & Phillips (1990) first derived a third-order ordinary differential equation in the surface displacement η of the short waves. This equation was then decomposed into a second-order ordinary differential equation in which all the coefficients are regular at the turning point. That this approach was successful was because in this case, expansion of the dispersion relation

$$n = (g'k + \gamma k^3)^{\frac{1}{2}} + Uk \tag{2.1}$$

takes the form

$$k^3 - \frac{U^2}{\gamma}k^2 + \frac{g' + 2nU}{\gamma}k - \frac{n^2}{\gamma} = 0,$$

which is also a third-order polynomial equation in the local wavenumber k and the coefficients coincide exactly with the leading terms in the differential equation governing η (see (2.18), Shyu & Phillips (1990)). In (2.1), g' is the effective gravitational acceleration suggested by Phillips (1981), n the observed frequency of the wave, γ the ratio of surface tension to water density, and U the local current velocity.

In case that the waves propagate obliquely upon the current and the effects of surface tension are neglected, the dispersion relation becomes

$$n = \left[g(k_x^2 + k_y^2)^{\frac{1}{2}}\right]^{\frac{1}{2}} + Uk_x, \tag{2.2}$$

where k_x and k_y represent respectively the x and y components of the wavenumber vector $\mathbf{k}$ while the x-axis is chosen to be exactly opposite to the current (so that in (2.2) U is always negative) and the z-axis in the vertically upward direction. An expansion yields

$$U^4k_x^4 - 4nU^3k_x^3 + (6n^2U^2 - g^2)k_x^2 - 4n^3Uk_x + (n^4 - g^2k_y^2) = 0, \tag{2.3}$$

which is a quartic equation. From it and from Shyu & Phillips' (1990) analysis, it is anticipated that a fourth-order ordinary differential equation is desired in order to eventually obtain a second-order equation by decomposition that can describe the reflection phenomenon as well as be uniformly valid.

To obtain such equation, certain results of the ray theory will be utilized. The latter is invalid in the vicinity of the turning point, but as long as we can prove that the resulting differential equation is regular at this point (meaning that the singularities inherent in the earlier ray solutions of the incident and reflected waves are completely offset from this equation), this equation can virtually be applicable everywhere, including the turning point.

For a slowly varying wavetrain, the distribution of $\mathbf{k}$ is irrotational (see e.g. Phillips 1977) so that

$$\frac{\partial k_y}{\partial x} - \frac{\partial k_x}{\partial y} = 0. \tag{2.4}$$

On the other hand, since the current velocity is independent of y, we have

$$\frac{\partial k_x}{\partial y} = 0 \qquad \text{and} \qquad \frac{\partial k_y}{\partial y} = 0. \tag{2.5}$$

From (2.4) and (2.5) it immediately follows that

$$k_y = \text{constant}$$

everywhere. Next, from the kinematic conservation equation,

$$\frac{\partial \mathbf{k}}{\partial t} + \nabla n = 0.$$

Thus, if the current field is steady, we also have

$$n = \text{constant} = n_0, \quad \text{say}$$

everywhere. Therefore, the ray solutions of the surface displacement η and the velocity potential ϕ of a single wave component can now be written as

$$\eta = a(x) \exp\left[i \int k_x(x)\, dx\right] \exp i(k_y y - n_0 t), \tag{2.6}$$

and

$$\phi = A(x) \exp\left[i \int k_x(x)\, dx + \int_0^z l(x,z)\, dz\right] \exp i(k_y y - n_0 t), \tag{2.7}$$

where $a(x), A(x)$ and $k_x(x)$ vary slowly in the x-direction and $l(x,z)$ varies slowly in both $x-$ and $z-$directions. For the sake of definiteness, we here take $z=0$ to be the mean water level. Notice that in the present case the roles played by k_y and n_0 in the solutions are quite similar.

The relation between k_x and l can be deduced from the 3D Laplace equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\phi=0. \tag{2.8}$$

Substitution of (2.7) into (2.8) yields

$$-k_x^2+i\frac{dk_x}{dx}+2ik_x\frac{1}{A}\frac{dA}{dx}-k_y^2+l^2+\frac{\partial l}{\partial z}=0 \qquad \text{at} \qquad z=0. \tag{2.9}$$

In (2.9), the higher-order term $(1/A)(d^2A/dx^2)$ has been neglected. In fact, to achieve the same level of accuracy as the ray solution, all derivatives with order higher than 1 of the slowly varying parameters can be neglected. In (2.9), since both l and $\partial l/\partial z$ are unknown, another equation is required. To obtain this equation, we may reconsider the simpler case when the waves are exactly opposite the current. In this case, (2.8) reduces to

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\phi=0 \tag{2.10}$$

or

$$\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial z}\right)\phi=0.$$

In addition, from the deep-water boundary condition and the fact that the phases of oscillation of both incident and reflected waves increase in the positive x-direction, it is clear that the present solution should satisfy

$$\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial z}\right)\phi=0 \tag{2.11}$$

only, otherwise ϕ will grow exponentially as $z\rightarrow-\infty$. Substituting (2.7) into (2.10) and (2.11) and setting $k_y=0$ and $k_x=k$, we have at the free surface

$$-k^2+i\frac{dk}{dx}+2ik\frac{1}{A}\frac{dA}{dx}+l^2+\frac{\partial l}{\partial z}=0 \tag{2.12}$$

and

$$ik + \frac{1}{A}\frac{dA}{dx} = il \tag{2.13}$$

respectively. Squaring both sides of (2.13), neglecting the higher order term, and then subtracting the result from (2.12), we obtain

$$\left.\frac{\partial l}{\partial z}\right|_{z=0} = -i\frac{dk}{dx}.$$

The above relation involves only smaller terms so that when the waves obliquely propagate upon a two-dimensional current, the small curvature of the wave crests induced in the present case will impose even smaller modification upon this relation, which is certainly negligible within the present approximation. Therefore, for the present case,

$$\left.\frac{\partial l}{\partial z}\right|_{z=0} = -i\frac{\partial k}{\partial x'}, \tag{2.14}$$

where $k = (k_x^2 + k_y^2)^{1/2}$ represents the magnitude of $\mathbf{k}$ and x' is the coordinate in the direction of $\mathbf{k}$. This expression in the present coordinate system can be written as

$$\left.\frac{\partial l}{\partial z}\right|_{z=0} = -i\frac{k_x^2}{k^2}\frac{dk_x}{dx}, \tag{2.15}$$

because k_y = constant. Substitution in (2.9) yields

$$l^2\big|_{z=0} = k^2 - 2ik_x\frac{1}{A}\frac{dA}{dx} - i(1 - \frac{k_x^2}{k^2})\frac{dk_x}{dx},$$

and its square root is

$$l\big|_{z=0} = k - i\frac{k_x}{k}\frac{A'}{A} - \frac{i}{2}(1 - \frac{k_x^2}{k^2})\frac{k_x'}{k} \tag{2.16}$$

consistently. (In the following discussion we shall use a prime to indicate differentiation with respect to x in certain circumstances.)

From (2.7) and (2.16), one can obtain at $z = 0$

$$\frac{\partial \phi}{\partial x} = i\frac{k_x}{k}[1 - ic_0\frac{A'}{A} + ic_1 k_x'] \frac{\partial \phi}{\partial z}, \tag{2.17a}$$

$$\frac{\partial^2 \phi}{\partial x^2} = i\frac{k_x}{k}[1 - ic_0\frac{A'}{A} + ic_2 k_x'] \frac{\partial^2 \phi}{\partial x \partial z}, \tag{2.17b}$$

$$\frac{\partial^3 \phi}{\partial x^3} = i\frac{k_x}{k}[1 - ic_0\frac{A'}{A} + ic_3 k_x'] \frac{\partial^3 \phi}{\partial x^2 \partial z}, \tag{2.17c}$$

$$\frac{\partial^4 \phi}{\partial x^4} = i\frac{k_x}{k}[1 - ic_0\frac{A'}{A} + ic_4 k_x'] \frac{\partial^4 \phi}{\partial x^3 \partial z}, \tag{2.17d}$$

where

$$\left.\begin{aligned}
c_0 &= \frac{1}{k_x}(1 - \frac{k_x^2}{k^2}), \\
c_1 &= \frac{1}{2k^2}(1 - \frac{k_x^2}{k^2}), \\
c_2 &= -\frac{1}{k_x^2}(1 - \frac{3}{2}\frac{k_x^2}{k^2} + \frac{1}{2}\frac{k_x^4}{k^4}), \\
c_3 &= -\frac{1}{k_x^2}(2 - \frac{5}{2}\frac{k_x^2}{k^2} + \frac{1}{2}\frac{k_x^4}{k^4}), \\
c_4 &= -\frac{1}{k_x^2}(3 - \frac{7}{2}\frac{k_x^2}{k^2} + \frac{1}{2}\frac{k_x^4}{k^4}).
\end{aligned}\right\} \tag{2.17e}$$

These will later be applied to combine the two surface boundary conditions into one equation. Note that when $k_y = 0$, (2.17a) reduces to (2.11).

Since the x-axis is taken in the direction exactly opposite the

current and the latter itself is steady and two-dimensional, the expressions for the approximate kinematic and dynamical free-surface conditions in the present case should take exactly the same form as those given in Shyu & Phillips (1990), which are

$$-in_0\eta + U\eta' + \eta U' = \frac{\partial \phi}{\partial z} \qquad \text{at} \qquad z = 0, \tag{2.18}$$

$$-in_0\phi + g\eta + U\frac{\partial \phi}{\partial x} = 0 \qquad \text{at} \qquad z = 0. \tag{2.19}$$

From (2.18) we obtain

$$(-in_0 + 2U')\eta' + U\eta'' = \frac{\partial^2\phi}{\partial x\partial z}, \tag{2.20a}$$

$$(-in_0 + 3U')\eta'' + U\eta''' = \frac{\partial^3\phi}{\partial x^2\partial z}, \tag{2.20b}$$

$$(-in_0 + 4U')\eta''' + U\eta^{IV} = \frac{\partial^4\phi}{\partial x^3\partial z}, \tag{2.20c}$$

and from (2.19)

$$(-in_0 + U')\frac{\partial\phi}{\partial x} + g\eta' + U\frac{\partial^2\phi}{\partial x^2} = 0, \tag{2.21a}$$

$$(-in_0 + 2U')\frac{\partial^2\phi}{\partial x^2} + g\eta'' + U\frac{\partial^3\phi}{\partial x^3} = 0, \tag{2.21b}$$

$$(-in_0 + 3U')\frac{\partial^3\phi}{\partial x^3} + g\eta''' + U\frac{\partial^4\phi}{\partial x^4} = 0. \tag{2.21c}$$

These equations can be combined into one equation in η by virtue of (2.17). Although there are many (actually infinite) ways to achieve this, according to Shyu & Phillips (1990), it will be more useful to derive a fourth-order ordinary differential equation with the coefficients coinciding with those in (2.3). Therefore, we first substitute (2.17c,d) and (2.20b,c) into (2.21c), obtaining

$$U^2\eta^{IV} + \eta'''\left[-i(2n_0U + \frac{gk}{k_x}) + 7UU' + c_0\frac{gk}{k_x}\frac{A'}{A} + (c_3 - c_4)n_0Uk_x' - c_4\frac{gk}{k_x}k_x'\right]$$
$$+ \eta''\left[-n_0^2 - in_0^2(c_3 - c_4)k_x' - 6in_0U'\right] = 0. \tag{2.22}$$

Next, substitution of (2.17b,c) and (2.20a,b) into (2.21b) yields

$$U^2\eta''' + \eta''\left[-i(2n_0U + \frac{gk}{k_x}) + 5UU' + c_0\frac{gk}{k_x}\frac{A'}{A} + (c_2 - c_3)n_0Uk_x' - c_3\frac{gk}{k_x}k_x'\right]$$
$$+ \eta'\left[-n_0^2 - in_0^2(c_2 - c_3)k_x' - 4in_0U'\right] = 0.$$

Multiplying it by $-2in_0/U$ and igk/U^2k_x separately, we have

$$-2in_0U\eta''' - \eta''\Big[\frac{2n_0}{U}(2n_0U + \frac{gk}{k_x}) + 10in_0U' + c_0\frac{2in_0}{U}\frac{gk}{k_x}\frac{A'}{A} + 2i(c_2 - c_3)n_0^2k_x'$$

$$- c_3\frac{2in_0}{U}\frac{gk}{k_x}k_x'\Big] - \eta'\Big[-\frac{2in_0^3}{U} + (c_2 - c_3)\frac{2n_0^3}{U}k_x' + 8n_0^2\frac{U'}{U}\Big] = 0 \qquad (2.23)$$

and

$$i\frac{gk}{k_x}\eta''' + \eta''\Big[\frac{gk}{U^2k_x}(2n_0U + \frac{gk}{k_x}) + 5i\frac{gk}{k_x}\frac{U'}{U} + i\frac{c_0}{U^2}(\frac{gk}{k_x})^2\frac{A'}{A} + i(c_2 - c_3)\frac{n_0}{U}\frac{gk}{k_x}k_x'$$

$$- i\frac{c_3}{U^2}(\frac{gk}{k_x})^2k_x'\Big] + \eta'\Big[-i\frac{n_0^2}{U^2}\frac{gk}{k_x} + (c_2 - c_3)\frac{n_0^2}{U^2}\frac{gk}{k_x}k_x' + 4\frac{n_0}{U^2}\frac{gk}{k_x}U'\Big] = 0 \qquad (2.24)$$

respectively. Also, substituting (2.17a,b), (2.18) and (2.20a) into (2.21a) and multiplying the result by $-n_0^2/U$, we obtain

$$-n_0^2\eta'' + \eta'\Big[i\frac{n_0^2}{U^2}(2n_0U + \frac{gk}{k_x}) - 3n_0^2\frac{U'}{U} - c_0\frac{n_0^2}{U^2}\frac{gk}{k_x}\frac{A'}{A} - (c_1 - c_2)\frac{n_0^3}{U}k_x'$$

$$+ c_2\frac{n_0^2}{U^2}\frac{gk}{k_x}k_x'\Big] + \eta\Big[\frac{n_0^4}{U^2} + i(c_1 - c_2)\frac{n_0^4}{U^2}k_x' + 2i\frac{n_0^3}{U^2}U'\Big] = 0. \qquad (2.25)$$

Since from (2.17e),

$$c_1 - c_2 = c_2 - c_3 = c_3 - c_4 = \frac{k_y^2}{k^2k_x^2}, \qquad (2.26)$$

the sum of (2.22)–(2.25) can be written as

$$U^2\eta^{IV} + \eta'''\Big[-4in_0U + 7UU' + c_0\frac{gk}{k_x}\frac{A'}{A} + (c_3 - c_4)n_0Uk_x' - c_4\frac{gk}{k_x}k_x'\Big]$$

$$+ \eta''\Big\{-6n_0^2 + (\frac{gk}{Uk_x})^2 - (16in_0 - 5i\frac{gk}{Uk_x})U' + ic_0\frac{gk}{Uk_x}(\frac{gk}{Uk_x} - 2n_0)\frac{A'}{A}$$

$$+ \Big[-3i(c_2 - c_3)n_0^2 - ic_3(\frac{gk}{Uk_x})^2 + i(c_2 + c_3)n_0\frac{gk}{Uk_x}\Big]k_x'\Big\}$$

$$+ \eta'\Big[\frac{4in_0^3}{U} - 11n_0^2\frac{U'}{U} + 4\frac{n_0}{U}\frac{gk}{Uk_x}U' - c_0\frac{n_0^2}{U}\frac{gk}{Uk_x}\frac{A'}{A} - 3(c_2 - c_3)\frac{n_0^3}{U}k_x'$$

$$+ (2c_2 - c_3)\frac{n_0^2}{U}\frac{gk}{Uk_x}k_x'\Big] + \eta\Big[\frac{n_0^4}{U^2} + i(c_1 - c_2)\frac{n_0^4}{U^2}k_x' + 2i\frac{n_0^3}{U^2}U'\Big] = 0. \qquad (2.27)$$

The dominant terms of (2.27) indeed agree with (2.3) if the operator $\partial/\partial x$ is replaced by ik_x at certain places. The minor terms in (2.27) can further be reduced and written in terms of U'. To achieve this goal, the following relations should be applied:

$$k_x' = -\frac{k_x}{V}U', \tag{2.28}$$

$$V = U + C_{gx} = U + \frac{1}{2}(gk)^{\frac{1}{2}}\frac{k_x}{k^2}, \tag{2.29}$$

$$A = -i\frac{(gk)^{\frac{1}{2}}}{k}a, \tag{2.30}$$

$$\frac{a'}{a} = \frac{A'}{A} + \frac{k_x k_x'}{2k^2}, \tag{2.31}$$

which can be derived straightforwardly from the dispersion relation (2.2) and the boundary condition (2.18) as well as (2.16). In (2.29), C_{gx} represents the x component of the group velocity of short waves. We also note that in (2.30), all terms containing a', k_x', and U' are negligible, because as the amplitude, they should be attributed to the next order term of the asymptotic expansion of the solution.

