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台中港附近波浪預報模式設計：

海流效應

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台中港附近波浪預報模式設計: 海流效應

前 言

本篇報告是將上一年度的「台中港近岸波浪預報模式研究」的報告加以修改, 並增加其內容而成。 在本年度裏我們針對各種可能進一步改善數值解準確度的方法加以研究和測試, 因而確定目前的方法乃為一最佳方案。 此外, 在本篇報告裏我們亦大幅修改其文字、公式與圖形, 使讀者更易瞭解其內容。 此項修改一部份是根據美國約翰霍普金斯大學的 Professor Owen M. Phillips 的建議, 並參考美國航空及太空總署的黃鍔教授的意見, 故作者在此特致謝忱。 本研究初期由行政院國家科學委員會提供經費, 後由台灣省政府交通處之交通建設基金支助完成, 特此誌謝。 本所楊怡芸小姐協助打字亦一併致謝。 本篇論文目前已投稿於 Journal of Fluid Mechanics.

波被流反射現象之解析解及其應用於數值計算

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摘　要

當行進於一大尺度流上的波之群速度在某一點和流速相平衡時, 波將被流阻塞於此點, 並發生反射現象。 本研究首先將 Shyu and Phillips (1990) 的理論延伸至深水重力波「斜行」於一穩定及單一方向非旋流上情況, 在此情況下, 我們應用拉普拉斯方程式及運動和動力邊界條件導出波被流反射現象之均勻漸近解及 WKBJ 解。 這些解, 除了其中屬於第二類項 (例如 (2.16) 式中的虛數項) 的表式外, 其形式和 Shyu & Phillips (1990) 的解皆相同。 另外, 我們經由考慮離散關係式和波作用守恆方程式 (兩者已由 Smith (1975) 証明可適用於焦線附近) 更証明, 即使在一個由三度空間和不穩定之非旋流所產生的彎曲及移動焦線附近, 且波爲中間水深波時, 解的形式仍和 Shyu & Phillips 的解相同, 雖然此時其第二類項的表式無法獲得, 而必須在數值計算中加以估計。

上述第二類項中的一部份將導致反射波的振幅在焦線附近和入射波的振幅不相等, 故如何在一般情況在焦線附近估計這些項的大小, 對於我們是否能在波被反射以後繼續計算其 ray solution 極爲重要。 此方法我們經由一些數值試驗加以解說, 其結果顯示, 當以往應用波作用守恆原理估計焦線附近的反射波會產生極嚴重的誤差放大現象時, 目前的方法由於有一明白形式的解可資應用, 故絕大部份誤差放大的現象可加以避免。 此項估計仍代表一漸近近似, 故其準確度被加以檢驗, 其結果令人滿意。

目　錄

Analytical solutions of the wave reflection phenomenon by currents and their application to numerical computations

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Abstract

Surface waves superimposed upon a larger-scale flow are blocked and reflected at the points where the group velocities balance the convection by the larger-scale flow. In this study, we first extend the theory of Shyu and Phillips (1990) to the situation when short deep-water gravity waves propagate *obliquely* upon a steady unidirectional irrotational current. In this case, the uniform asymptotic and the WKBJ solutions of the wave reflection phenomenon by currents are derived from the Laplace equation and the kinematical and dynamical boundary conditions. These solutions, except the expressions of the Class 2 terms (e.g., the imaginary terms in (2.16)), take the same forms as those derived by Shyu and Phillips (1990). Furthermore, from considerations of the dispersion relation and the action conservation equation, the validity of which in the vicinity of the caustic in a general situation has been verified by Smith (1975), we demonstrate that even for waves in an intermediate-depth region and near a curved moving caustic induced by an unsteady multidirectional irrotational current, the solutions still take the same forms as those in Shyu & Phillips (1990), although in this general case the expressions of the Class 2 terms in these solutions cannot be obtained so that their values must be estimated in a numerical calculation.

Some of these Class 2 terms are responsible for the amplitude of the reflected wave being unequal to that of the incident wave in the vicinity of the caustic, therefore the estimation of these terms in this region in a general situation is crucial for a continuation of the ray solution after reflection. The algorithm for this estimation is illustrated through numerical tests, which indicate that while the error magnification phenomenon is very serious in the pre-

vious estimates of the reflected wave in the vicinity of the caustic from a consideration of the action conservation principle, this phenomenon can mostly be avoided by the present algorithm by taking advantage of the explicit forms of the present solutions. These estimates still represent an asymptotic approximation so that their accuracy is examined and the results are encouraging.

1. Introduction

The modulations and reflection of short surface waves by a variable current or a long wave are of importance for predication of the wave fields in the regions with strong non-uniform currents (see, for example, Peregrine 1976; Smith 1976; Mei 1983; Holthuijsen & Tolman 1991), and for interpretation of the remote sensing records (see Phillips (1988) for a review). Modern theories on the dynamics of short waves on larger-scale currents were begun by Longuet-Higgins & Stewart (1960, 1961), Whitham (1965), and Bretherton & Garrett (1968) (see Peregrine (1976) for a review of these theories and many other developments), in which the idea of radiation stress was introduced and the action conservation equation established so that the evolution of the short waves can be determined rigorously until a caustic is met, at which these theories characteristic of a ray description all predict a singularity in the wave height and therefore are not applicable there.

The short waves blocked at the caustic will virtually be reflected at a different wavelength, which leads to a more drastic change of wave slopes. Therefore it is important to determine the amplitude of the reflected wave in terms of that of the incident wave in the vicinity of the caustic which permits us to continue the calculations of the ray solution after reflection. To achieve this goal, Smith (1975) has derived a uniform asymptotic solution of short surface gravity waves near a curved moving caustic induced by an unsteady multidirectional irrotational current. This solution can be expressed as

$$u = \{A\,\mathrm{Ai}(\rho) + iC\,\mathrm{Ai}'(\rho)\}\exp(is) \tag{1.1}$$

with

$$\left.\begin{aligned} \rho = -\left[\tfrac{3}{4}(\chi_1 - \chi_2)\right]^{\frac{2}{3}}, \quad & s = \tfrac{1}{2}(\chi_1 + \chi_2), \\ A = \pi^{\frac{1}{2}}(-\rho)^{\frac{1}{4}}(a_1 + a_2), \quad & C = \pi^{\frac{1}{2}}(-\rho)^{-\frac{1}{4}}(a_1 - a_2), \end{aligned}\right\} \tag{1.2}$$

where u denotes any instantaneous property of the waves, $\mathrm{Ai}(\rho)$ and $\mathrm{Ai}'(\rho)$ represent respectively the Airy function and its derivative, and a_1, a_2, χ_1 and χ_2 correspond to the local amplitudes and phases of the incident and reflected waves which even near the caustic have been proved by Smith (1975) to fulfill the action conservation equation and the local dispersion relation. The above unified formulae were summarized by Peregrine & Smith (1979).