Using (2.28)–(2.31), replacing η'' with

$$\left[2ik_x\frac{A'}{A} + ik_x'(1 + \frac{k_x^2}{k^2}) - k_x^2\right]\eta$$

when necessary, and neglecting the higher order terms, (2.27) may finally be reduced to

$$U^2\eta^{IV} + \eta'''\left[-4in_0U\right] + \eta''\left[-6n_0^2 + \frac{g^2}{U^2}\right] + \eta'\left[4i\frac{n_0^3}{U}\right]$$
$$+ \eta\left[\frac{n_0^4}{U^2} - \frac{g^2k_y^2}{U^2} + iU'(-6\frac{gkk_x}{U} + 2\frac{(gk)^{\frac{3}{2}}}{U^2} + 4\frac{g^{\frac{3}{2}}k_x^2}{U^2k^{\frac{1}{2}}})\right] = 0. \tag{2.32}$$

Since in deriving (2.32), only a single wave component is under consideration and k_x cannot be eliminated from (2.32) through any further transfer of terms, it is apparent that this equation can truly describe only one individual component (though from the dominant terms it seems to have four independent solutions corresponding to the k_{x1}, k_{x2}, k_{x3} and k_{x4} components in figure 1) and be singular at the turning point if this component is either the k_1 or the k_2 component. However, this equation will later be decomposed into a first-order differential equation, which either for the incident or for the reflected wave is again singular at the turning point, but a combination of them can result in a uniformly valid second-order ordinary differential equation.

3. The equation coupling the incident and reflected waves

The technique for decomposing a higher-order equation into a lower-order one in a general asymptotic analysis was first given by Turrittin (1952) (also see Wasow 1985). For the special case to decompose (2.32), one may refer to Shyu & Phillips (1990). Since each time the procedure can decrease the order of equation only by one, this procedure must be conducted three times. The result is

$$\eta' - \eta[ik_{x1} + iR_1] = 0, \tag{3.1}$$

where

$$R_1 = \frac{1}{k_{x1} - k_{x2}}(\hat{P}_1 + ik_{x1}\hat{Q}_1 + ik'_{x1}), \tag{3.2}$$

$$\begin{aligned}\hat{P}_1 =& \frac{1}{(k_{x3} - k_{x1})(k_{x3} - k_{x2})}\left\{\frac{-b_1}{(k_{x2} - k_{x4})(k_{x4} - k_{x1})}\Big[k_{x3}k_{x4} - (k_{x1} + k_{x2})(k_{x3} + k_{x4})\right.\\ &+ k_{x1}k_{x2} + (k_{x1}^2 + k_{x2}^2)\Big] + i\frac{k_{x3} - k_{x2}}{k_{x4} - k_{x1}}(k_{x3} + k_{x4} - 2k_{x1})k_{x2}k'_{x1}\\ &\left. + i\frac{k_{x3} - k_{x1}}{k_{x4} - k_{x2}}(k_{x3} + k_{x4} - 2k_{x2})k_{x1}k'_{x2}\right\},\end{aligned} \tag{3.3}$$

$$\begin{aligned}\hat{Q}_1 =& \frac{1}{(k_{x3} - k_{x1})(k_{x3} - k_{x2})}\left\{\frac{ib_1}{(k_{x2} - k_{x4})(k_{x4} - k_{x1})}(k_{x3} + k_{x4} - k_{x2} - k_{x1}) - \frac{k_{x3} - k_{x2}}{k_{x4} - k_{x1}}\right.\\ &\left.\cdot (k_{x3} + k_{x4} - 2k_{x1})k'_{x1} - \frac{k_{x3} - k_{x1}}{k_{x4} - k_{x2}}(k_{x3} + k_{x4} - 2k_{x2})k'_{x2}\right\},\end{aligned} \tag{3.4}$$

and

$$b_1 = iU'\left[6\frac{gk_1k_{x1}}{U^3} - 2\frac{(gk_1)^{\frac{3}{2}}}{U^4} - 4\frac{g^{\frac{3}{2}}k_{x1}^2}{U^4k_1^{\frac{1}{2}}}\right]. \tag{3.5}$$

To obtain (3.1), the parameter k_x in (2.32) has been fixed as the wavenumber component k_{x1} of the incident wave such that (3.1), including its major and minor terms, can truly describe the incident wave in the regions far from the turning point. The corresponding equation for the reflected wave can directly be obtained from interchange of k_{x1} and k_{x2} in (3.1)–(3.4) and from replacement of k_1 and k_{x1} by k_2 and k_{x2} respectively in (3.5), giving

$$\eta' - \eta[ik_{x2} + iR_2] = 0 \tag{3.6}$$

with

$$R_2 = \frac{1}{k_{x2} - k_{x1}}(\hat{P}_2 + ik_{x2}\hat{Q}_2 + ik'_{x2}), \tag{3.7}$$

where $\hat{P}_2$ and $\hat{Q}_2$ take the same forms as (3.3) and (3.4) except that b_1 is replaced by

$$b_2 = iU'\Big[6\frac{gk_2k_{x2}}{U^3} - 2\frac{(gk_2)^{\frac{3}{2}}}{U^4} - 4\frac{g^{\frac{3}{2}}k_{x2}^2}{U^4k_2^{\frac{1}{2}}}\Big]. \tag{3.8}$$

Both (3.1) and (3.6) are singular at the turning point where $k_{x1} = k_{x2}$ (see figure 1), which is not unexpected as the reflection phenomenon cannot be described by a first-order differential equation. Nevertheless, a combination of them into a second-order equation can couple the incident wave with the reflected wave and in the meantime cancel out the singularities from the equation. Note that during the decomposition we have already obtained a second-order equation before (3.1) was reached, but this equation cannot describe the incident and reflected waves simultaneously, therefore is singular at the turning point.

Intuitively, we may combine (3.1) with (3.6) as

$$\left\{\frac{\partial}{\partial x} - i[k_{x2} + R_2]\right\}\left\{\frac{\partial}{\partial x} - i[k_{x1} + R_1]\right\}\eta = 0, \tag{3.9}$$

but it can describe only the k_1 component, since the coefficient k_{x1} in (3.9) is not constant (the differentiation of the smaller term R_1 with respect to x is however negligible). An expansion of (3.9) yields

$$\eta'' - \big[ik_{x1} + ik_{x2} + i(R_1 + R_2)\big]\eta' - \big[k_{x1}k_{x2} + (k_{x2}R_1 + k_{x1}R_2) + ik'_{x1}\big]\eta = 0, \tag{3.10}$$

which is obviously not symmetric with respect to k_{x1} and k_{x2}. To solve this problem, we add, from (3.1), neglecting the minor term in it,

$$\frac{k'_{x1} - k'_{x2}}{k_{x2} - k_{x1}}\eta' - ik_{x1}\frac{k'_{x1} - k'_{x2}}{k_{x2} - k_{x1}}\eta = 0 \tag{3.11}$$

to (3.10), resulting in

$$\eta'' + [-i(k_{x1} + k_{x2}) + Q]\eta' + [-k_{x1}k_{x2} + P]\eta = 0, \tag{3.12}$$

where

$$\left.\begin{aligned} P &= -(k_{x2}R_1 + k_{x1}R_2) - i\frac{k_{x2}k'_{x1} - k_{x1}k'_{x2}}{k_{x2} - k_{x1}}, \\ Q &= -i(R_1 + R_2) + \frac{k'_{x1} - k'_{x2}}{k_{x2} - k_{x1}}. \end{aligned}\right\} \tag{3.13}$$

Since (3.10) and (3.11) can both be fulfilled by the k_1 component within the present approximation, and on the other hand, (3.12) together with (3.13) is symmetric with respect to k_{x1} and k_{x2}, it is obvious that the two independent solutions of (3.12) will correspond to the k_1 and k_2 components. In the following section, we shall prove that the singularities in (3.1) and (3.6) can completely be cancelled out from (3.12) so that as a multiple-scale asymptotic approximation, this equation can virtually be valid everywhere including the turning point.

4. Proof of regularity

The sufficient condition for (3.12) being regular at the turning point is that the coefficients in (3.12) are all regular at this point. Since k_{x1} and k_{x2} are a pair of the solutions of the quartic equation (2.3), they will represent two branches of a doublevalued function with the branch point at the turning point. Thus, they can be divided into two parts:

$$k_{x1} = M - N, \qquad k_{x2} = M + N, \tag{4.1}$$

where N and $-N$ are two branches of a doublevalued function, say $\psi^{1/2}$, which equals zero at the turning point, and M and ψ are both regular at this point. From the above,

$$N^2 = \psi, \qquad NN' = \frac{1}{2}\psi', \tag{4.2}$$

so that N^2, NN', N^4, N^3N', etc. are all regular at the turning point. Therefore we shall later prove that when (4.1) are substituted in (3.12) and (3.13), only this kind of terms and the terms not containing N can survive cancellation. Also the term $k_{x2} - k_{x1}$ in the denominators of P and Q in (3.13) can be eliminated.

First, from (4.1),

$$k_{x1} + k_{x2} = 2M, \qquad k_{x1}k_{x2} = M^2 - N^2.$$

Therefore the dominant terms in (3.12) are obviously regular. Next, from (3.13), (3.2) and (3.7), we have

$$\left.\begin{aligned} P &= \frac{-1}{k_{x1} - k_{x2}}\left[(k_{x2}\hat{P}_1 - k_{x1}\hat{P}_2) + ik_{x1}k_{x2}(\hat{Q}_1 - \hat{Q}_2)\right], \\ Q &= \frac{-1}{k_{x1} - k_{x2}}\left[i(\hat{P}_1 - \hat{P}_2) - (k_{x1}\hat{Q}_1 - k_{x2}\hat{Q}_2)\right]. \end{aligned}\right\} \tag{4.3}$$

Recall that the only difference between $\hat{P}_1$ and $\hat{P}_2$ or between $\hat{Q}_1$ and $\hat{Q}_2$ is that the former involves b_1 while the latter involves b_2 in (3.3) or (3.4). Thus we have

$$\hat{P}_1 - \hat{P}_2 = \frac{1}{L}\left[-k_{x3}k_{x4} + (k_{x1} + k_{x2})(k_{x3} + k_{x4}) - k_{x1}k_{x2} - (k_{x1}^2 + k_{x2}^2)\right](b_1 - b_2), \tag{4.4a}$$

$$\hat{Q}_1 - \hat{Q}_2 = \frac{i}{L}(k_{x3} + k_{x4} - k_{x2} - k_{x1})(b_1 - b_2), \tag{4.4b}$$

$$k_{x2}\hat{P}_1 - k_{x1}\hat{P}_2 = \frac{1}{L}\Big\{ \Big[-k_{x3}k_{x4} + (k_{x1} + k_{x2})(k_{x3} + k_{x4}) - k_{x1}k_{x2} - (k_{x1}^2 + k_{x2}^2)\Big](k_{x2}b_1 - k_{x1}b_2)$$
$$+ i(k_{x1} - k_{x2})\Big[(k_{x3} + k_{x4} - 2k_{x1})(k_{x3} - k_{x2})(k_{x4} - k_{x2})k_{x2}k'_{x1}$$
$$+ (k_{x3} + k_{x4} - 2k_{x2})(k_{x3} - k_{x1})(k_{x4} - k_{x1})k_{x1}k'_{x2}\Big]\Big\}, \tag{4.4c}$$

and

$$k_{x1}\hat{Q}_1 - k_{x2}\hat{Q}_2 = \frac{1}{L}\Big\{ i(k_{x3} + k_{x4} - k_{x2} - k_{x1})(k_{x1}b_1 - k_{x2}b_2) + (k_{x1} - k_{x2})\Big[(k_{x3} + k_{x4} - 2k_{x1})$$
$$\cdot (k_{x3} - k_{x2})(k_{x4} - k_{x2})k'_{x1} + (k_{x3} + k_{x4} - 2k_{x2})(k_{x3} - k_{x1})(k_{x4} - k_{x1})k'_{x2}\Big]\Big\}, \tag{4.4d}$$

where

$$L = (k_{x3} - k_{x1})(k_{x3} - k_{x2})(k_{x2} - k_{x4})(k_{x4} - k_{x1}). \tag{4.4e}$$

From (3.5) and (3.8) and by using (2.2), we also have

$$b_1 - b_2 = i(k_{x1} - k_{x2})\frac{U'}{U^2}\Big\{ \frac{12}{U}n_0^2 + 8UT - 18n_0(k_{x1} + k_{x2}) - \frac{1}{k_1^2k_2^2}\Big[4k_y^2\frac{n_0^3}{U^2}(k_{x1} + k_{x2})$$
$$+ 12n_0k_y^2S - 12k_y^2\frac{n_0^2}{U}T - 4Uk_y^2(k_{x1}^4 + k_{x2}^4) - 4Uk_y^2k_{x1}k_{x2}T$$
$$+ 12n_0k_{x1}^2k_{x2}^2(k_{x1} + k_{x2} - \frac{n_0}{U}) - 4Uk_{x1}^2k_{x2}^2T\Big]\Big\}, \tag{4.5a}$$

$$k_{x2}b_1 - k_{x1}b_2 = i(k_{x1} - k_{x2})\frac{U'}{U^2}\Big\{ 8Uk_{x1}k_{x2}(k_{x1} + k_{x2}) - 18n_0k_{x1}k_{x2} + 2\frac{n_0^3}{U^2} - \frac{1}{k_1^2k_2^2}$$
$$\cdot \Big[4k_y^2\frac{n_0^3}{U^2}k_{x1}k_{x2} + 12n_0k_y^2k_{x1}k_{x2}T - 12k_y^2\frac{n_0^2}{U}k_{x1}k_{x2}(k_{x1} + k_{x2}) - 4Uk_y^2k_{x1}k_{x2}S$$
$$- 4\frac{n_0^3}{U^2}k_{x1}^2k_{x2}^2 + 12n_0k_{x1}^3k_{x2}^3 - 4Uk_{x1}^3k_{x2}^3(k_{x1} + k_{x2})\Big]\Big\}, \tag{4.5b}$$

except d_0 are slowly varying according to the explanation following (6.3). On substitution χ_1 into (6.3) we have

$$\left.\begin{aligned} k_{x1} &= [d_1 + 2d_2 x + O(x^2)] - [\psi_1 x + O(x^2)]^{\frac{1}{2}} \\ h_y k_{y1} &= \left[\frac{\partial d_0}{\partial y} + \frac{\partial d_1}{\partial y} x + O(x^2)\right] - \frac{1}{3}\frac{\partial \psi_1}{\partial y}\left[(-\psi_1)^{-1}(-x)^3 + O(x^4)\right]^{\frac{1}{2}} \end{aligned}\right\} \quad (6.12)$$

The other branch of (6.11) and (6.12) then provide the phase and wave-number components of the reflected wave:

$$\chi_2 = [d_0 + d_1 x + d_2 x^2 + \cdots] - \frac{2}{3}\left[(-\psi_1)^{\frac{1}{3}}(-x) + O(x^2)\right]^{\frac{3}{2}} \quad (6.13)$$

$$\left.\begin{aligned} k_{x2} &= [d_1 + 2d_2 x + O(x^2)] + [\psi_1 x + O(x^2)]^{\frac{1}{2}} \\ h_y k_{y2} &= \left[\frac{\partial d_0}{\partial y} + \frac{\partial d_1}{\partial y} x + O(x^2)\right] + \frac{1}{3}\frac{\partial \psi_1}{\partial y}\left[(-\psi_1)^{-1}(-x)^3 + O(x^4)\right]^{\frac{1}{2}} \end{aligned}\right\} \quad (6.14)$$

The above phase functions not only lead to the right forms of k_{x1} and k_{x2}, but also ensure that $\nabla \times \mathbf{k} = 0$ for both waves. However, the values of the series coefficients d_1, d_2, ψ_1, etc. can be determined only from the dispersion relation. We notice in passing that for a curved caustic, even though k_{y1} is unequal to k_{y2} when $x \neq 0$ (if $\partial\psi_1/\partial y \neq 0$), their difference is vanishingly small and proportional to $x^{3/2}$ only (also with a very small coefficient $(2/3\psi_1)\partial\psi_1/\partial y$) when the caustic is approached. A similar situation also occurs to the observed frequencies n_1 and n_2 for an unsteady caustic. These can benefit the numerical computations of the reflected wave significantly as illustrated in the next section.