In (1.2), the requirement that C remains finite and analytic at caustics implies that a_1 and a_2 have equal singularities there. This, together with the action conservation equation 'enables us to conclude that the flux of wave action normal to the caustic carried by the incident and by the reflected waves are equal and opposite' (Smith 1975). Thus, near the caustic, the amplitude of the reflected wave relative to that of the incident wave can be determined in theory. However, in the immediate vicinity of the caustic, the amplitude of the incident wave itself cannot be solved accurately from the action conservation equation by using any numerical methods (ray-tracing or gridded method), because a_1 and χ_1 are singular at the caustic. In the regions at a certain distance from the caustic, the numerical solution of a_1 becomes reliable and the difference between a_1 and a_2, though small, is not negligible. Therefore from the action conservation principle, we have in these regions,

$$(U_x + C_{gx2})\frac{a_2^2}{\sigma_2} = -(U_x + C_{gx1})\frac{a_1^2}{\sigma_1} + \cdots, \tag{1.3}$$

where σ_1 and σ_2 are the intrinsic frequencies of the incident and reflected waves, U_x and C_{gx} the x components of the local current and group velocities, and the x-axis (which might be curvilinear) is perpendicular to the caustic. In (1.3) the extra terms denoted by dots (the contents of which will become clear in section 7) are small compared with each of the two terms shown explicitly, but otherwise are not negligible in a general situation in which the convergence of the action flux in the y-direction or the local rate of change of wave action is significant. From (1.3) it follows that

$$\frac{a_2}{a_1} = \left[-\frac{(U_x + C_{gx1})/\sigma_1}{(U_x + C_{gx2})/\sigma_2}\right]^{\frac{1}{2}} + \cdots = 1 + \varepsilon, \tag{1.4}$$

in which ε again represents a small quantity, because

$$(U_x + C_{gx1})/\sigma_1 \approx -(U_x + C_{gx2})/\sigma_2$$

in these regions. The relation (1.4) permits us, in theory, to determine the amplitude of the reflected wave in terms of that of the incident in the vicinity of the caustic. However, since for both incident and reflected waves, $U_x + C_{gx} = 0$ at the caustic, we have

$$\left. \begin{aligned} |U_x + C_{gx1}| \ll |U_y + C_{gy1}| \\ |U_x + C_{gx2}| \ll |U_y + C_{gy2}| \end{aligned} \right\} \tag{1.5}$$

in the regions not too far from the caustic provided that the components of action fluxes in the y-direction are significant, which often occurs in the situation when the waves and currents are not collinear. Consequently, slight misalignment of the co-ordinate lines can cause large changes in $U_x + C_{gx1}$ and $U_x + C_{gx2}$ in opposite directions, that will produce even larger percent change in ε in (1.4), because $\varepsilon \ll 1$. Therefore a very serious error magnification phenomenon will occur in the estimates of the difference between a_1 and a_2 in these regions in a general situation if the action conservation principle is utilized directly. This phenomenon will certainly become less severe in the regions far away from the caustic, but in these regions the values of a_2/a_1 in general cannot be determined without a knowledge of a_2 itself, because in these regions the one-term asymptotic approximation of the parameter ε in (1.4) which can be inferred from the solution values of a_1 alone will become invalid. Therefore it is of practical importance to develop another theory for estimates of a_2/a_1 in the vicinity of the caustic that can avoid the error magnification phenomenon.

The blockage and reflection of short waves by currents or long waves can also occur to capillary waves with opposite characteristics as suggested by Phillips (1981). The uniformly valid solutions of this capillary blockage phenomenon were given by Shyu & Phillips (1990) and by Trulsen & Mei (1993) who even derived a uniform solution near a triple turning point at which the two kinds of reflection points coalesce to one. In these two theories, the expressions for a_1 and a_2 take an explicit form instead of being described by the action conservation equation or its equivalent, but these analyses were restricted to the case when both the wave and current are unidirectional and are in the same or opposite directions. (Shyu & Phillips' (1990) theory also requires that the underlying current is steady in a moving frame of reference.) Thus an extension of the theories to a more general situation is desired for practical applications.

In this paper we shall extend Shyu & Phillips' (1990) theory first to the case when short deep-water gravity waves propagate obliquely upon a steady unidirectional irrotational current. In this case, a second-order ordinary differential equation for the surface displacement of the short waves is again deduced from the Laplace equation and the kinematical and dynamical boundary conditions, which can be written as

$$\frac{\partial^2 \eta}{\partial x^2} + [-i(k_{x1} + k_{x2}) + Q]\frac{\partial \eta}{\partial x} + [-k_{x1}k_{x2} + P]\eta = 0. \tag{1.6}$$

The similarity between the forms of this equation and the equation (6.3) in Shyu & Phillips (1990) is remarkable, although k_1 and k_2 in the latter are replaced respectively by k_{x1} and k_{x2} in (1.6) due to the fact that k_1 and k_2 in the present case also contain k_{y1} and k_{y2} respectively, which are irrelevant to the variation of η in the $x-$direction.

In (1.6), the terms $-i(k_{x1}+k_{x2})$ and $-k_{x1}k_{x2}$ arise from the fluctuation of the waves and therefore specify their basic properties, while P and Q account for the modulations of these local properties, including the local amplitudes. Thus the terms in the coefficients of (1.6) (and in the analysis) can be divided into two classes accordingly, and if the current field and therefore the local properties of the wave trains vary slowly, one may expect that the Class 2 terms should be small compared to the Class 1 terms in the same equation (for example, $|P|$ is smaller than $|k_{x1}k_{x2}|$), but since these Class 2 terms have major effects on the modulations of wave trains, they are not negligible in the present theory.

The expressions of P and Q in the present case are much more complicated than those in Shyu & Phillips (1990), but still their regularities at the caustic will be shown in section 4. On the other hand, although k_{x1} and k_{x2} are the two branches of a doublevalued function and therefore are singular at the caustic, their singularities can apparently be cancelled out from $-i(k_{x1}+k_{x2})$ and $-k_{x1}k_{x2}$. Therefore equation (1.6) is regular at the caustic and its uniform asymptotic solution and the corresponding WKBJ solution are derived in section 5, which except the expressions of the Class 2 terms again take the same forms as those in Shyu & Phillips (1990). All of these similarities lead us to hope that even for waves in an intermediate-depth region and near a curved moving caustic induced by an unsteady multidirectional irrotational current, the forms of the solutions may still be the same. This anticipation, especially the regularities of the resulting equation and its uniform solution, will in section 6 be verified through considerations of the dispersion relation and the action conservation equation, the validity of which in the vicinity of the caustic has been demonstrated by Smith (1975) in exactly the same circumstance.