From (6.11) and (6.13) it is immediately clear that $\chi_1 + \chi_2$ in (6.5) is regular at the caustic. On the other hand, from (6.12) and (6.14), we have in the vicinity of caustic

$$H = \frac{1}{4}(k_{x2} - k_{x1})^2 = \psi_1 x + O(x^2) \quad (6.15)$$

Consequently

$$-\int_0^x H^{\frac{1}{2}}\, dx = \frac{2}{3}(-\psi_1)^{\frac{1}{2}}(-x)^{\frac{3}{2}}[1 + O(x)] \quad (6.16)$$

After crossing out the denominator $k_{x1} - k_{x2}$ of the smaller coefficients P and Q in (3.12), they can be calculated straightforwardly by using (4.3), (4.4) and (4.5), but in order to completely avoid singularity at the turning point, it is necessary to substitute (4.1) into (4.4c) and (4.4d) to eliminate k'_{x1} and k'_{x2} in favour of M' and NN'. Both M' and NN' are regular at the turning point as mentioned before, and since at this point,

$$U + C_{gx1} = U + C_{gx2} = 0,$$

it is not difficult to obtain from (2.28) and (2.2)

$$NN' = \frac{n_0 - U_0 M_0}{n_0 + 2U_0 M_0} \frac{M_0^2}{U_0} U', \tag{4.6}$$

$$M' = \left[-\frac{8}{3} \frac{M_0^2}{n_0 + 2U_0 M_0} + \frac{2}{3} \frac{M_0}{U_0} \frac{n_0^2 - U_0^2 M_0^2}{(n_0 + 2U_0 M_0)^2} \right] U', \tag{4.7}$$

at this point, where U_0 and M_0 are the values of U and M at the same point. Hence the calculations of singularities can now be avoided everywhere.

5. Solutions of reflection phenomenon

The equation (3.12) in terms of the parameters k_{x1}, k_{x2}, P and Q takes exactly the same form as that given by Shyu & Phillips (1990) so that its uniformly valid asymptotic solution can similarly be derived using the treatment described in Shyu & Phillips (1990), as suggested by the results of Smith (1975).

Using this procedure, we obtain

$$\eta \approx v(x) \exp\left\{-\frac{1}{2}\int_0^x [-i(k_{x1}+k_{x2})+Q]\, dx\right\} e^{i(k_y y - n_0 t)} \tag{5.1}$$

with

$$v(x) = A_0 \mathrm{Ai}(-r) - C_0 \mathrm{Ai}'(-r), \tag{5.2}$$

where $\mathrm{Ai}'(-r) = \{d\mathrm{Ai}(x)/dx\}_{x=-r}$ and

$$\left.\begin{aligned} \tfrac{2}{3} r^{\frac{3}{2}} &= -\textstyle\int_0^x H^{\frac{1}{2}}\, dx, \\ A_0 = \left(\tfrac{r}{H}\right)^{\frac{1}{4}} \cos\left(-\textstyle\int_0^x \tfrac{1}{2} G/H^{\frac{1}{2}}\, dx\right), &\quad C_0 = r^{-\frac{1}{4}} H^{-\frac{1}{4}} \sin\left(-\textstyle\int_0^x \tfrac{1}{2} G/H^{\frac{1}{2}}\, dx\right), \end{aligned}\right\} \tag{5.3}$$

in which

$$H = \frac{1}{4}(k_{x2} - k_{x1})^2, \tag{5.4}$$

$$G = P + \frac{i}{2}(k_{x1}+k_{x2})Q + \frac{i}{2}(k'_{x1}+k'_{x2}). \tag{5.5}$$

For the sake of definiteness, we have taken $x = 0$ to be the turning point and assumed that $H > 0$ for $x < 0$, corresponding to a situation in which the reflected wave is shorter and its group velocity smaller than the incident wave (see figure 1 and recall that $C_g \equiv \partial\sigma/\partial k$).

In the above solution, the parameters A_0, C_0, H, G and Q are all regular and slowly varying at the turning point; it possesses a more explicit form than those in (1.1) and (1.2). The implications of this fact, as they relate to the extension of the solutions to a curved and/or unsteady caustic, will be discussed in the last section.

In the present case, A_0 and C_0 in (5.3) can be calculated straightforwardly and everywhere; at the turning point where $H = 0$ and in the immediate vicinity of this point, the one-term Taylor-series approximations of A_0 and C_0 may be employed. From (4.1) we have in the vicinity of the turning point

$$H = \frac{1}{4}(k_{x2} - k_{x1})^2 = N^2 \approx \psi_1 x, \quad \text{say,} \tag{5.6}$$

where $\psi_1 = d\psi/dx|_{x=0}$ and from (4.2) and (4.6) it immediately follows that

$$\psi_1 = 2\frac{n_0 - U_0 M_0}{n_0 + 2U_0 M_0}\frac{M_0^2}{U_0}U'. \tag{5.7}$$

Consequently,

$$-\int_0^x H^{\frac{1}{2}}\,dx \approx \frac{2}{3}(-\psi_1)^{\frac{1}{2}}(-x)^{\frac{3}{2}}, \tag{5.8}$$

$$-\int_0^x \frac{1}{2}G/H^{\frac{1}{2}}\,dx \approx G_0(-\psi_1)^{-\frac{1}{2}}(-x)^{\frac{1}{2}}, \tag{5.9}$$

where $G_0 = G(x = 0)$. Substituting (5.8) in (5.3), we obtain

$$r \approx (-\psi_1)^{\frac{1}{3}}(-x) \tag{5.10}$$

in this region. Thus even both r and H as well as the integral $-\int_0^x \frac{1}{2}G/H^{\frac{1}{2}}\,dx$ go to zero as $x \to 0$, we have

$$\left.\begin{aligned} A_0 &= (-\psi_1)^{-\frac{1}{6}}, \\ C_0 &= G_0(-\psi_1)^{-\frac{5}{6}}, \end{aligned}\right\} \tag{5.11}$$

at the turning point, which are finite. Since (5.11) represent the first term of the Taylor-series expansions about $x = 0$, the above results show that A_0 and C_0 are indeed regular at the turning point.

At points away from the turning point, $\mathrm{Ai}(-r)$ and $\mathrm{Ai}'(-r)$ can be replaced by their asymptotic approximations, which for r large and positive are

$$\mathrm{Ai}(-r) \approx \pi^{-\frac{1}{2}}r^{-\frac{1}{4}}\sin\left(\frac{2}{3}r^{\frac{3}{2}} + \frac{1}{4}\pi\right), \tag{5.12}$$

$$\mathrm{Ai}'(-r) \approx -\pi^{-\frac{1}{2}}r^{\frac{1}{4}}\cos\left(\frac{2}{3}r^{\frac{3}{2}} + \frac{1}{4}\pi\right). \tag{5.13}$$

(For r large and negative, both $\mathrm{Ai}(-r)$ and $\mathrm{Ai}'(-r)$ have a decreasing exponential behavior; therefore (5.1) and (5.2) indeed represent the acceptable solution of (3.12).) Thus from (5.1), (5.2), (5.3) and the above approximations, we have

$$\eta \approx H^{-\frac{1}{4}} \exp\left[\int_0^x \frac{1}{2}(-Q - iG/H^{\frac{1}{2}})\,dx\right] \exp i\left[\int_0^x k_{x1}\,dx + k_y y - n_0 t - \frac{1}{4}\pi\right]$$
$$+ H^{-\frac{1}{4}} \exp\left[\int_0^x \frac{1}{2}(-Q + iG/H^{\frac{1}{2}})\,dx\right] \exp i\left[\int_0^x k_{x2}\,dx + k_y y - n_0 t + \frac{1}{4}\pi\right] \quad (5.14)$$

for $x \ll 0$. This solution represents the WKBJ approximation; it obviously fails at the turning point where $H = 0$, but can nevertheless indicate the existence of the incident and reflected waves, as well as show their relative amplitudes and phases (an irrelevant constant common factor was neglected from (5.14)). From (5.14), we have the local amplitudes

$$a = \begin{cases} H^{-1/4} \exp\left[\int_0^x \frac{1}{2}(-Q - iG/H^{\frac{1}{2}})\,dx\right] & (k_1 \text{ component}); \\ H^{-1/4} \exp\left[\int_0^x \frac{1}{2}(-Q + iG/H^{\frac{1}{2}})\,dx\right] & (k_2 \text{ component}). \end{cases} \quad (5.15)$$

(notice that G is pure imaginary while H and Q are real), which have been proved to satisfy the action conservation principle.

6. Extensions to general cases

The above analyses provide explicit solutions for the special case when deep-water gravity waves obliquely propagating upon a steady, two-dimensional irrotational flow. These solutions will certainly become invalid when the water is of intermediate depth and the underlying larger-scale irrotational flow is completely three-dimensional and/or unsteady. However, following the same procedures as those in sections 3 and 5, we may still reach similar results for the general cases except that the smaller parameters G and Q must be determined from the action conservation equation, which has previously been justified in many other works (e.g. Smith, 1975). Since these results provide a far more detailed account of the variations of the amplitudes, a feasible scheme for numerical calculation of the wave field near the caustic, including the reflected wave, can subsequently be developed in the next section.

When the larger-scale flows are not uni-directional in the (x, y)–plane, the caustics are unlikely to be straight. Thus it is necessary and convenient to derive the solutions in a set of orthogonal curvilinear co-ordinates, in which the x co-ordinate is measured perpendicular to the caustic. Therefore we define all the lines $x =$ constant are parallel curves while $x = 0$ corresponds to the caustic. Thus the scale factor h_x in the x direction is independent of the position, and on the other hand, if at the caustic we set the scale factor in the y direction $h_y = 1$, the variation of h_y in the x direction has the simple relation

$$h_y = 1 - \frac{x}{R(y, t)} \tag{6.1}$$

where R is the radius of curvature of the caustic which is large compared with the wavelength. This co-ordinate system will certainly produce singular points of the differential equations at certain positions far away from the caustic, but since it is the present purpose to determine the reflected wave from the incident wave in the vicinity of the caustic, these singular points can be avoided in the present analysis.

The above co-ordinate system can also be applied to an unsteady flow field. In such case, the co-ordinate lines will move with the caustic so that the position with fixed coordinates may vary slowly from time to time in a fixed reference frame. This variation, however, will have no influence on the following analysis, because the major concern in this analysis is only about the variations in the x direction.

In this curvilinear co-ordinate system, the WKBJ solution of each wave component can still be written as

$$\eta = a(x, y, t)e^{i\chi(x,y,t)} \tag{6.2}$$

from which the x and y components of the wavenumber and the frequency observed in this moving frame are

$$k_x = \frac{\partial \chi}{\partial x}, \quad k_y = \frac{1}{h_y}\frac{\partial \chi}{\partial y}, \quad n = -\frac{\partial \chi}{\partial t}, \tag{6.3}$$

where the dependence of a, k_x, k_y and n on x, y and t are expected to be slow except that in the vicinity of the caustic, the variations of a and k_x with x will be rapid owing to the singularities at the caustic. Notice that the variations of h_y with x, y, t are also slow in view of (6.1).

From (6.2) and (6.3), equations (3.1) and (3.6) immediately follow with

$$\left.\begin{aligned} R_1 &= -i\frac{a_1'}{a_1}, \\ R_2 &= -i\frac{a_2'}{a_2}. \end{aligned}\right\} \tag{6.4}$$

The values of a_1'/a_1 and a_2'/a_2 cannot be determined without a consideration of the dynamics, which in the present problem is described by the action conservation principle. Next, following the same procedure as that in section 3, we again obtain (3.12) together with (3.13). Thus if we choose

$$\eta = v(x, y, t)\exp\left\{\frac{i}{2}(\chi_1 + \chi_2) - \int_0^x \frac{Q}{2}\,dx\right\} \tag{6.5}$$

which equivalent to (5.1), we can similarly achieve

$$v(x, y, t) = A_0\mathrm{Ai}(-r) - C_0\mathrm{Ai}'(-r) \tag{6.6}$$

with

$$\frac{2}{3}r^{\frac{3}{2}} = -\int_0^x H^{\frac{1}{2}}\,dx, \tag{6.7}$$

$$A_0 = \left(\frac{r}{H}\right)^{\frac{1}{4}} \cos\left(-\int_0^x \frac{1}{2}G/H^{\frac{1}{2}}\,dx\right), \quad C_0 = r^{-\frac{1}{4}}H^{-\frac{1}{4}} \sin\left(-\int_0^x \frac{1}{2}G/H^{\frac{1}{2}}\,dx\right), \tag{6.8}$$

in which

$$H = \frac{1}{4}(k_{x2} - k_{x1})^2, \tag{6.9}$$

$$G = P + \frac{i}{2}(k_{x1} + k_{x2})Q + \frac{i}{2}(k'_{x1} + k'_{x2}). \tag{6.10}$$

The new variable r now depends on x, y and t, but its variations with respect to y and t will be slow.

The adequacy of (6.5)–(6.10) as a uniformly valid solution for the case of a curved and/or unsteady caustic which occurred in a deep or intermediate-depth region, depends on whether the singularities at the caustic have been cancelled out from $\chi_1 + \chi_2, Q, G$, etc., otherwise the above solution will become singular and the coefficients A_0 and C_0 are no longer slowly varying in the vicinity of the caustic (which will decline the use of the approximation implied by (4.8) in Shyu & Phillips (1990)). Therefore it is required in the following to demonstrate the above-mentioned regularity.

Since even for a curved and/or unsteady caustic in an intermediate-depth region, from the dispersion relation and the fact that $U_x + C_{gx} = 0$ at $x = 0$, one can always prove that k_{x1} and k_{x2} represent two branches of a double-valued function with the branch point at the caustic. Therefore the phase function of the incident wave can be written as

$$\chi_1 = [d_0 + d_1 x + d_2 x^2 + \cdots] + \frac{2}{3}\left[(-\psi_1)^{\frac{1}{3}}(-x) + O(x^2)\right]^{\frac{3}{2}} \tag{6.11}$$

(also see Smith (1975)), where the coefficients of the Taylor series expansions about $x = 0$ in the two square brackets are functions of y and t, and all of them except d_0 are slowly varying according to the explanation following (6.3). On substitution χ_1 into (6.3) we have

$$\left.\begin{aligned} k_{x1} &= [d_1 + 2d_2 x + O(x^2)] - [\psi_1 x + O(x^2)]^{\frac{1}{2}} \\ h_y k_{y1} &= \left[\frac{\partial d_0}{\partial y} + \frac{\partial d_1}{\partial y} x + O(x^2)\right] - \frac{1}{3}\frac{\partial \psi_1}{\partial y}\left[(-\psi_1)^{-1}(-x)^3 + O(x^4)\right]^{\frac{1}{2}} \end{aligned}\right\} \tag{6.12}$$

The other branch of (6.11) and (6.12) then provide the phase and wave-number components of the reflected wave:

$$\chi_2 = [d_0 + d_1 x + d_2 x^2 + \cdots] - \frac{2}{3}\left[(-\psi_1)^{\frac{1}{3}}(-x) + O(x^2)\right]^{\frac{3}{2}} \tag{6.13}$$

$$\left.\begin{aligned} k_{x2} &= [d_1 + 2d_2 x + O(x^2)] + [\psi_1 x + O(x^2)]^{\frac{1}{2}} \\ h_y k_{y2} &= \left[\frac{\partial d_0}{\partial y} + \frac{\partial d_1}{\partial y} x + O(x^2)\right] + \frac{1}{3}\frac{\partial \psi_1}{\partial y}\left[(-\psi_1)^{-1}(-x)^3 + O(x^4)\right]^{\frac{1}{2}} \end{aligned}\right\} \tag{6.14}$$

The above phase functions not only lead to the right forms of k_{x1} and k_{x2}, but also ensure that $\nabla \times \mathbf{k} = 0$ for both waves. However, the values of the series coefficients d_1, d_2, ψ_1, etc. can be determined only from the dispersion relation. We notice in passing that for a curved caustic, even though k_{y1} is unequal to k_{y2} when $x \neq 0$ (if $\partial\psi_1/\partial y \neq 0$), their difference is vanishingly small and proportional to $x^{3/2}$ only (also with a very small coefficient $(2/3\psi_1)\partial\psi_1/\partial y$) when the caustic is approached. A similar situation also occurs to the observed frequencies n_1 and n_2 for an unsteady caustic. These can benefit the numerical computations of the reflected wave significantly as illustrated in the next section.

From (6.11) and (6.13) it is immediately clear that $\chi_1 + \chi_2$ in (6.5) is regular at the caustic. On the other hand, from (6.12) and (6.14), we have in the vicinity of caustic

$$H = \frac{1}{4}(k_{x2} - k_{x1})^2 = \psi_1 x + O(x^2) \tag{6.15}$$

Consequently

$$-\int_0^x H^{\frac{1}{2}}\, dx = \frac{2}{3}(-\psi_1)^{\frac{1}{2}}(-x)^{\frac{3}{2}}[1 + O(x)] \tag{6.16}$$

and

$$-\int_0^x \frac{1}{2} G/H^{\frac{1}{2}}\,dx = G_0(-\psi_1)^{-\frac{1}{3}}(-x)^{\frac{1}{2}}[1+O(x)] \tag{6.17}$$

if G is regular at the caustic and $G_0 = G(x=0)$. Substitution of (6.16) into (6.7) results in

$$r = (-\psi_1)^{\frac{1}{3}}(-x)[1+O(x)] \tag{6.18}$$

so that r is regular at the caustic. Furthermore, on substituting (6.15), (6.17) and (6.18) into (6.8), we have

$$\left.\begin{aligned} A_0 &= (-\psi_1)^{-\frac{1}{6}}[1+O(x)], \\ C_0 &= G_0(-\psi_1)^{-\frac{5}{6}}[1+O(x)]. \end{aligned}\right\} \tag{6.19}$$

in the vicinity of the caustic. Therefore A_0 and C_0 are also regular (and therefore slowly varying) at the caustic. All of these findings enable us to conclude that the singularities at the caustic have been cancelled out from the solution (6.5)–(6.6), provided that G and Q are also regular here, which will be demonstrated as follows.

Since the parameters G and Q also occur to the WKBJ solution, their regularity can be proved by consideration of the action conservation principle which is fulfilled by the WKBJ solution even at the caustic as demonstrated by Smith (1975). Now, substituting (5.12) and (5.13) into (6.6), using (6.7) and (6.9), and also taking (6.3) into consideration, we obtain the WKBJ solution

$$\begin{aligned} \eta =& C(y,t)H^{-\frac{1}{4}} \exp\left[\int_0^x \frac{1}{2}(-Q - iG/H^{\frac{1}{2}})\,dx\right] \exp i\left(\chi_1 - \frac{1}{4}\pi\right) \\ &+ C(y,t)H^{-\frac{1}{4}} \exp\left[\int_0^x \frac{1}{2}(-Q + iG/H^{\frac{1}{2}})\,dx\right] \exp i\left(\chi_2 + \frac{1}{4}\pi\right) \end{aligned} \tag{6.20}$$

where the common factor $C(y,t)$ is independent of x. From (6.20), the local amplitudes of the incident and reflected waves are

$$\left.\begin{aligned} a_1 &= CH^{-\frac{1}{4}} \exp\left[\int_0^x \frac{1}{2}(-Q - iG/H^{\frac{1}{2}})\,dx\right], \\ a_2 &= CH^{-\frac{1}{4}} \exp\left[\int_0^x \frac{1}{2}(-Q + iG/H^{\frac{1}{2}})\,dx\right] \end{aligned}\right\} \tag{6.21}$$

Therefore substitution of (6.15) for H and differentiation yield

$$\left.\begin{aligned}\frac{a_1'}{a_1} &= \frac{1}{2}(-Q - iG/H^{\frac{1}{2}}) - \frac{1}{2}\frac{k_{x2}' - k_{x1}'}{k_{x2} - k_{x1}} \\ \frac{a_2'}{a_2} &= \frac{1}{2}(-Q + iG/H^{\frac{1}{2}}) - \frac{1}{2}\frac{k_{x2}' - k_{x1}'}{k_{x2} - k_{x1}}\end{aligned}\right\} \qquad (6.22a,b)$$

The above expressions for a_1'/a_1 and a_2'/a_2 are exactly identical with those in (6.4) as the expression (3.13) and the definition (6.10) for G are substituted into (6.4) for R_1 and R_2. This coincidence will put even more confidence in the assumption that if the singularities at the caustic are completely cancelled out from (3.12), this equation and the solution (6.5)–(6.10) will remain valid uniformly in a region containing the caustic, since (6.22a,b) represent a rigorous asymptotic approximation of the solution (6.5)–(6.10). Also, when (6.22a,b) are identical with those in (6.4) in the ray solution, the regularity of the parameters Q, G (and therefore P in virtue of (6.10)) in (6.22a,b) can therefore be demonstrated through a consideration of the action conservation principle.

From (6.12) and (6.14) it immediately follows that

$$\frac{1}{2}\frac{k_{x2}' - k_{x1}'}{k_{x2} - k_{x1}} = \frac{1}{4x}[1 + O(x)] \qquad (6.23)$$

near the caustic. Next, from the action conservation principle

$$\frac{\partial}{\partial t}\left(\frac{E_1}{\sigma_1}\right) + \frac{\partial}{\partial x}\left[(U_x + C_{gx1})\frac{E_1}{\sigma_1}\right] + \frac{\partial}{\partial y}\left[(U_y + C_{gy1})\frac{E_1}{\sigma_1}\right] = 0 \qquad (6.24)$$

in which the wave action density of the incident wave

$$\frac{E_1}{\sigma_1} = \frac{1}{2}\rho g\frac{a_1^2}{\sigma_1}$$

where E_1 represents its energy density, σ_1 the intrinsic frequency, $\mathbf{C_{g1}}$ the group velocity, ρ the density of water, and g the gravitational acceleration. Therefore

$$\frac{a_1'}{a_1} = -\frac{\sigma_1}{2(U_x + C_{gx1})}\frac{\partial}{\partial x}\left(\frac{U_x + C_{gx1}}{\sigma_1}\right) - \frac{\sigma_1}{2a_1^2(U_x + C_{gx1})}\frac{\partial}{\partial t}\left(\frac{a_1^2}{\sigma_1}\right)$$
$$- \frac{\sigma_1}{2a_1^2(U_x + C_{gx1})}\frac{\partial}{\partial y}\left[(U_y + C_{gy1})\frac{a_1^2}{\sigma_1}\right] \qquad (6.25)$$

In (6.24) and (6.25), the scale factor h_y of the curvilinear co-ordinates has been neglected as it will affect only the higher powers of x in the following Taylor series expansions.

First, the relation between σ_1 and k_1 differs according to whether the water is deep or of moderate depth, but in any cases, by using (6.12) and (6.14), one can always obtain the form of the expansion

$$\frac{U_x + C_{gx1}}{\sigma_1} = \sqrt{\psi_1 x}[\alpha_0 + O(x)] + [e_1 x + O(x^2)] \tag{6.26}$$

in the vicinity of the caustic, where α_0 and e_1 are the coefficients of the two Taylor series in the square brackets. The absence of e_0 from the second series is simply due to the fact that $U_x + C_{gx1} = 0$ at the caustic. Thus from (6.26), the first term on the right-hand side of (6.25)

$$-\frac{\sigma_1}{2(U_x + C_{gx1})}\frac{\partial}{\partial x}\left(\frac{U_x + C_{gx1}}{\sigma_1}\right) = -\frac{1}{4x}[1 + O(x)] + \frac{1}{\sqrt{\psi_1 x}}[\beta_0 + O(x)] \tag{6.27}$$

near the caustic. Since in the vicinity of the caustic, variations of wave properties perpendicular to the caustic are large compared to variations along the caustic, (6.27) represents the major contribution to a_1'/a_1 in this region. Therefore, from a comparison between (6.27) and (6.23) one may conclude that the term $H^{-\frac{1}{4}}$ (or more precisely, $x^{-\frac{1}{4}}$) in a_1, a_2 in the present solution (6.21) is indeed consistent with the predication by the action conservation principle as far as their first approximations are concerned.

The values of G and Q cannot be obtained without further evaluation of the second and third terms on the right-hand side of (6.25), which represent the higher-order modification. Since the first approximation $a_1 \approx CH^{-\frac{1}{4}}$ has been justified, we have

$$a_1^2 \approx \frac{C^2}{\sqrt{\psi_1 x}} \tag{6.28}$$

near the caustic. Also, from the dispersion relation, it is not difficult to see that the series expansions of σ_1 and $U_y + C_{gy1}$ will possess the same form as that of

k_{x1} in (6.12). Thus, using these series and (6.28) as well as (6.26), and recalling that C and ψ_1 are slow functions of y and t, we obtain

$$-\frac{\sigma_1}{2a_1^2(U_x + C_{gx1})}\frac{\partial}{\partial t}\left(\frac{a_1^2}{\sigma_1}\right) \approx \frac{\xi_0}{\sqrt{\psi_1 x}} \tag{6.29}$$

$$-\frac{\sigma_1}{2a_1^2(U_x + C_{gx1})}\frac{\partial}{\partial y}\left[(U_y + C_{gy1})\frac{a_1^2}{\sigma_1}\right] \approx \frac{\zeta_0}{\sqrt{\psi_1 x}} \tag{6.30}$$

Consequently, a combination of (6.27), (6.29) and (6.30) results in

$$-iG_0 = 2(\beta_0 + \xi_0 + \zeta_0), \tag{6.31}$$

if (6.22a) and (6.25) are equal.

After the first term of the expansion for G was found, the next order terms in (6.29) and (6.30) can be pursued by substitution, which are featured of zeroth power of x. These terms and the corresponding term in (6.27), excluding those attributed to the series in (6.23), can be identified with $Q_0 \equiv Q(x = 0)$ in (6.22a). This procedure can be continued to determine subsequent terms in the expansions for G and Q, and the results show that these expansions indeed take the form of a power series with center at $x = 0$. Although there is no way to estimate the radii of convergence of these two series in the present situation, since the variation of the underlying current is slow, one may expect that these series will be uniformly convergent in a large (relative to the wavelength) area centering at the caustic. Therefore we conclude that G, Q (and P) are regular at the caustic. This conclusion can also be drawn from a consideration of the action conservation equation for the reflected wave, since in this case, we can obtain the same results except that the signs of the terms containing $\sqrt{\psi_1 x}$ in (6.27)–(6.30) become opposite, which also occurs to (6.22b) compared with (6.22a), so that the parameters G and Q in (6.22b) have the same values as those in (6.22a) and are indeed regular at the caustic.

In summary, by investigating the series expansion and therefore the regularity of each quantity evaluated in the curvilinear co-ordinates, we have demonstrated that even for a curved and/or unsteady caustic and for waves in an intermediate-depth region, the uniform asymptotic and the WKBJ solutions in the vicinity of the caustic take the same form as those in Shyu & Phillips (1990) and those derived in the foregoing section. These solutions have provided explicit expressions for the amplitudes, although their minor terms G and Q can be determined only from the action conservation equation and the dispersion relation which themselves have been proved by Smith (1975) to be valid in the vicinity of the caustic in the same circumstances. The explicitness of these solutions will in the next section prove of great use to a practical numerical computation of the reflection phenomenon.

7. Application to numerical computations

In this section, we shall conduct numerical simulations in two cases: a straight caustic and a curved caustic. Since it will later become clear that estimates of the reflected wave from the incident wave in the vicinity of the caustic at each instant involves only the instantaneous values of various variables and the derivatives with respect to x, and furthermore the relation between n_1 and n_2 near an unsteady caustic is analogous to that between k_{y1} and k_{y2} near a curved caustic (see the discussion following (6.14)), any conclusions from the present simulations will have implications for the case of an unsteady caustic. To eliminate other complication and without loss of generality, we also assume that all waves are in deep water.

Since we have in the previous sections derived the analytical solutions for the case when a straight caustic is caused by a deep-water gravity wave propagating obliquely upon a steady, two-dimensional current, the results of the present numerical computations for this case can be compared with the analytical solutions to show the accuracy of the present numerical schemes. To achieve this goal, even for a straight caustic, we deliberately take the x and y directions (denoted by x' and y' instead) of the computational grid not along $\mathbf{U}$ so that no simplifications, which may originally suitable to this special case, will be made and the extension of the numerical model to the case of a curved caustic is straightforward. Also we note that even though the analysis in the foregoing section was made in a curvilinear co-ordinate system, without a prior knowledge of the location of the caustic, the differential equations can conveniently be solved only on a rectangular grid, after which and after the caustic was determined numerically, the components of each vector relative to the co-ordinate system defined in section 6 can be calculated from those referred to (x', y').

Determinations of the incident wave and the caustic

Since in the present simulations the underlying currents are steady, the action conservation equation can be reduced to

$$\frac{\partial}{\partial x'}\left[(U_{x'} + C_{gx'})\frac{a^2}{\sigma}\right] + \frac{\partial}{\partial y'}\left[(U_{y'} + C_{gy'})\frac{a^2}{\sigma}\right] = 0 \tag{7.1}$$

and the apparent frequency n remains constant everywhere (denoted by n_0 again) so that the wave-number components $k_{x'}$ and $k_{y'}$ can be determined entirely from the irrotationality

$$\frac{\partial k_{y'}}{\partial x'} - \frac{\partial k_{x'}}{\partial y'} = 0 \tag{7.2}$$

and the dispersion relation

$$n_0 = \left[g\sqrt{k_{x'}^2 + k_{y'}^2}\right]^{\frac{1}{2}} + U_{x'}k_{x'} + U_{y'}k_{y'}. \tag{7.3}$$

After their determination, the intrinsic frequency σ and the group velocity components $C_{gx'}$ and $C_{gy'}$ can also be calculated.

The partial differential equations (7.1) and (7.2) are solved by using a finite difference scheme. Since it is not the present purpose to develop an efficient model, an explicit difference equation of first-order accuracy is used to approximate (7.1) and (7.2). As this solution scheme marches towards caustic (in the y' direction, say) one row at a time, the derivatives with respect to y' are replaced by the forward difference. On the other hand, the derivatives with respect to x' are replaced by the forward difference or the backward difference, depending on whether $U_{x'} + C_{gx'}$ is negative or positive. This choice is important to the stability of the present different scheme, and according to the Courant-Friedrichs-Lewy condition, this scheme will be stable only if the ratio of the grid spacing $\Delta y'$ to $\Delta x'$ is smaller than the ratio of $|U_{y'} + C_{gy'}|$ to $|U_{x'} + C_{gx'}|$. Therefore, to ensure that the numerical solutions will be accurate even in the vicinity of the caustic, the grid spacings $\Delta x' = 20$ cm, $\Delta y' = 0.0625$ cm are chosen for both cases of a straight and a curved caustics.

When the difference formula for (7.2) is solved at each mesh point, we first obtain the solution of $k_{x'}$ at this point. Then the numerical solution of $k_{y'}$ at the same point is calculated from (7.3) using the Secant Method. Consequently, all the quantities related to the kinematics of the incident wave, including the characteristic velocity $\mathbf{U}+\mathbf{C_g}$, can be calculated, after which the action conservation equation (7.1) can subsequently be solved for the amplitude a at this point.

The above computations can be continued until at a certain point,the subroutine for the Secant Method fails to return a reliable and real root of (7.3) which signifies the occurrence of the blockage phenomenon at this point. If this occur to point A in figure 2, the solutions at the points on the same row but on the right side of A can still be pursued. However, since point A is excluded from the integration domain,the solutions at point D on the next row cannot be computed with the present difference scheme as $U_{x'} + C_{gx'} > 0$ at this point. This difficulty can be overcome by using the forward difference instead of the backward difference to approximate $\partial k_{y'}/\partial x'$ in (7.2) at this particular point and at the points above D and in the same column. This enables us to continue the calculations of $\mathbf{k}$ (but not the amplitude a) on the above rows until the Secant Method fails again or $U_x + C_{gx}$ became negative at another point, E say, which always occurs in the next column. Note that $U_x + C_{gx}$ represents the component of $\mathbf{U} + \mathbf{C_g}$ in the direction perpendicular to the estimate caustic.

The line AE in figure 2 can approximate the true caustic satisfactorily if $\Delta y'$ is sufficiently small. Furthermore, the numerical solution of (7.1) at point F (and at other points on the same row) can now be calculated reliably by using the present difference scheme and the solutions at points B and C which have previously been obtained without difficulty. As such, we have all the informations required for the subsequent calculations using the same procedure.

The above strategy for estimates of the location of the caustic can be justified directly by a comparison between its numerical and analytical solutions in the case of a straight caustic (an indirect justification for both cases will also be given later). In figure 3, the conditions of the incoming wave prescribed on the boundaries AB and BC are determined from the requirement that $k_y = -0.2$ rad/m everywhere and the component of the action flux in the x direction is equal to 1 everywhere. The velocity distribution of the underlying current is

$$U_x = -1.818 - 0.01x \text{ (m/s)}, \qquad U_y = 0,$$

while the apparent frequency $n_0 = 1.4$ rad/s. In this situation, the turning point C in figure 3 is first detected when the subroutine for the Secant Method fails to estimate k_x at this point. Then by using the above strategy, the other turning points can successively be located, which as shown in figure 3 coincide with the true caustic very well.