In this general situation, the expressions of the Class 2 terms in the solutions cannot be obtained, though their Taylor series expansions with center at the caustic are proved to exist in section 6. Since some of these Class 2 terms are responsible for a_2 being unequal to a_1 in the vicinity of the caustic, their estimates in this region in a numerical computation is crucial for a continuation of the ray solution after reflection. The algorithm for this estimation is developed and tested in section 7 through numerical simulations of a straight and a curved caustics, but its validity in the case of a moving caustic is also obvious. In this algorithm, by taking advantage of the explicit forms of the expressions for a_1 and a_2, the aforementioned error magnification phenomenon can mostly be avoided. These estimates still represent an asymptotic approximation so that their accuracy is examined through a comparison between the analytical and numerical solutions in the case of a straight caustic in which the expressions of the Class 2 terms exist.

2. The ordinary differential equation for a single wave component

In this section we shall derive an ordinary differential equation for short deep-water gravity waves propagating obliquely on a unidirectional irrotational current $U(x)\mathbf{i}$ where $\mathbf{i}$ denotes the unit vector in the direction of increase of x. For a gravity-capillary wave propagating in the direction parallel to this current, Shyu & Phillips (1990) have derived a third-order ordinary differential equation in the surface displacement η of the short waves. This equation was then decomposed into a second-order ordinary differential equation in which all the coefficients are regular at the caustic so that a uniformly valid solution of the wave reflection phenomenon by currents can be obtained. That this approach was successful was because in this case, expansion of the dispersion relation

$$n = (g'k + \gamma k^3)^{\frac{1}{2}} + Uk \tag{2.1}$$

takes the form

$$k^3 - \frac{U^2}{\gamma}k^2 + \frac{g' + 2nU}{\gamma}k - \frac{n^2}{\gamma} = 0,$$

which is a third-order polynomial equation in the local wavenumber k and its coefficients coincide exactly with the Class 1 terms in the coefficients of the above-mentioned third-order differential equation (see (2.18), Shyu & Phillips (1990)). In (2.1), g' is the effective gravitational acceleration suggested by Phillips (1981), n the observed frequency of the wave, and γ the ratio of surface tension to water density.

In case that the waves propagate obliquely upon the current and the effects of surface tension are neglected, the dispersion relation becomes

$$n = \left[g(k_x^2 + k_y^2)^{\frac{1}{2}}\right]^{\frac{1}{2}} + Uk_x, \tag{2.2}$$

where k_x and k_y represent respectively the x and y components of the wavenumber vector $\mathbf{k}$ while the x-axis is chosen to be exactly opposite to the current (so that in (2.2) U is always negative) and the y-axis be horizontal and perpendicular to the x-axis. Notice that if the slope and curvature of the mean free surface becomes significant, the x-axis is also curved and the gravitational acceleration g in (2.2) should be replaced by g' according to Phillips (1981), Longuet-Higgins (1985, 1987) and Henyey *et al.* (1988). An expansion of (2,2) yields

$$U^4k_x^4 - 4nU^3k_x^3 + (6n^2U^2 - g^2)k_x^2 - 4n^3Uk_x + (n^4 - g^2k_y^2) = 0, \tag{2.3}$$

which is a quartic equation. From it and from Shyu & Phillips' (1990) analysis, it is anticipated that a fourth-order ordinary differential equation is desired in order to eventually obtain a second-order equation by decomposition that can describe the reflection phenomenon as well as be uniformly valid.

To obtain such equation, certain results of the ray theory will be utilized. The latter is invalid in the immediate vicinity of the caustic, but as long as we can prove that the resulting differential equation is regular at this point (meaning that the singularities inherent in the ray solutions of the incident and reflected waves are completely offset from this equation), this equation can virtually be applicable everywhere, including the caustic.

Also we emphasize that both the uniformly valid solution and the ray solution represent the first-order approximations of asymptotic expansions and in these expansions each differentiation of the slowly varying parameters increases the order by one (see e.g. Whitham 1974). Therefore, in the following discussion, the derivatives of these parameters, except their first derivatives which corresponding to the Class 2 terms in the solutions, and the products of any two derivatives of these parameters can all be neglected, and if following this ordering, there is no need to introduce explicitly an ordering parameter in the analysis.

For a slowly varying wavetrain, the distribution of $\mathbf{k}$ is irrotational (see e.g. Phillips 1977) so that

$$\frac{\partial k_y}{\partial x} - \frac{\partial k_x}{\partial y} = 0. \tag{2.4}$$

On the other hand, since the current velocity is independent of y, we have

$$\frac{\partial k_x}{\partial y} = 0 \qquad \text{and} \qquad \frac{\partial k_y}{\partial y} = 0. \tag{2.5}$$

From (2.4) and (2.5) it immediately follows that

$$k_y = \text{constant}$$

everywhere. Next, from the kinematic conservation equation,

$$\frac{\partial \mathbf{k}}{\partial t} + \nabla n = 0.$$

Thus, if the current field is steady, we also have

$$n = \text{constant} = n_0, \quad \text{say}$$

everywhere. Therefore, the ray solutions of the surface displacement η and the velocity potential ϕ of a single wave component can now be written as

$$\eta = a(x) \exp\left[i \int k_x(x)\, dx\right] \exp i(k_y y - n_0 t), \tag{2.6}$$

and

$$\phi = A(x) \exp\left[i \int k_x(x)\, dx + \int_0^z l(x,z)\, dz\right] \exp i(k_y y - n_0 t), \tag{2.7}$$

where $a(x), A(x)$ and $k_x(x)$ vary slowly in the x-direction and $l(x,z)$ varies slowly in both $x-$ and $z-$directions. For the sake of definiteness, we here take $z = 0$ to be the mean water level. Notice that in the present case the roles played by k_y and n_0 in the solutions are quite similar.