On the other hand, to simulate a curved caustic, we assume that the streamlines of the underlying larger-scale current are circles and the magnitude of the velocity at each point

$$|U| = -4.0\frac{3\pi/2 - \theta}{\pi}\frac{100}{r} \quad (\text{m/s}),$$

where (r, θ) represents the polar coordinates of this point (see figure 4). This velocity distribution has zero vorticity everywhere except at the point $r = 0$ which represents a singular point but will be excluded from the integration domain because of the wave blockage phenomenon. Another feature of this distribution is that when r is very large, $|U|$ becomes vanishingly small. Therefore a regular deep-water wavetrain with frequency $n_0 = 1.7$ rad/s propagating in a single direction can be prescribed at every grid point on the boundaries in figure 4 which are far from the origin. From these boundary conditions and using the numerical scheme, the variation of the wave-number of the incident wave in the interior was solved until a turning point was first met. After this, we restrict ourselves to calculate the incident wave field and the location of the caustic in a small area in figure 5, in which the turning point A has been located and the numerical solutions of the wave-number components at each point on the line AB have been estimated in the previous stages. Thus, to calculate the wave-number as well as the amplitude of the incident wave in this area, it is only necessary to prescribe the value of a_1 at each point on line AB in figure 5. In consideration of (6.21) and (6.15), the particular boundary condition of a_1 is chosen as

$$a_1 = (-\bar{x})^{-0.25}$$

where $\bar{x}$ represents the distance between each point on AB and the dashed line which approximates the estimate caustic. Note that since a slowly modulated incoming wave is allowed by the theories, the above arrangement is particularly suitable for tests of the present algorithm.

Determinations of the reflected wave

After the incident wave field and the position of the caustic were determined, we proceed to estimate the reflected wave in the vicinity of the caustic using Smith's (1975) theory and the present theory.

Since the difference between k_{y1} and k_{y2} is very small near a curved caustic (see the remark following (6.14)) and is zero everywhere in the case of a straight caustic, we let $k_{y2} = k_{y1}$ at each point in the vicinity of the caustic and then calculate the value of k_{x2} as another root of equation (7.3) which will coalesce with k_{x1} at the caustic. The accuracy of these estimates can more or less be seen by calculations of the vorticity of the resulting $\mathbf{k_2}$ as shown in figure 6. (In this figure and in the following figures, the results presented are along the dotted lines in figures 3 and 5 for a straight and curved caustics respectively, and also the values of $\sqrt{\psi_{1x}}$ estimated from (6.15) by neglecting the higher powers of x are chosen as the abscissas.) The results in figure 6 indicate that the irrotationality is approximately fulfilled by the estimates of $\mathbf{k_2}$ in the case of a curved caustic and this fulfillment is even more satisfactory in the case of a straight caustic as might be expected.

Next, we shall calculate a_2 in terms of a_1 in the vicinity of the caustic, which can in principle be achieved by using Smith's (1975) theory or by using the present theory. According to Smith (1975), the flux of wave action normal to the caustic carried by the incident and by the reflected waves are equal and opposite at the caustic, so that we have

$$\left[(U_x + C_{gx1})\frac{a_1^2}{\sigma_1}\right]_{x=0} = -\left[(U_x + C_{gx2})\frac{a_2^2}{\sigma_2}\right]_{x=0}. \tag{7.5}$$

Also integration of (6.24) with respect to x yields

$$\left.\begin{aligned}\left[(U_x + C_{gx1})\frac{a_1^2}{\sigma_1}\right] - \left[(U_x + C_{gx1})\frac{a_1^2}{\sigma_1}\right]_{x=0} &= \int_0^x F_1(x,y,t)\,dx\\ \left[(U_x + C_{gx2})\frac{a_2^2}{\sigma_2}\right] - \left[(U_x + C_{gx2})\frac{a_2^2}{\sigma_2}\right]_{x=0} &= \int_0^x F_2(x,y,t)\,dx\end{aligned}\right\} \tag{7.6}$$

where

$$\left.\begin{aligned} F_1 &= -\frac{\partial}{\partial t}\left(\frac{a_1^2}{\sigma_1}\right) - \frac{\partial}{\partial y}\left[(U_y + C_{gy1})\frac{a_1^2}{\sigma_1}\right] \\ F_2 &= -\frac{\partial}{\partial t}\left(\frac{a_2^2}{\sigma_2}\right) - \frac{\partial}{\partial y}\left[(U_y + C_{gy2})\frac{a_2^2}{\sigma_2}\right] \end{aligned}\right\} \tag{7.7}$$

Here the scale factor h_y has again been neglected as it affects only the higher powers of x in the Taylor series expansions of the minor terms.

From (6.28) and because of $U_y + C_{gy} \neq 0$ at $x = 0$, we have the first approximation of (7.7):

$$F_1 \approx F_2 \approx \frac{\tau_0}{\sqrt{\psi_1 x}}. \tag{7.8}$$

Thus, substituting (7.5) and (7.8) into (7.6) and carrying out the integration, we obtain

$$\left[(U_x + C_{gx2})\frac{a_2^2}{\sigma_2}\right] = -\left[(U_x + C_{gx1})\frac{a_1^2}{\sigma_1}\right] - 4\frac{\tau_0}{\psi_1}\sqrt{\psi_1 x} \tag{7.9}$$

at each point in the vicinity of the caustic. Therefore, near a curved and/or unsteady caustic, if F_1 in (7.7) can reliably be estimated from the numerical solutions of the incident wave, the ratio a_2/a_1 in the vicinity of the caustic can be calculated from the solutions of k_{x1} and k_{x2} using (7.9). However, in practical applications, the errors inherent in the difference scheme will drastically increase when the caustic is approached, owing to the singularities at the caustic. Therefore, in the immediate vicinity of the caustic, even the estimates of a_1 will become unreliable. On the other hand, away from the caustic, the values of a_2/a_1 are significantly different from 1, but the values of $|U_x + C_{gx1}|$ and $|U_x + C_{gx2}|$ in (7.9) are small and their difference which responsible for a_2/a_1 being unequal to 1 are even smaller. Therefore the errors due to the misalignment of the curvilinear co-ordinate lines, which will not decrease when the distance from the caustic increases, will be magnified significantly in the estimates of a_2/a_1. Lastly, in the region far away from the caustic, the one-term approximations in (7.8) and in other relations are no longer valid. All of these have forbidden the use of (7.9) in a general situation.

Next, we consider the applications of the present theory. From (6.21) and (6.17) it immediately follows that

$$\frac{a_2}{a_1} = \exp\left[\int_0^x iG/H^{\frac{1}{2}}\, dx\right] \approx \exp\left[2\frac{iG_0}{\psi_1}\sqrt{\psi_1 x}\right]. \tag{7.10}$$

Remark that since

$$\exp\left[2\frac{iG_0}{\psi_1}\sqrt{\psi_1 x}\right] \approx 1 - 2\frac{iG_0}{\psi_1}\sqrt{\psi_1 x} \tag{7.11}$$

the parameter G_0 is closely related to the difference between the amplitudes of the incident and reflected waves in the vicinity of the caustic. From (6.22a,b) and (6.15), we also have

$$\frac{a_2'}{a_2} - \frac{a_1'}{a_1} = iG/H^{\frac{1}{2}} \approx iG_0/\sqrt{\psi_1 x} \tag{7.12}$$

so that the value of G_0 can be estimated from (6.15) and the solutions of a_1'/a_1 and a_2'/a_2, which themselves can be calculated from (6.25) and the corresponding equation for a_2'/a_2.

The first terms on the right-hand side of (6.25) and the corresponding equation for a_2'/a_2 can be computed solely from the numerical solutions of $\mathbf{k_1}$ and $\mathbf{k_2}$. On the other hand, the second and third terms represent the minor terms and their one-term approximations are proportional to $1/\sqrt{\psi_1 x}$ according to (6.29) and (6.30). Thus the values of these terms for a_1'/a_1 and for a_2'/a_2 are equal in magnitude and opposite in sign within the present approximation. Therefore, even without solving (7.1) for a_2, the approximation of a_2'/a_2 at each point near the caustic can still be estimated in theory. Nevertheless, since all three terms in (6.25) involve $U_x + C_{gx}$ and since $\partial a_1/\partial y << \partial a_1/\partial x$, the error magnification phenomenon occurred in the application of the Smith's (1975) theory also will happen here.

To clarify and remedy this problem, we temporarily neglect the second and third terms on the right-hand side of (6.25) and neglect F_1 and F_2 in (7.6) correspondingly. Therefore, if the resulting quantities are designated by a bar, we have

$$\left(\overline{\frac{a_2}{a_1}}\right)_{SM} \approx \left[-\frac{(U_x + C_{gx1})/\sigma_1}{(U_x + C_{gx2})/\sigma_2}\right]^{\frac{1}{2}} \tag{7.13}$$

from Smith's (1975) theory, and

$$\left(\overline{\frac{a_2}{a_1}}\right)_{ST} \approx \exp\left[2\frac{i\overline{G_0}}{\psi_1}\sqrt{\psi_{1x}}\right] \tag{7.14}$$

from the present theory, in which

$$i\overline{G_0}/\sqrt{\psi_{1x}} \approx \overline{\frac{a_2'}{a_2}} - \overline{\frac{a_1'}{a_1}} \tag{7.15}$$

where

$$\left.\begin{aligned} \overline{\frac{a_1'}{a_1}} &= -\frac{\sigma_1}{2(U_x + C_{gx1})}\frac{\partial}{\partial x}\left(\frac{U_x + C_{gx1}}{\sigma_1}\right) \\ \overline{\frac{a_2'}{a_2}} &= -\frac{\sigma_2}{2(U_x + C_{gx2})}\frac{\partial}{\partial x}\left(\frac{U_x + C_{gx2}}{\sigma_2}\right) \end{aligned}\right\} \tag{7.16}$$

To determine the exponent in (7.14), the estimates of (7.15) should be multiplied by $2x$, where the values of x can be measured from the estimate caustic to the point under consideration, however for a reason to do with the cancellation of errors, the approximation

$$x \approx \frac{1}{2}\frac{k_{x2} - k_{x1}}{k'_{x2} - k'_{x1}} = \hat{x}, \quad \text{say} \tag{7.17}$$

is used in virtue of (6.23). Therefore we have

$$2\frac{i\overline{G_0}}{\psi_1}\sqrt{\psi_{1x}} \approx 2\frac{i\overline{G_0}}{\sqrt{\psi_{1x}}}\hat{x} \tag{7.18}$$

Both $(\overline{a_2/a_1})_{SM}$ and $(\overline{a_2/a_1})_{ST}$ can be estimated from the numerical solutions of $\mathbf{k_1}$ and $\mathbf{k_2}$ only, which have been obtained earlier. Figure 7 shows the results of the simulation for a straight caustic, in which $(\overline{a_2/a_1})_{SM}$ and $(\overline{a_2/a_1})_{ST}$ coincide with each other very well. However, since the estimates of $\sqrt{\psi_{1x}}$ may contain errors, the accuracy of these calculations is best seen from the comparison of the numerical values of $\overline{G_0}$ and $\overline{Q_0}$ with their analytical solutions, where the numerical values of $\overline{Q_0}$ are calculated from the relation

$$\overline{Q_0} = -2\overline{\frac{a_1'}{a_1}} - \frac{1}{2\hat{x}} - \frac{i\overline{G_0}}{\sqrt{\psi_1 x}}$$

derived from (6.22) and (6.23). This comparison can be made because for a straight and steady caustic, the second and third terms on the right-hand side of (6.25) are essentially zero so that the true values of $\overline{G_0}$ and $\overline{Q_0}$ should be identical with $G(x = 0)$ and $Q(x = 0)$ in (5.5) and (4.3) respectively. The results shown in figure 8 are indeed very satisfactory except in the immediate vicinity of the caustic in which the discretization errors become large due to the singularities at the caustic.

We note that in order to obtain $\overline{G_0}$, the values of (7.15) have been multiplied by $\sqrt{\psi_1 x}$ which as mentioned before was estimated from the relation

$$\sqrt{\psi_1 x} \approx \frac{1}{2}(k_{x2} - k_{x1}) \tag{7.19}$$

in accordance with (6.15). Therefore it is also desired to check the accuracy of the approximation in (7.19), that can be achieved by comparing $\hat{x}$ with the true distance x measured from the estimate caustic to the point considered, since the one-term approximations in both (7.17) and (7.19) originate from the same function. The results of this comparison are also shown in figure 7, which are not as satisfactory as the results of $\overline{G_0}$ in figure 8, indicating that the errors in the estimates of $i\overline{G_0}/\sqrt{\psi_1 x}$ and $\sqrt{\psi_1 x}$ have been cancelled out in the calculations of $\overline{G_0}$.

In figure 8, we also estimate ψ_1 by using the following relation

$$\psi_1 \approx \frac{1}{4}(k_{x2} - k_{x1})^2/x. \tag{7.20}$$

The results are again less satisfactory, but their approach to a constant is obvious so that from this constant and the estimate of $\overline{G_0}$ one may determine a_2/a_1 at each point in the vicinity of the caustic, from which the whole reflected wave field can be calculated numerically by solving (7.1)–(7.3).

Before we proceed to present the results of the simulation for a curved caustic, it is interesting to estimate the coefficient in (6.30) for a straight caustic. The results are also shown in figure 8, which are indeed very small compared with $|\overline{G_0}|$, meaning that the misalignment of the co-ordinate lines and the error magnification phenomenon are not severe in the case of a straight caustic.

In case that the caustic is curved, the coefficient ζ_0 in (6.30) and τ_0 in (7.8) are in general nonzero, but when a particular distribution of a_1 is prescribed on the boundary, it is still possible for ζ_0 and τ_0 to vanish. In such case, G_0 is identical with $\overline{G_0}$ and more importantly, the value of ψ_1 upon determination can remain valid for any different boundary conditions of a_1. Therefore it is still worthwhile to consider $(\overline{a_2/a_1})_{SM}$, $(\overline{a_2/a_1})_{ST}$ and $\overline{G_0}$ in this case.

Figure 9 shows the results similar to figure 7 but for a curved caustic. The discrepancy between the estimates of $(\overline{a_2/a_1})_{SM}$ and $(\overline{a_2/a_1})_{ST}$ is not unusual, but the values of $\hat{x}/x$ in figure 9 do not approach to 1 when x increases, implying that the errors due to the misalignment of the curvilinear co-ordinate lines have been magnified significantly. In figure 9, we also show the different estimates of $(\overline{a_2/a_1})_{ST}$ when $\hat{x}$ in (7.18) is replaced by x, which are much worse than the original ones. Therefore the errors in $i\overline{G_0}/\sqrt{\psi_1 x}$ and $\hat{x}$ have for the most part been cancelled out in (7.18), which is not surprising in consideration of the first approximations of (6.26) and (7.19). This situation, though less dramatic, can even be seen in figure 7 for a straight caustic.

The estimates of $\sqrt{\psi_1 x}$ are completely obscured by the errors in the case of a curved caustic. This situation can be illuminated by calculations of ψ_1 from (7.20), which as shown in figure 10 are indeed far from been constant. Therefore, even the relation between $(\overline{a_2/a_1})_{SM}$ and $\sqrt{\psi_1 x}$ in figure 9 is perfectly linear, these estimates may individually contain unproportionately large errors and therefore lead to erroneous prediction of the amplitude of the reflected wave even when $\zeta_0, \tau_0 = 0$.

However, since the estimates of $\overline{a_2/a_1}$ in figure 9 vary linearly with those of $\sqrt{\psi_1 x}$, the values of $i\overline{G_0}/\psi_1$ in (7.14) calculated from the expression

$$i\frac{\overline{G_0}}{\psi_1} = \left(\frac{\overline{a_2}}{a_1} - 1\right) / (2\sqrt{\psi_1 x})$$

remain nearly constant as shown in figure 10, contrary to the results of $\overline{G_0}/i$ in the same figure. The constancy of the estimates of $i\overline{G_0}/\psi_1$ from both $(\overline{a_2/a_1})_{SM}$ and $(\overline{a_2/a_1})_{ST}$ is certainly encouraging, but without knowing $\overline{G_0}$ and ψ_1 separately, no reliable solutions of $\overline{a_2/a_1}$ can be obtained in the vicinity of the caustic. This difficulty can however be solved by using the following error-reducing strategy.

When the misalignment of the co-ordinate lines occur, each quantity or derivative is effectively calculated in a 'new' co-ordinate system so that its influences on certain quantities might be small, though large on others. This can explain why the estimates of $i\overline{G_0}/\psi_1$ remain nearly constant. The development of an analytical theory for these phenomena would be extremely difficult if not impossible; therefore we rest content with the discussion of the consistency of the results.