The relation between k_x and l can be deduced from the three-dimensional Laplace equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \phi = 0. \tag{2.8}$$

Substitution of (2.7) into (2.8) yields

$$-k_x^2 + i\frac{dk_x}{dx} + 2ik_x \frac{1}{A}\frac{dA}{dx} - k_y^2 + l^2 + \frac{\partial l}{\partial z} = 0 \qquad \text{at} \qquad z = 0, \tag{2.9}$$

in which the higher-order term $(1/A)(d^2A/dx^2)$ has been neglected. In (2.9), both l and $\partial l/\partial z$ are unknown so that another equation is required for their determinations. To obtain this equation, we may reconsider the simpler case when the waves are exactly opposite the current. In this case, (2.8) reduces to

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) \phi = 0 \tag{2.10}$$

or

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial z}\right) \phi = 0.$$

In addition, from the deep-water boundary condition and the fact that the phases of oscillation of both incident and reflected waves increase in the positive x-direction (because both k_1 and k_2 are positive), it is clear that the present solution should satisfy

$$\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial z}\right)\phi = 0 \tag{2.11}$$

only, otherwise ϕ will grow exponentially as $z \to -\infty$ (for a rigorous analysis, see Shyu & Phillips 1990). Substituting (2.7) into (2.10) and (2.11) and setting $k_y = 0$ and $k_x = k$, we have at the free surface

$$-k^2 + i\frac{dk}{dx} + 2ik\frac{1}{A}\frac{dA}{dx} + l^2 + \frac{\partial l}{\partial z} = 0 \tag{2.12}$$

and

$$ik + \frac{1}{A}\frac{dA}{dx} = il. \tag{2.13}$$

Squaring both sides of (2.13), neglecting the higher-order term $(1/A)^2(dA/dx)^2$, and then subtracting the result from (2.12), we obtain

$$\left.\frac{\partial l}{\partial z}\right|_{z=0} = -i\frac{dk}{dx}.$$

The above relation involves only smaller terms so that when the waves propagate obliquely upon a unidirectional current, the small curvature of the wave crests induced in this case will impose even smaller modification upon the above relation, which is certainly negligible within the present approximation. Therefore, for the present case,

$$\left.\frac{\partial l}{\partial z}\right|_{z=0} = -i\frac{\partial k}{\partial x'}, \tag{2.14}$$

where $k = (k_x^2 + k_y^2)^{1/2}$ represents the magnitude of $\mathbf{k}$ and x' is the co-ordinate in the direction of $\mathbf{k}$. This expression in the present co-ordinate system can be written as

$$\left.\frac{\partial l}{\partial z}\right|_{z=0} = -i\frac{k_x^2}{k^2}\frac{dk_x}{dx}, \tag{2.15}$$

because k_y = constant and k_x is independent of y. Substitution in (2.9) yields

$$l^2\big|_{z=0} = k^2 - 2ik_x \frac{1}{A}\frac{dA}{dx} - i(1 - \frac{k_x^2}{k^2})\frac{dk_x}{dx},$$

and its square root is

$$l\big|_{z=0} = k - i\frac{k_x}{k}\frac{A'}{A} - \frac{i}{2}(1 - \frac{k_x^2}{k^2})\frac{k_x'}{k} \tag{2.16}$$

within the present approximation. (In the following discussion we shall use a prime to indicate differentiation with respect to x in certain circumstances.) Notice that (2.16) can be reduced to (2.13) when $k_y = 0$.

From (2.7) and (2.16) and neglecting the higher-order terms, one can obtain at $z = 0$

$$\frac{\partial \phi}{\partial x} = i\frac{k_x}{k}[1 - ic_0\frac{A'}{A} + ic_1 k_x']\frac{\partial \phi}{\partial z}, \tag{2.17a}$$

$$\frac{\partial^2 \phi}{\partial x^2} = i\frac{k_x}{k}[1 - ic_0\frac{A'}{A} + ic_2 k_x']\frac{\partial^2 \phi}{\partial x \partial z}, \tag{2.17b}$$

$$\frac{\partial^3 \phi}{\partial x^3} = i\frac{k_x}{k}[1 - ic_0\frac{A'}{A} + ic_3 k_x']\frac{\partial^3 \phi}{\partial x^2 \partial z}, \tag{2.17c}$$

$$\frac{\partial^4 \phi}{\partial x^4} = i\frac{k_x}{k}[1 - ic_0\frac{A'}{A} + ic_4 k_x']\frac{\partial^4 \phi}{\partial x^3 \partial z}, \tag{2.17d}$$

where

$$\left.\begin{aligned} c_0 &= \frac{1}{k_x}(1 - \frac{k_x^2}{k^2}), \\ c_1 &= \frac{1}{2k^2}(1 - \frac{k_x^2}{k^2}), \\ c_2 &= -\frac{1}{k_x^2}(1 - \frac{3}{2}\frac{k_x^2}{k^2} + \frac{1}{2}\frac{k_x^4}{k^4}), \\ c_3 &= -\frac{1}{k_x^2}(2 - \frac{5}{2}\frac{k_x^2}{k^2} + \frac{1}{2}\frac{k_x^4}{k^4}), \\ c_4 &= -\frac{1}{k_x^2}(3 - \frac{7}{2}\frac{k_x^2}{k^2} + \frac{1}{2}\frac{k_x^4}{k^4}). \end{aligned}\right\} \tag{2.17e}$$

These will later be applied to combine the two surface boundary conditions into one equation. Note that when $k_y = 0$, (2.17a) also reduces to (2.11).

Since the x-axis is taken in the direction exactly opposite the current and the latter itself is steady and unidirectional, the expressions for the approximate kinematic and dynamical free-surface conditions in the present case take exactly the same form as those derived in Shyu & Phillips (1990), which when $\gamma = 0$ are

$$-in_0\eta + U\eta' + \eta U' = \frac{\partial\phi}{\partial z} \qquad \text{at} \qquad z = 0, \tag{2.18}$$

$$-in_0\phi + g\eta + U\frac{\partial\phi}{\partial x} = 0 \qquad \text{at} \qquad z = 0. \tag{2.19}$$

These equations represent a linear wave approximation but otherwise are exact. If the two-scale approximation is imposed further, we have

$$(-in_0 + 2U')\eta' + U\eta'' = \frac{\partial^2\phi}{\partial x\partial z}, \tag{2.20a}$$

$$(-in_0 + 3U')\eta'' + U\eta''' = \frac{\partial^3\phi}{\partial x^2\partial z}, \tag{2.20b}$$

$$(-in_0 + 4U')\eta''' + U\eta^{IV} = \frac{\partial^4\phi}{\partial x^3\partial z}, \tag{2.20c}$$

from (2.18), and

$$(-in_0 + U')\frac{\partial\phi}{\partial x} + g\eta' + U\frac{\partial^2\phi}{\partial x^2} = 0, \tag{2.21a}$$

$$(-in_0 + 2U')\frac{\partial^2\phi}{\partial x^2} + g\eta'' + U\frac{\partial^3\phi}{\partial x^3} = 0, \tag{2.21b}$$

$$(-in_0 + 3U')\frac{\partial^3\phi}{\partial x^3} + g\eta''' + U\frac{\partial^4\phi}{\partial x^4} = 0. \tag{2.21c}$$

from (2.19). These equations can be combined into one equation in η by virtue of (2.17) to eliminate ϕ from the equations. Although there are many (actually infinite) ways to achieve this, according to Shyu & Phillips (1990), it will be more useful to derive a fourth-order ordinary differential equation with the Class 1 terms in the coefficients coinciding with those in (2.3). Therefore, we first substitute (2.17c,d) and (2.20b,c) into (2.21c), obtaining

$$U^2\eta^{IV} + \eta'''\Big[-i(2n_0U + \frac{gk}{k_x}) + 7UU' + c_0\frac{gk}{k_x}\frac{A'}{A} + (c_3 - c_4)n_0Uk_x' - c_4\frac{gk}{k_x}k_x'\Big]$$
$$+ \eta''\Big[-n_0^2 - in_0^2(c_3 - c_4)k_x' - 6in_0U'\Big] = 0. \tag{2.22}$$