From the numerical solutions in figure 9 and the expressions (6.26), (7.16), (7.17) and (7.19), it is clear that the relative error in $\overline{a_2'/a_2} - \overline{a_1'/a_1}$ is equal to the inverse of that in $\hat{x}$. This relation may actually occur to each of $\overline{a_1'/a_1}$ and $\overline{a_2'/a_2}$ for the same reason. Therefore, using the ratio $\hat{x}/x$, the errors in $\overline{a_1'/a_1}$ and $\overline{a_2'/a_2}$ can be remedied. Figure 11 shows the results of $\overline{a_1'/a_1}/(1+\varepsilon)$ and $\overline{a_2'/a_2}/(1+\varepsilon)$ where $1+\varepsilon = \hat{x}/x$, which (especially those of the former) are much closer to the values of $-1/4x$ than the original estimates. The small difference between $\overline{a_1'/a_1}/(1+\varepsilon)$ and $-1/4x$ can be expected to equal $-i\overline{G_0}/(2\sqrt{\psi_1 x})$ approximately according to (6.22a), where $\sqrt{\psi_1 x} \approx (k_{x2} - k_{x1})/2$. Therefore the values of $\overline{G_0}$ at each point can now be estimated, from which and from the estimates of $i\overline{G_0}/\psi_1$ obtained earlier, the solution values of ψ_1 immediately follow. These new estimates of $\overline{G_0}$ and ψ_1 are also shown in figure 10 for comparison, which indeed approach to a constant while $\overline{a_1'/a_1}/(1+\varepsilon) + 1/4x$ get smaller and smaller.

From figure 10, the value $\psi_1 \approx -0.21$ can be anticipated which is valid for any wave-amplitude distributions, but the result $\overline{G_0}/i \approx 0.02$ is applicable to the estimates of a_2/a_1 using (7.10) only if the action flux in the y direction is independent of y and the current field and the wave properties do not vary with time, otherwise the second and third terms on the right-hand side of

(6.25) should be included in the calculations of a_1'/a_1 to estimate G_0 in (7.10). However, these two terms in general cannot reliably be estimated owing to the error magnification phenomenon. This situation can be illustrated in figure 12 by comparing the sum of the estimates of (6.27) and (6.30) with the values of a_1'/a_1 obtained directly from the numerical differentiation of a_1, which itself has been calculated from (7.1) using the present difference scheme.

Since equation (7.1) was solved on a rectilinear grid, the misalignment of the curvilinear co-ordinate lines did not affect the numerical solutions of a_1. Furthermore, $a' \equiv \partial a_1/\partial x >> \partial a_1/\partial y$. Therefore the estimates of a_1'/a_1 will introduce only very small errors. On the other hand, although in the 'new' co-ordinate system the value of each term on the right-hand side of (6.25) will change, their sum will remain the same so that the very large 'error' in the first term should be balanced by those in the second and third terms. This situation can be clearly seen in figure 12. Also we emphasize that the small difference between the estimates of a_1'/a_1 and $-1/4x$ in figure 12 (and in figure 11) has provided the evidence that the location of the curved caustic had been determined very accurately.

Again, the small difference between a_1'/a_1 and $-1/4x$ will contribute to $-iG_0/(2\sqrt{\psi_1 x})$ approximately, but to further estimate G_0, the values of $\sqrt{\psi_1 x}$ should now be calculated directly from the estimates of ψ_1 and x instead of (7.19), because the errors in the approximation

$$-\frac{iG_0}{2\sqrt{\psi_1 x}} = \frac{a_1'}{a_1} + \frac{1}{4x}$$

can be small (though not very small) and on the other hand, their cancellation by those in (7.19) cannot be expected. The resulting estimates of G_0 are also shown in figure 10, which indeed approach to a constant and are not very different from those of $\overline{G_0}$.

After determinations of G_0 and ψ_1, the values of a_2/a_1 (and therefore a_2) at each point in the vicinity of the caustic can be calculated from (7.10). These values and the values of $\mathbf{k}_2$ obtained earlier in the same region can serve as the boundary conditions for calculations of the reflected wave in the region far away from the caustic. This work is just routine and therefore requires no elaboration here

8. Conclusions

When short deep-water gravity waves propagate obliquely upon a steady, two-dimensional, and irrotational current and are reflected by the latter, a second-order ordinary differential equation for the surface displacement of the short wave is derived from the Laplace equation and the kinematical and dynamical boundary conditions. This equation is similar to that derived by Shyu & Phillips (1990) but with the coefficients much more complicated than those in the latter. The regularity of this equation at the caustic is demonstrated and its uniform asymptotic solution and the corresponding WKBJ solution are subsequently derived. The satisfaction of the action conservation principle by this WKBJ solution at every point including the caustic has also been proved elsewhere.

Except the expressions of the minor terms, Shyu & Phillips' (1990) solutions and the present solutions take the forms valid even for a curved and/or unsteady caustic induced by a three-dimensional and/or unsteady irrotational flow, and also valid for waves in an intermediate-depth region, in the vicinity of the caustic. This suggestion is verified in a curvilinear co-ordinate system from considerations of the dispersion relation and the action conservation equation which themselves have been deduced by Smith (1975) in the same situation in the vicinity of the caustic. In this general situation, the minor terms in these solutions which are responsible for the difference between the amplitudes of the incident and reflected waves in the vicinity of the caustic, are solved numerically by taking advantage of the explicit forms of these solutions to avoid the error magnification phenomenon due to the singularities at the caustic. This algorithm is developed and tested in the numerical simulations of a straight and a curved caustics but its validity in the case of an unsteady caustic is also obvious.

The results of these simulations indicate that for a curved caustic, while the errors due to the misalignment of the curvilinear co-ordinate lines are magnified very seriously in the previous estimates of the amplitude of the reflected wave in the vicinity of the caustic, this situation can be improved significantly by using the present algorithm.

Finally, since the analysis in section 6 is based on the dispersion relation and the action conservation equation and since the properties of these two equations, especially those which are essential to the analysis, are common in many situations, the conclusion in section 6 about the forms of the solutions might also be drawn for the capillary blockage phenomenon (Phillips, 1981) and for waves propagating on a rotational current with uniform vorticity, or for an even more general situation. However, to verify these conjectures rigorously, the validity of these two equations in the vicinity of the caustic in these situations should be demonstrated by an extension of Smith's (1975) theory.

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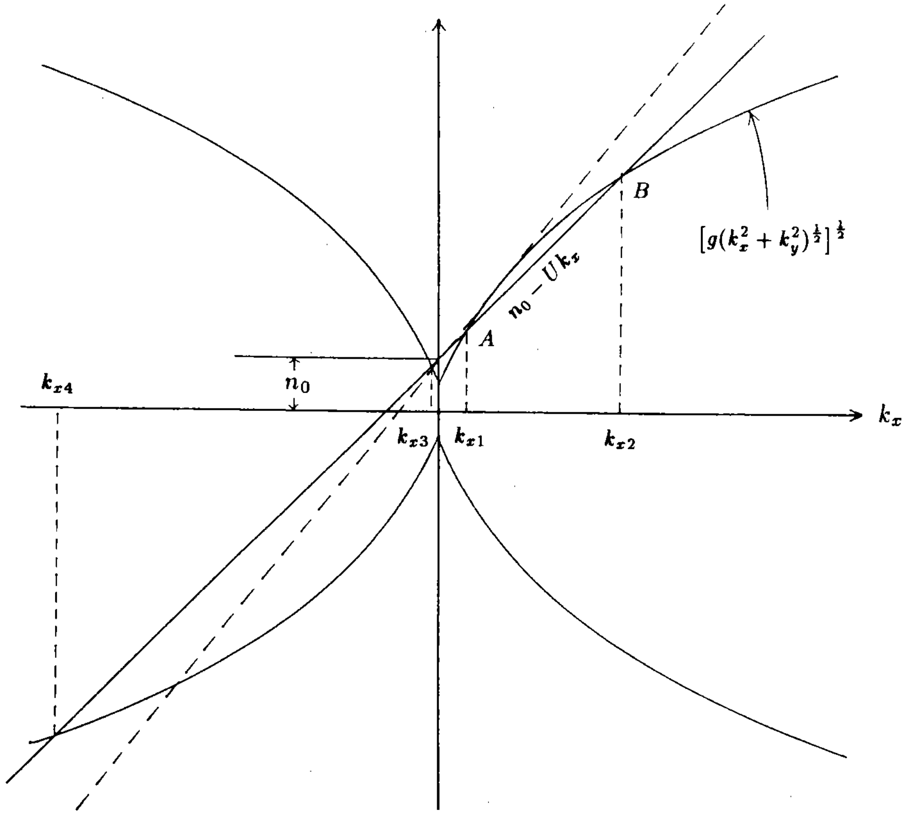

Figure 1. Solutions of the dispersion relation (2.3) for given n_0. The dashed line represents the situation occurred at the turning point where the solution points A and B coalesce and therefore $k_{x1} = k_{x2}$.

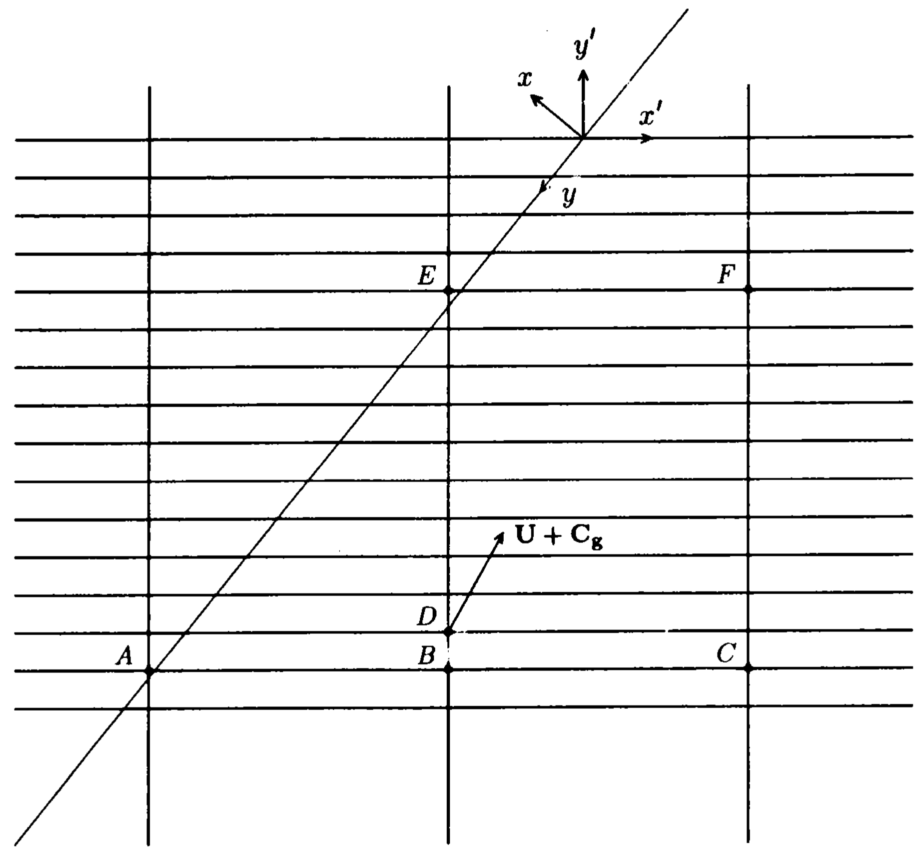

Figure 2. The strategy to locate the caustic and compute the incident wave near the caustic.

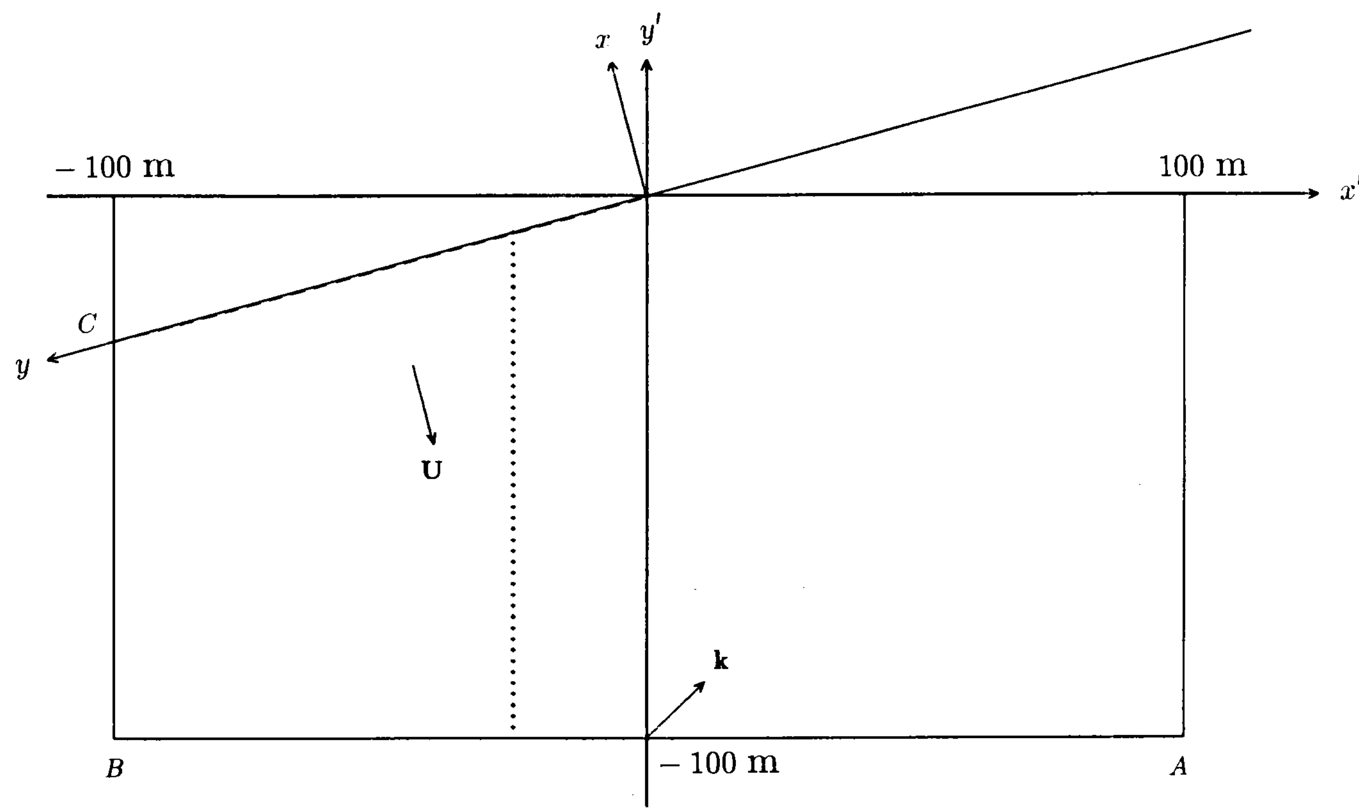

Figure 3. The domain of integration and the true (solid line) and predicated (dashed line) locations of the caustic in the simulation of a straight caustic. All test results described below are along the dotted line.