Next, substitution of (2.17b,c) and (2.20a,b) into (2.21b) yields

$$U^2\eta''' + \eta''\Big[-i(2n_0U + \frac{gk}{k_x}) + 5UU' + c_0\frac{gk}{k_x}\frac{A'}{A} + (c_2 - c_3)n_0Uk_x' - c_3\frac{gk}{k_x}k_x'\Big]$$

$$+ \eta'\Big[-n_0^2 - in_0^2(c_2 - c_3)k_x' - 4in_0U'\Big] = 0.$$

Multiplying it by $-2in_0/U$ and igk/U^2k_x separately, we have

$$-2in_0U\eta''' - \eta''\Big[\frac{2n_0}{U}(2n_0U + \frac{gk}{k_x}) + 10in_0U' + c_0\frac{2in_0}{U}\frac{gk}{k_x}\frac{A'}{A} + 2i(c_2 - c_3)n_0^2k_x'$$

$$- c_3\frac{2in_0}{U}\frac{gk}{k_x}k_x'\Big] - \eta'\Big[-\frac{2in_0^3}{U} + (c_2 - c_3)\frac{2n_0^3}{U}k_x' + 8n_0^2\frac{U'}{U}\Big] = 0 \tag{2.23}$$

and

$$i\frac{gk}{k_x}\eta''' + \eta''\Big[\frac{gk}{U^2k_x}(2n_0U + \frac{gk}{k_x}) + 5i\frac{gk}{k_x}\frac{U'}{U} + i\frac{c_0}{U^2}(\frac{gk}{k_x})^2\frac{A'}{A} + i(c_2 - c_3)\frac{n_0}{U}\frac{gk}{k_x}k_x'$$

$$- i\frac{c_3}{U^2}(\frac{gk}{k_x})^2k_x'\Big] + \eta'\Big[-i\frac{n_0^2}{U^2}\frac{gk}{k_x} + (c_2 - c_3)\frac{n_0^2}{U^2}\frac{gk}{k_x}k_x' + 4\frac{n_0}{U^2}\frac{gk}{k_x}U'\Big] = 0 \tag{2.24}$$

respectively. Also, substituting (2.17a,b), (2.18) and (2.20a) into (2.21a) and multiplying the result by $-n_0^2/U$, we obtain

$$-n_0^2\eta'' + \eta'\Big[i\frac{n_0^2}{U^2}(2n_0U + \frac{gk}{k_x}) - 3n_0^2\frac{U'}{U} - c_0\frac{n_0^2}{U^2}\frac{gk}{k_x}\frac{A'}{A} - (c_1 - c_2)\frac{n_0^3}{U}k_x'$$

$$+ c_2\frac{n_0^2}{U^2}\frac{gk}{k_x}k_x'\Big] + \eta\Big[\frac{n_0^4}{U^2} + i(c_1 - c_2)\frac{n_0^4}{U^2}k_x' + 2i\frac{n_0^3}{U^2}U'\Big] = 0. \tag{2.25}$$

Since from (2.17e),

$$c_1 - c_2 = c_2 - c_3 = c_3 - c_4 = \frac{k_y^2}{k^2k_x^2}, \tag{2.26}$$

the sum of (2.22)–(2.25) can be written as

$$U^2\eta^{IV} + \eta'''\Big[-4in_0U + 7UU' + c_0\frac{gk}{k_x}\frac{A'}{A} + (c_3 - c_4)n_0Uk_x' - c_4\frac{gk}{k_x}k_x'\Big]$$

$$+ \eta''\Big\{-6n_0^2 + (\frac{gk}{Uk_x})^2 - (16in_0 - 5i\frac{gk}{Uk_x})U' + ic_0\frac{gk}{Uk_x}(\frac{gk}{Uk_x} - 2n_0)\frac{A'}{A}$$

$$+ \Big[-3i(c_2 - c_3)n_0^2 - ic_3(\frac{gk}{Uk_x})^2 + i(c_2 + c_3)n_0\frac{gk}{Uk_x}\Big]k_x'\Big\}$$

$$+ \eta'\Big[\frac{4in_0^3}{U} - 11n_0^2\frac{U'}{U} + 4\frac{n_0}{U}\frac{gk}{Uk_x}U' - c_0\frac{n_0^2}{U}\frac{gk}{Uk_x}\frac{A'}{A} - 3(c_2 - c_3)\frac{n_0^3}{U}k_x'$$

$$+ (2c_2 - c_3)\frac{n_0^2}{U}\frac{gk}{Uk_x}k_x'\Big] + \eta\Big[\frac{n_0^4}{U^2} + i(c_1 - c_2)\frac{n_0^4}{U^2}k_x' + 2i\frac{n_0^3}{U^2}U'\Big] = 0. \tag{2.27}$$

Notice that if (2.6) is substituted into all the above ordinary differential equations in η, from the Class 1 terms of the resulting equations, one can always obtain the dispersion relation (2.2). This is because the local dispersion relation can essentially be achieved from the zeroth-order equation in the hierarchy arising from an asymptotic expansion. Also we note that all of the above equations in η can specify the variation of a single wave component, including the modulation of its amplitude, in the regions away from the caustic. However, in order to eventually obtain a uniformly valid second-order ordinary differential equation which can thus describe the wave reflection phenomenon by currents, we shall in the following restrict our consideration only to (2.27).

The Class 1 terms in the coefficients of (2.27) indeed coincide with the coefficients of the quartic equation (2.3) if the operator $\partial/\partial x$ is replaced by ik_x at certain places in (2.27). The Class 2 terms in (2.27) can further be reduced and written in terms of U'. To achieve this goal, the following relations should be applied:

$$k_x' = -\frac{k_x}{V}U', \tag{2.28}$$

$$V = U + C_{gx} = U + \frac{1}{2}(gk)^{\frac{1}{2}}\frac{k_x}{k^2}, \tag{2.29}$$

$$A = -i\frac{(gk)^{\frac{1}{2}}}{k}a, \tag{2.30}$$

$$\frac{a'}{a} = \frac{A'}{A} + \frac{k_x k_x'}{2k^2}, \tag{2.31}$$

which can be derived straightforwardly from the dispersion relation (2.2) and the boundary condition (2.18) as well as (2.16). In (2.30), all terms containing a', k_x', and U' have been neglected, because in the present analysis, the relation (2.30) will always be substituted into the Class 2 terms so that the terms neglected will contribute to the even higher order terms only.