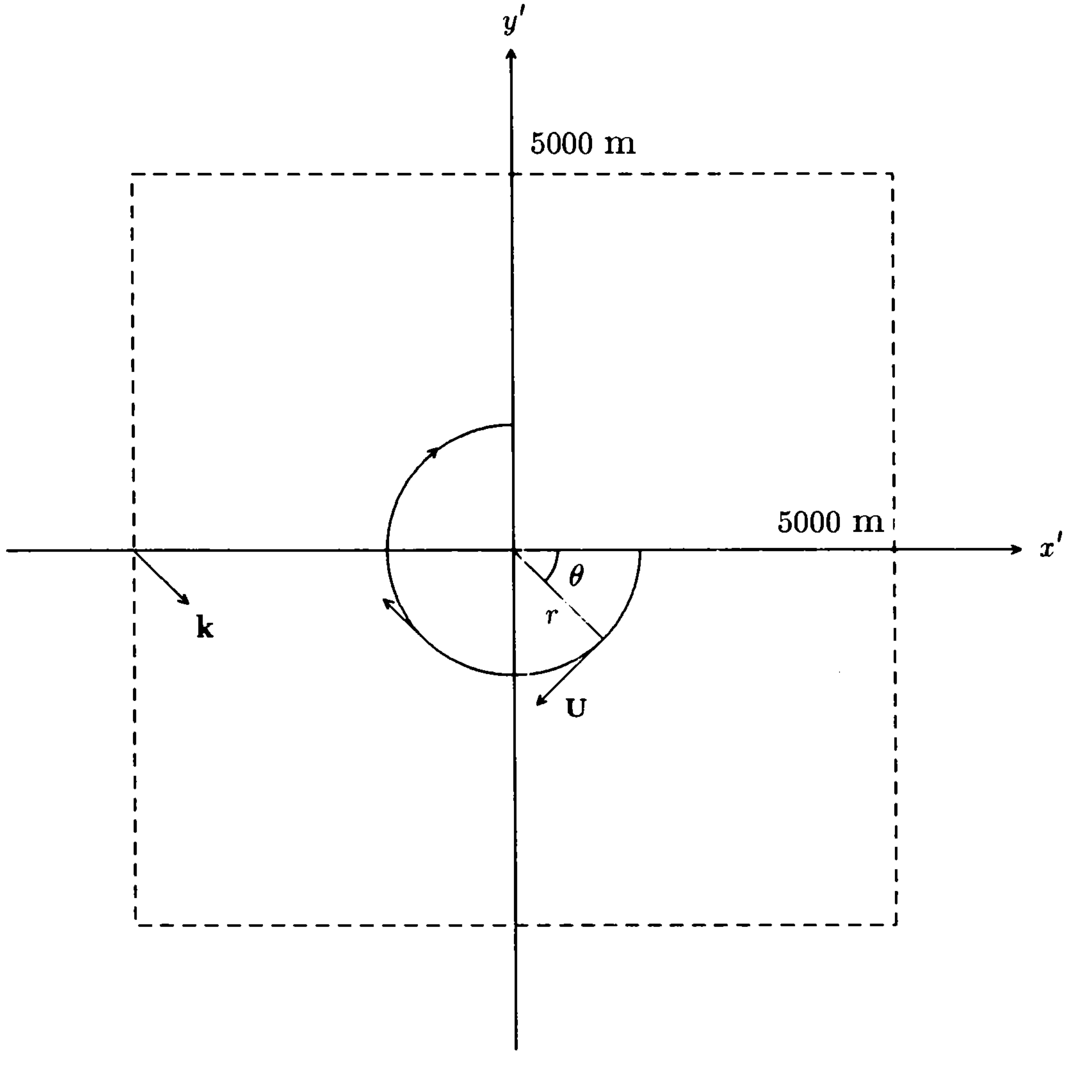

Figure 4. The domain of integration and the directions of the incoming wave and the current field in the simulation of a curved caustic.

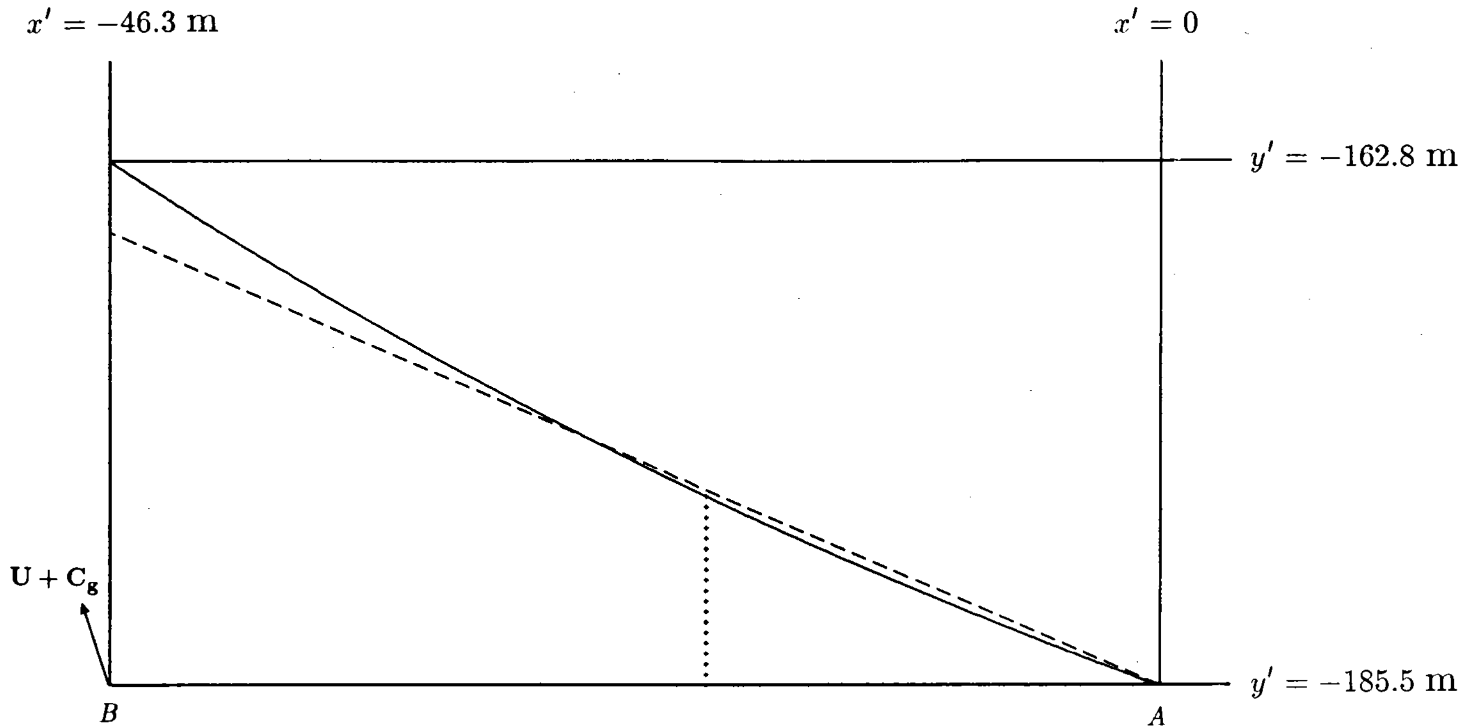

Figure 5. The part of the integration domain in figure 4 in which a caustic (solid curve) exists so that the wave reflection process is computed. All test results described below are along the dotted line.

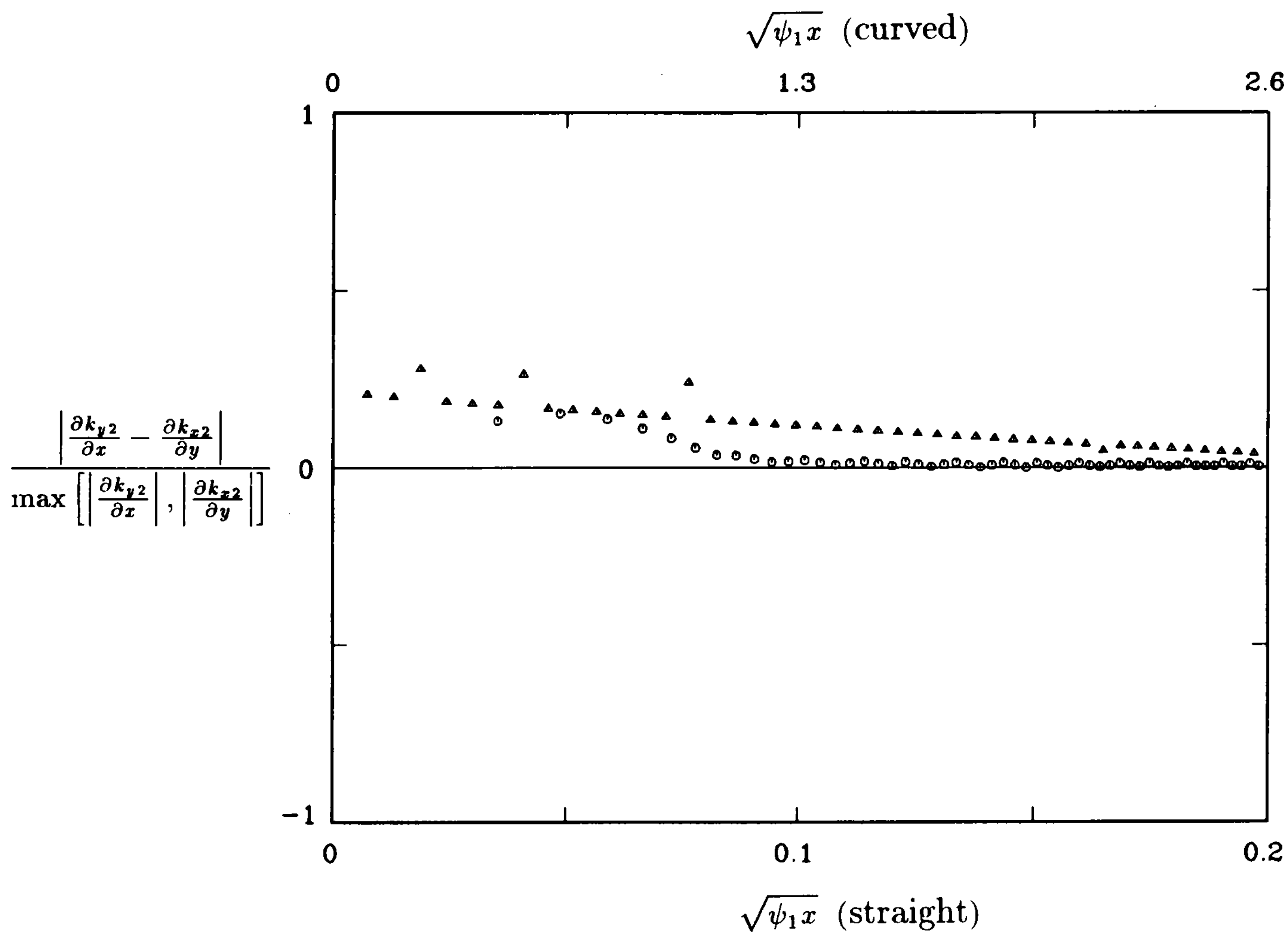

Figure 6. The vorticity of the wave-number estimates of the reflected wave in the simulations of a straight caustic (circles) and a curved caustic (triangles).

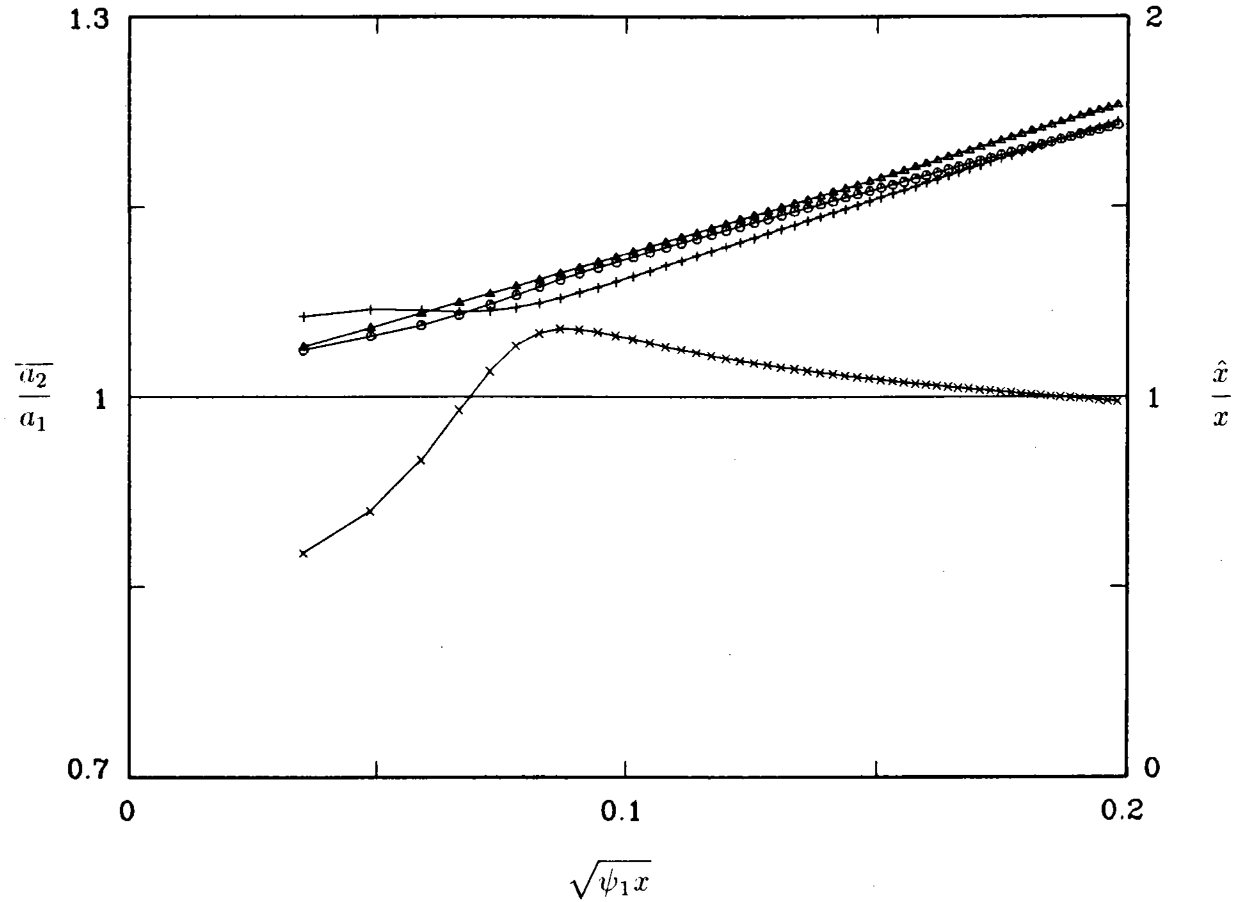

Figure 7. **Estimates of $\overline{a_2/a_1}$ and $\hat{x}/x$ in the simulation of a straight caustic,** $\Delta = (\overline{a_2/a_1})_{SM}$; $\bigcirc = (\overline{a_2/a_1})_{ST}$ (using $\hat{x}$); $+ = (\overline{a_2/a_1})_{ST}$ (using x); $\times = \hat{x}/x$.

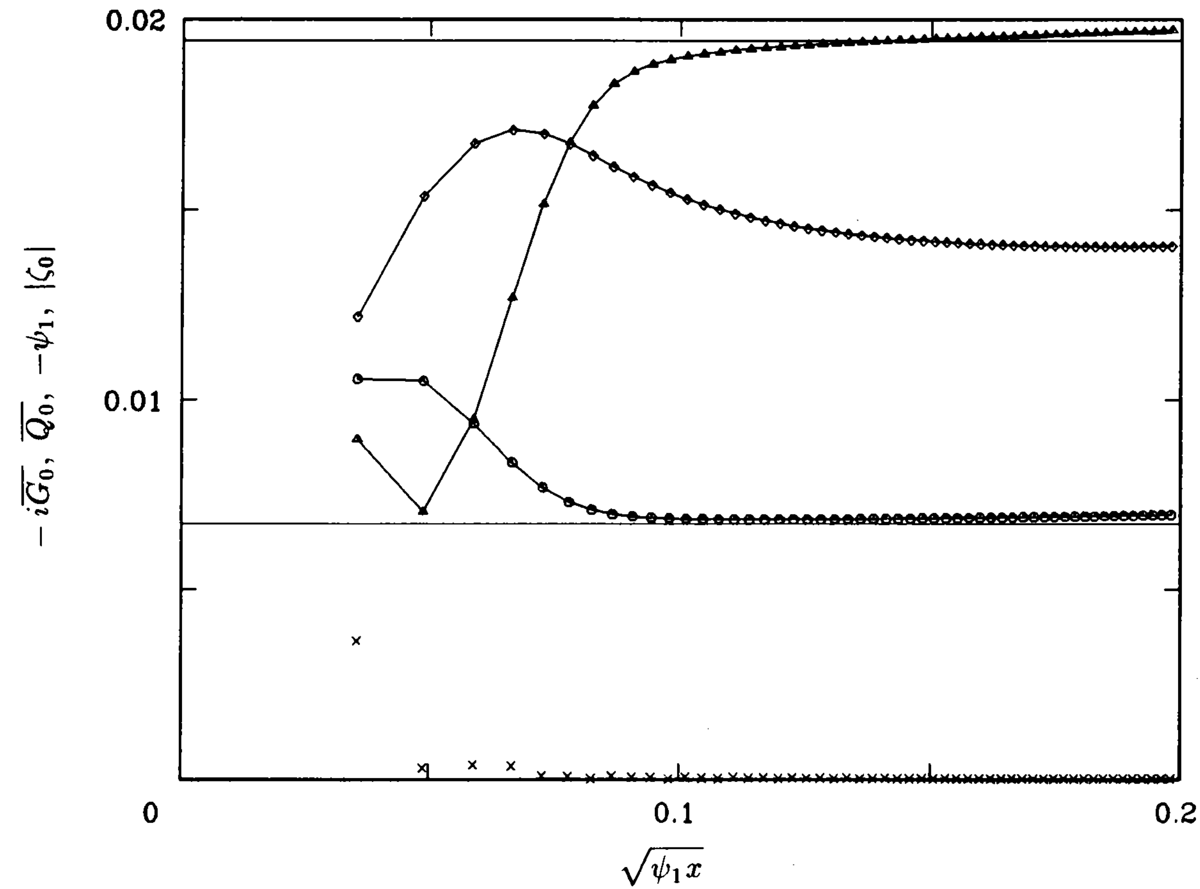

Figure 8. **Estimates of the various parameters in the simulation of a straight caustic and comparisons with the analytical solutions, $\bigcirc = \overline{G_0}/i$; $\Delta = \overline{Q_0}/i$; $\Diamond = -\psi_1$; $\times = |\zeta_0|$. The solid horizontal lines represent the analytical solutions.**

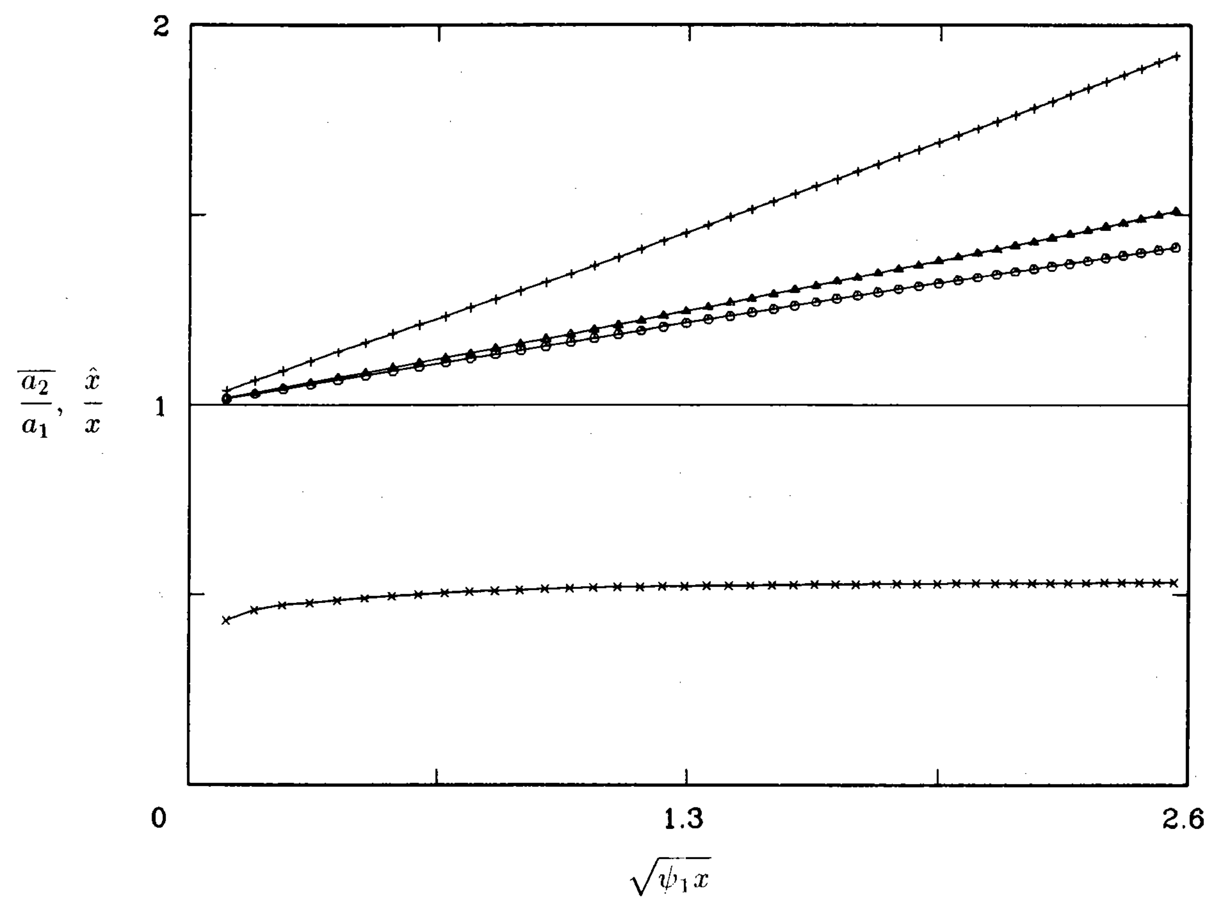

Figure 9. Estimates of $\overline{a_2/a_1}$ and $\hat{x}/x$ in the simulation of a curved caustic, $\Delta = (\overline{a_2/a_1})_{SM}$; $\bigcirc = (\overline{a_2/a_1})_{ST}$ (using $\hat{x}$); $+ = (\overline{a_2/a_1})_{ST}$ (using x); $\times = \hat{x}/x$.

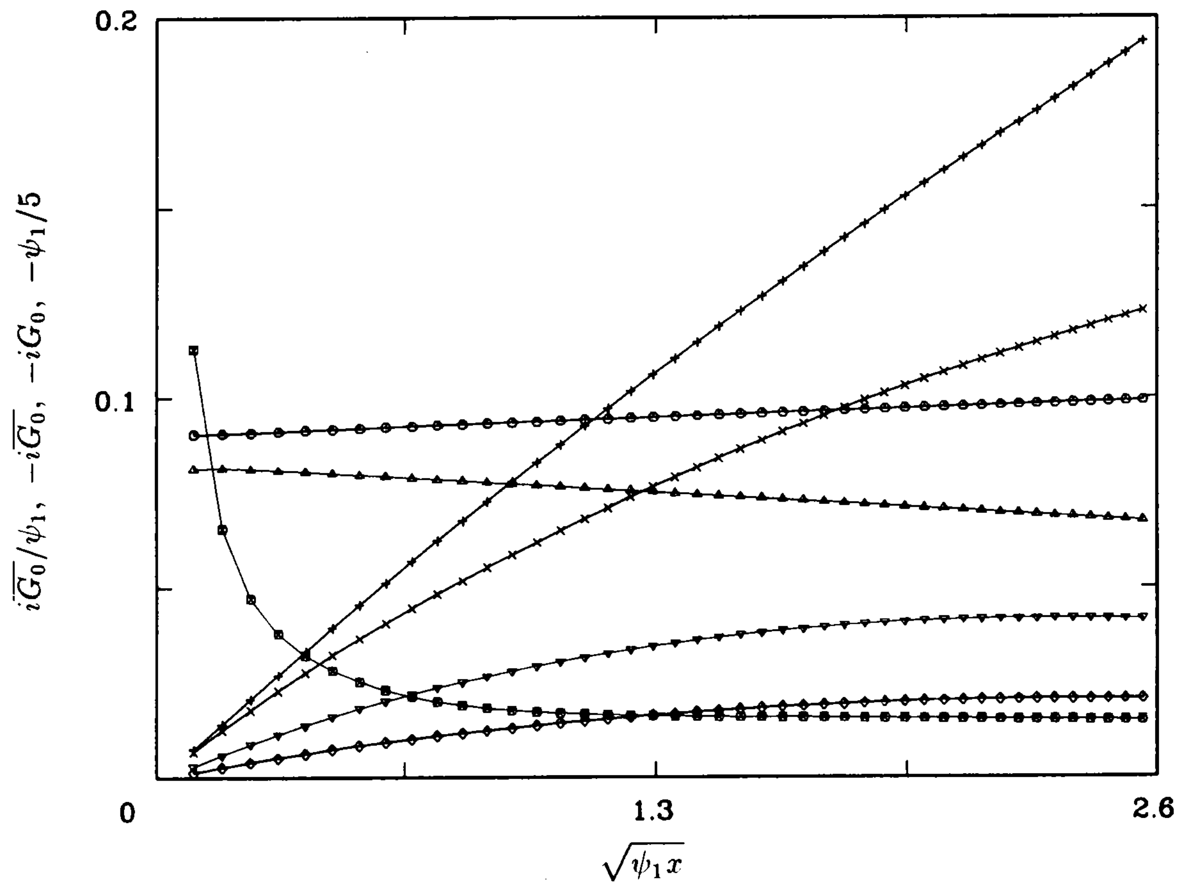

Figure 10. **Estimates of the various parameters in the simulation of a curved caustic,** $\bigcirc = i\overline{G_0}/\psi_1$ (using $(\overline{a_2/a_1})_{SM}$); $\Delta = i\overline{G_0}/\psi_1$ (using $(\overline{a_2/a_1})_{ST}$); $\times = \overline{G_0}/i$ (without using the error reducing strategy (e.r.s.)); $\diamond = \overline{G_0}/i$ (using e.r.s.); $+ = -\psi_1/5$ (without using e.r.s.); $\nabla = -\psi_1/5$ (using e.r.s.); $\otimes = G_0/i$.

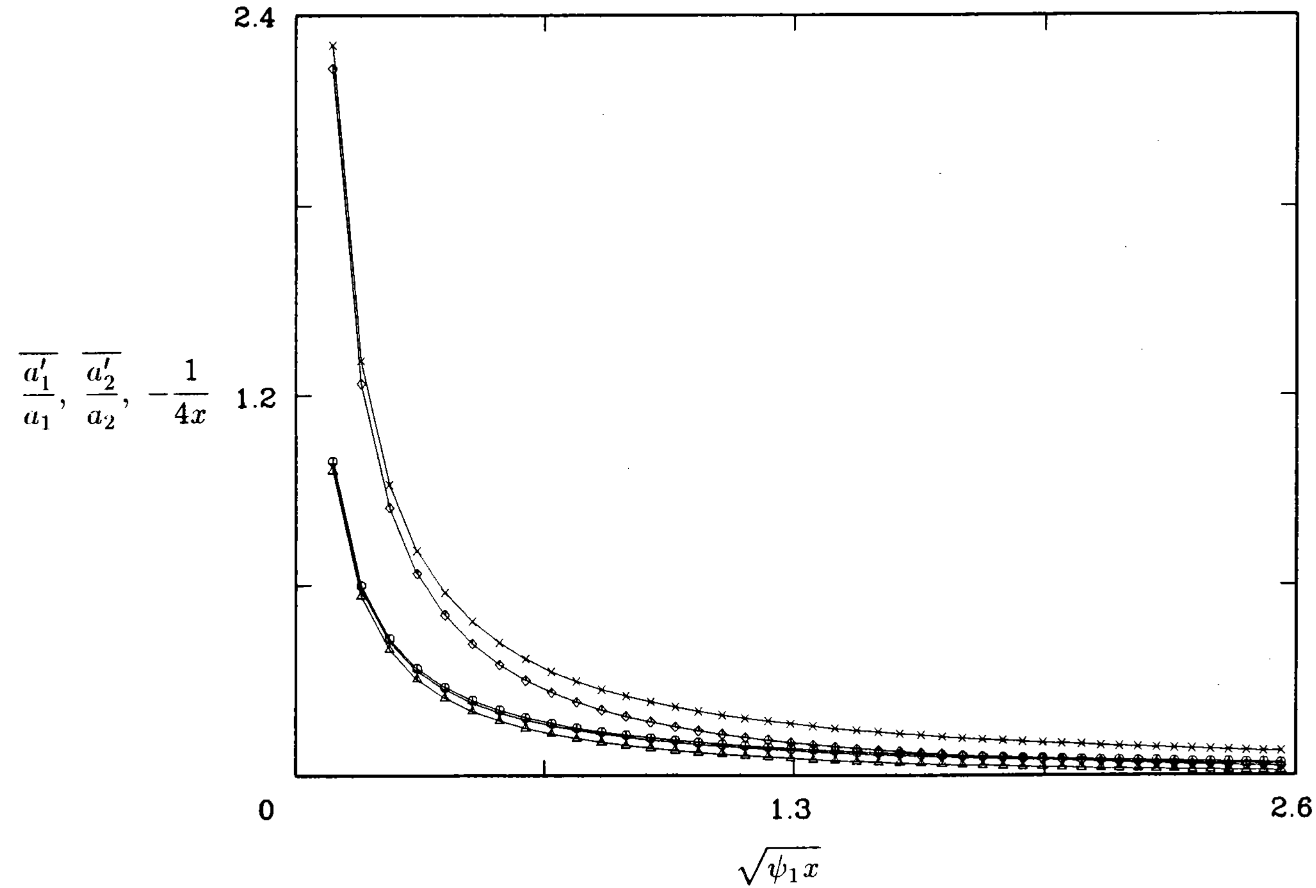

Figure 11. Estimates of $\overline{a'_1/a_1}$ and $\overline{a'_2/a_2}$ with and without using the error reducing strategy (e.r.s.) and comparison with $-1/4x$, $\times = \overline{a'_1/a_1}$ (without using e.r.s.); $\bigcirc = \overline{a'_1/a_1}$ (using e.r.s.); $\diamond = \overline{a'_2/a_2}$ (without using e.r.s.); $\Delta = \overline{a'_2/a_2}$ (using e.r.s.); $+ = -1/4x$.

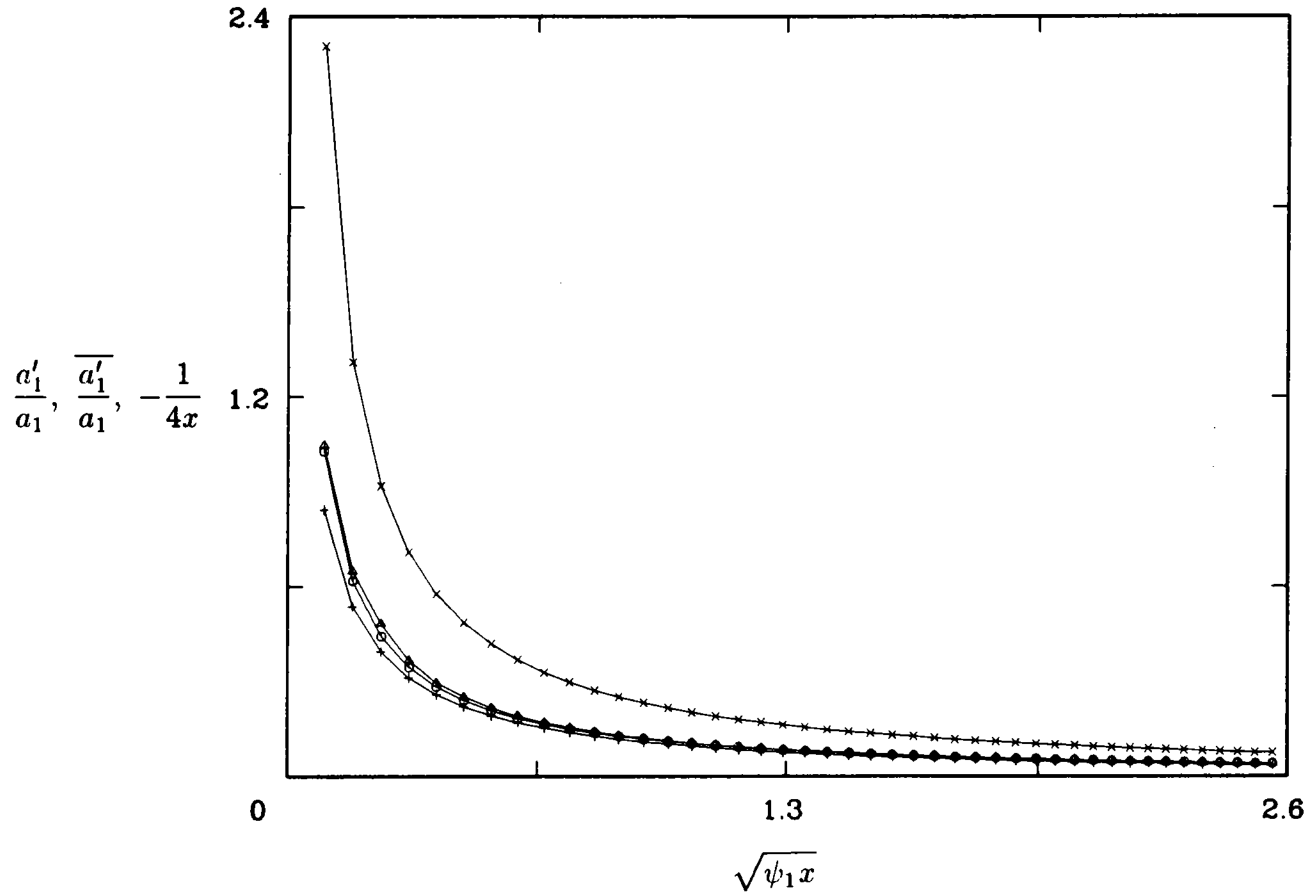

Figure 12. Comparisons between two kinds of estimates of a_1'/a_1 and between a_1'/a_1 and $-1/4x$. The circles and the triangles are the estimates of a_1'/a_1 obtained from the numerical differentiation of a_1 and from the summation of the estimates of $\overline{a_1'/a_1}$ (×) and (6.30), respectively.

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