Using (2.28)–(2.31), replacing η'' with

$$\left[2ik_x\frac{A'}{A} + ik_x'\left(1 + \frac{k_x^2}{k^2}\right) - k_x^2\right]\eta$$

when necessary, and neglecting the higher order terms, (2.27) may finally be reduced to

$$U^2\eta^{IV} + \eta'''\left[-4in_0U\right] + \eta''\left[-6n_0^2 + \frac{g^2}{U^2}\right] + \eta'\left[4i\frac{n_0^3}{U}\right]$$
$$+\eta\left[\frac{n_0^4}{U^2} - \frac{g^2k_y^2}{U^2} + iU'(-6\frac{gkk_x}{U} + 2\frac{(gk)^{\frac{3}{2}}}{U^2} + 4\frac{g^{\frac{3}{2}}k_x^2}{U^2k^{\frac{1}{2}}})\right] = 0. \tag{2.32}$$

Since in deriving (2.32), only a single wave component is under consideration and k_x cannot be eliminated from (2.32) through any further transfer of terms, it is apparent that this equation can truly describe only one individual component (though from the Class 1 terms it seems to have four independent solutions corresponding to the k_{x1}, k_{x2}, k_{x3} and k_{x4} components in figure 1) and is singular at the caustic. Nevertheless, this equation will later be decomposed into a first-order differential equation, which either for the incident or for the reflected wave is again singular at the caustic, but a combination of them can result in a uniformly valid second-order ordinary differential equation.

3. The equation coupling the incident and reflected waves

The technique for decomposing a higher-order differential equation into a lower-order one in a general asymptotic analysis was first given by Turrittin (1952) (also see Wasow 1985). For the special case to decompose (2.32), one may refer to Shyu & Phillips (1990). Since each time the procedure can decrease the order of equation only by one, this procedure must be conducted three times. The result is

$$\eta' - \eta[ik_{x1} + iR_1] = 0, \tag{3.1}$$

where

$$R_1 = \frac{1}{k_{x1} - k_{x2}}(\hat{P}_1 + ik_{x1}\hat{Q}_1 + ik'_{x1}), \tag{3.2}$$

$$\begin{aligned}\hat{P}_1 =& \frac{1}{(k_{x3} - k_{x1})(k_{x3} - k_{x2})}\left\{\frac{-b_1}{(k_{x2} - k_{x4})(k_{x4} - k_{x1})}\Big[k_{x3}k_{x4} - (k_{x1} + k_{x2})(k_{x3} + k_{x4})\right.\\ &+ k_{x1}k_{x2} + (k_{x1}^2 + k_{x2}^2)\Big] + i\frac{k_{x3} - k_{x2}}{k_{x4} - k_{x1}}(k_{x3} + k_{x4} - 2k_{x1})k_{x2}k'_{x1}\\ &\left. + i\frac{k_{x3} - k_{x1}}{k_{x4} - k_{x2}}(k_{x3} + k_{x4} - 2k_{x2})k_{x1}k'_{x2}\right\},\end{aligned} \tag{3.3}$$

$$\begin{aligned}\hat{Q}_1 =& \frac{1}{(k_{x3} - k_{x1})(k_{x3} - k_{x2})}\left\{\frac{ib_1}{(k_{x2} - k_{x4})(k_{x4} - k_{x1})}(k_{x3} + k_{x4} - k_{x2} - k_{x1}) - \frac{k_{x3} - k_{x2}}{k_{x4} - k_{x1}}\right.\\ &\left.\cdot (k_{x3} + k_{x4} - 2k_{x1})k'_{x1} - \frac{k_{x3} - k_{x1}}{k_{x4} - k_{x2}}(k_{x3} + k_{x4} - 2k_{x2})k'_{x2}\right\},\end{aligned} \tag{3.4}$$

and

$$b_1 = iU'\left[6\frac{gk_1k_{x1}}{U^3} - 2\frac{(gk_1)^{\frac{3}{2}}}{U^4} - 4\frac{g^{\frac{3}{2}}k_{x1}^2}{U^4k_1^{\frac{1}{2}}}\right]. \tag{3.5}$$

Since the four eigenvalues of the matrix composed of the Class 1 terms in the coefficients of (2.32) are identical with the four roots k_{x1}, k_{x2}, k_{x3} and k_{x4} of (2.3) multiplied by $i \equiv \sqrt{-1}$, these four wavenumber components have entered into (3.1) symmetrically.

To obtain (3.1), the parameter k_x in (2.32) has been fixed as the wavenumber component k_{x1} of the incident wave such that (3.1), including its Class 1 and Class 2 terms, can truly describe the incident wave in the regions away from the caustic. The corresponding equation for the reflected wave can directly be obtained from interchange of k_{x1} and k_{x2} in (3.1)–(3.4) and from replacement of k_1 and k_{x1} by k_2 and k_{x2} respectively in (3.5), giving

$$\eta' - \eta[ik_{x2} + iR_2] = 0 \tag{3.6}$$

with

$$R_2 = \frac{1}{k_{x2} - k_{x1}}(\hat{P}_2 + ik_{x2}\hat{Q}_2 + ik'_{x2}), \tag{3.7}$$

where $\hat{P}_2$ and $\hat{Q}_2$ take the same forms as (3.3) and (3.4) except that b_1 is replaced by

$$b_2 = iU'\left[6\frac{gk_2k_{x2}}{U^3} - 2\frac{(gk_2)^{\frac{3}{2}}}{U^4} - 4\frac{g^{\frac{3}{2}}k_{x2}^2}{U^4k_2^{\frac{1}{2}}}\right]. \tag{3.8}$$

Both (3.1) and (3.6) are singular at the caustic where $k_{x1} = k_{x2}$ (see figure 1), which is not unexpected as the reflection phenomenon cannot be described by a first-order differential equation. Nevertheless, a combination of them into a second-order equation can couple the incident wave with the reflected wave and in the meantime cancel out the singularities from the equation. Note that during the decomposition we have already obtained a second-order equation before (3.1) was reached, but this equation cannot describe the incident and reflected waves simultaneously, therefore is singular at the caustic.

Intuitively, we may combine (3.1) with (3.6) as

$$\left\{\frac{\partial}{\partial x} - i[k_{x2} + R_2]\right\}\left\{\frac{\partial}{\partial x} - i[k_{x1} + R_1]\right\}\eta = 0, \tag{3.9}$$

but it can describe only the k_1 component, because the coefficient k_{x1} in (3.9) is not constant (the differentiation of the Class 2 term R_1 with respect to x is however negligible within the present approximation). An expansion of (3.9) yields

$$\eta'' - \left[ik_{x1} + ik_{x2} + i(R_1 + R_2)\right]\eta' - \left[k_{x1}k_{x2} + (k_{x2}R_1 + k_{x1}R_2) + ik'_{x1}\right]\eta = 0, \tag{3.10}$$

which is obviously not symmetric with respect to k_{x1} and k_{x2}. To solve this problem, we add, from (3.1) and neglecting the products of two Class 2 terms,

$$\frac{k'_{x1} - k'_{x2}}{k_{x2} - k_{x1}}\eta' - ik_{x1}\frac{k'_{x1} - k'_{x2}}{k_{x2} - k_{x1}}\eta = 0 \tag{3.11}$$

to (3.10), resulting in

$$\eta'' + [-i(k_{x1} + k_{x2}) + Q]\eta' + [-k_{x1}k_{x2} + P]\eta = 0, \tag{3.12}$$

where

$$\left.\begin{aligned} P &= -(k_{x2}R_1 + k_{x1}R_2) - i\frac{k_{x2}k'_{x1} - k_{x1}k'_{x2}}{k_{x2} - k_{x1}}, \\ Q &= -i(R_1 + R_2) + \frac{k'_{x1} - k'_{x2}}{k_{x2} - k_{x1}}. \end{aligned}\right\} \tag{3.13}$$

Since (3.10) and (3.11) can both be fulfilled by the k_1 component and on the other hand, (3.12) together with (3.13) is symmetric with respect to k_{x1} and k_{x2}, it is obvious that the two independent solutions of (3.12) will correspond to the k_1 and k_2 components. Furthermore, in the following section we shall prove that all singularities can be cancelled out from (3.12) so that as a multiple-scale asymptotic approximation, this equation can virtually be valid everywhere including the caustic. Remark that in view of (3.2)–(3.5), (3.7), (3.8) and (3.13), the parameters P and Q in (3.12) again represent the Class 2 terms of the coefficients.

4. Proof of regularity

The sufficient condition for (3.12) being regular at the caustic is that the coefficients in (3.12) are all regular at this point. Since k_{x1} and k_{x2} are a pair of the solutions of the quartic equation (2.3) which become identical with each other at the simple turning point (the following argument will not hold for a triple turning point suggested and investigated by Trulsen & Mei (1993)), they can be divided into two parts:

$$k_{x1} = M - N, \qquad k_{x2} = M + N, \tag{4.1}$$

where N and $-N$ are two branches of a doublevalued function, say $\psi^{1/2}$, which equals zero at the caustic, and M and ψ are both regular at this point. From the above,

$$N^2 = \psi, \qquad NN' = \frac{1}{2}\psi', \tag{4.2}$$

so that N^2, NN', N^4, N^3N', etc. are all regular at the caustic. Therefore we shall in the following prove that when (4.1) are substituted in (3.12) and (3.13), only this kind of terms and the terms devoid of N can survive cancellation.

First, from (4.1),

$$k_{x1} + k_{x2} = 2M, \qquad k_{x1}k_{x2} = M^2 - N^2.$$

Therefore the Class 1 terms in (3.12) are obviously regular. Next, from (3.13), (3.2) and (3.7), we have

$$\left.\begin{aligned} P &= \frac{-1}{k_{x1} - k_{x2}}\left[(k_{x2}\hat{P}_1 - k_{x1}\hat{P}_2) + ik_{x1}k_{x2}(\hat{Q}_1 - \hat{Q}_2)\right], \\ Q &= \frac{-1}{k_{x1} - k_{x2}}\left[i(\hat{P}_1 - \hat{P}_2) - (k_{x1}\hat{Q}_1 - k_{x2}\hat{Q}_2)\right]. \end{aligned}\right\} \tag{4.3}$$

Recall that the only difference between $\hat{P}_1$ and $\hat{P}_2$ or between $\hat{Q}_1$ and $\hat{Q}_2$ is that the former involves b_1 while the latter involves b_2 in (3.3) or (3.4). Thus we have

$$\hat{P}_1 - \hat{P}_2 = \frac{1}{L}\left[-k_{x3}k_{x4} + (k_{x1} + k_{x2})(k_{x3} + k_{x4}) - k_{x1}k_{x2} - (k_{x1}^2 + k_{x2}^2)\right](b_1 - b_2), \tag{4.4a}$$

$$\hat{Q}_1 - \hat{Q}_2 = \frac{i}{L}(k_{x3} + k_{x4} - k_{x2} - k_{x1})(b_1 - b_2), \tag{4.4b}$$

$$k_{x2}\hat{P}_1 - k_{x1}\hat{P}_2 = \frac{1}{L}\Big\{\Big[-k_{x3}k_{x4} + (k_{x1}+k_{x2})(k_{x3}+k_{x4}) - k_{x1}k_{x2} - (k_{x1}^2+k_{x2}^2)\Big](k_{x2}b_1 - k_{x1}b_2)$$

$$+ i(k_{x1}-k_{x2})\Big[(k_{x3}+k_{x4}-2k_{x1})(k_{x3}-k_{x2})(k_{x4}-k_{x2})k_{x2}k'_{x1}$$

$$+ (k_{x3}+k_{x4}-2k_{x2})(k_{x3}-k_{x1})(k_{x4}-k_{x1})k_{x1}k'_{x2}\Big]\Big\}, \qquad (4.4c)$$

and

$$k_{x1}\hat{Q}_1 - k_{x2}\hat{Q}_2 = \frac{1}{L}\Big\{i(k_{x3}+k_{x4}-k_{x2}-k_{x1})(k_{x1}b_1 - k_{x2}b_2) + (k_{x1}-k_{x2})\Big[(k_{x3}+k_{x4}-2k_{x1})$$

$$\cdot (k_{x3}-k_{x2})(k_{x4}-k_{x2})k'_{x1} + (k_{x3}+k_{x4}-2k_{x2})(k_{x3}-k_{x1})(k_{x4}-k_{x1})k'_{x2}\Big]\Big\}, \qquad (4.4d)$$

where

$$L = (k_{x3}-k_{x1})(k_{x3}-k_{x2})(k_{x2}-k_{x4})(k_{x4}-k_{x1}). \qquad (4.4e)$$

From (3.5) and (3.8) and by using (2.2), we also have

$$b_1 - b_2 = i(k_{x1}-k_{x2})\frac{U'}{U^2}\Big\{\frac{12}{U}n_0^2 + 8UT - 18n_0(k_{x1}+k_{x2}) - \frac{1}{k_1^2k_2^2}\Big[4k_y^2\frac{n_0^3}{U^2}(k_{x1}+k_{x2})$$

$$+ 12n_0k_y^2S - 12k_y^2\frac{n_0^2}{U}T - 4Uk_y^2(k_{x1}^4+k_{x2}^4) - 4Uk_y^2k_{x1}k_{x2}T$$

$$+ 12n_0k_{x1}^2k_{x2}^2(k_{x1}+k_{x2}-\frac{n_0}{U}) - 4Uk_{x1}^2k_{x2}^2T\Big]\Big\}, \qquad (4.5a)$$

$$k_{x2}b_1 - k_{x1}b_2 = i(k_{x1}-k_{x2})\frac{U'}{U^2}\Big\{8Uk_{x1}k_{x2}(k_{x1}+k_{x2}) - 18n_0k_{x1}k_{x2} + 2\frac{n_0^3}{U^2} - \frac{1}{k_1^2k_2^2}$$

$$\cdot \Big[4k_y^2\frac{n_0^3}{U^2}k_{x1}k_{x2} + 12n_0k_y^2k_{x1}k_{x2}T - 12k_y^2\frac{n_0^2}{U}k_{x1}k_{x2}(k_{x1}+k_{x2}) - 4Uk_y^2k_{x1}k_{x2}S$$

$$- 4\frac{n_0^3}{U^2}k_{x1}^2k_{x2}^2 + 12n_0k_{x1}^3k_{x2}^3 - 4Uk_{x1}^3k_{x2}^3(k_{x1}+k_{x2})\Big]\Big\}, \qquad (4.5b)$$

and

$$k_{x1}b_1 - k_{x2}b_2 = i(k_{x1}-k_{x2})\frac{U'}{U^2}\Big\{12\frac{n_0^2}{U}(k_{x1}+k_{x2}) + 8US - 18n_0T - 2\frac{n_0^3}{U^2} - \frac{1}{k_1^2k_2^2}$$

$$\Big[4k_y^2\frac{n_0^3}{U^2}T + 12n_0k_y^2(k_{x1}^4+k_{x2}^4) + 12n_0k_y^2k_{x1}k_{x2}T - 12k_y^2\frac{n_0^2}{U}S + 4\frac{n_0^3}{U^2}k_{x1}^2k_{x2}^2$$

$$- 4Uk_y^2(k_{x1}^3+k_{x2}^3)T + 12n_0k_{x1}^2k_{x2}^2T - 12\frac{n_0^2}{U}k_{x1}^2k_{x2}^2(k_{x1}+k_{x2}) - 4Uk_{x1}^2k_{x2}^2S\Big]\Big\}, \qquad (4.5c)$$

where

$$\left.\begin{aligned} S &= (k_{x1} + k_{x2})(k_{x1}^2 + k_{x2}^2), \\ T &= k_{x1}^2 + k_{x2}^2 + k_{x1}k_{x2}. \end{aligned}\right\} \tag{4.5d}$$

Therefore, from (4.4) and (4.5) it is clear that the denominator $k_{x1} - k_{x2}$ in (4.3) can be eliminated from both P and Q, after which they become symmetric about k_{x1} and k_{x2} so that when (4.1) is employed, all terms containing odd power of N (including N') will be cancelled out, ensuring that (3.12) is regular at the turning point. Hence one may expect that (3.12) is uniformly valid near and away from the turning point.

The values of P and Q at the caustic can be calculated exactly by using (4.3), (4.4) and (4.5), but in order to do that, it is necessary to substitute (4.1) into (4.4c) and (4.4d) to eliminate k'_{x1} and k'_{x2} (which become infinite at the caustic) in favour of M' and NN'. Both M' and NN' are regular at the caustic as mentioned before, and since at this point,

$$U + C_{gx1} = U + C_{gx2} = 0,$$

it is not difficult to obtain from (2.28), (2.29) and (2.2)

$$NN' = \frac{n_0 - U_0 M_0}{n_0 + 2U_0 M_0} \frac{M_0^2}{U_0} U', \tag{4.6}$$

$$M' = \left[\frac{1}{3}\left(\frac{n_0}{U_0} - 8M_0\right)\frac{M_0}{n_0 + 2U_0M_0} + \frac{2}{3}\frac{M_0}{U_0}\frac{n_0^2 - U_0^2M_0^2}{(n_0 + 2U_0M_0)^2}\right] U', \tag{4.7}$$

at this point, where U_0 and M_0 are the values of U and M at the same point. Hence the calculations of singularities can now be avoided completely.

Finally, we note that from (2.17a), (2.18), (2.19) and the relation

$$\left.\frac{\partial \phi}{\partial x}\right|_{z=0} = \left(ik_x + \frac{A'}{A}\right)\phi$$

resulting from (2.7), one may directly obtain two first-order equations for the incident and reflected waves, which are much simpler than (3.1) and (3.6). These two equations can also be combined into a second-order equation and proven regular at the caustic. However, in this second-order equation, the Class 2 terms of the coefficients corresponding to P and Q in (3.12) cannot be calculated accurately at the caustic, because after all substitutions and reductions, they still contain two terms which are both singular at the caustic, though their singularities can balance each other. Another disadvantage of this simpler equation is that when $k_y = 0$, the expressions of the Class 2 terms in this equation cannot be reduced to those in (6.3) of Shyu & Phillips (1990), but (3.13) can be reduced to (6.4) in Shyu & Phillips (1990).

5. Solutions of reflection phenomenon

The equation (3.12) in terms of the parameters k_{x1}, k_{x2}, P and Q takes the same form as those derived by Shyu & Phillips (1990) so that the uniformly valid asymptotic solution of (3.12) can similarly be derived using the treatment suggested by the results of Smith (1975).

Following the precedent of Shyu & Phillips (1990), we first eliminate the first derivative term from (3.12) by a change of the dependent variable η, such that

$$\eta = v(x)\exp\left\{-\frac{1}{2}\int_0^x [-i(k_{x1}+k_{x2})+Q]\,dx\right\} e^{i(k_y y - n_0 t)}, \tag{5.1}$$

where $v(x)$ is the new dependent variable. Next, substituting (5.1) into (3.12) and neglecting the higher-order terms involving Q' and Q^2, we obtain

$$v'' + v(H+G) = 0, \tag{5.2}$$

where

$$H = \frac{1}{4}(k_{x2}-k_{x1})^2, \tag{5.3}$$

$$G = P + \frac{i}{2}(k_{x1}+k_{x2})Q + \frac{i}{2}(k'_{x1}+k'_{x2}). \tag{5.4}$$

Equation (5.2) is still regular at the caustic, and from Smith's (1975) results, we may expect that

$$v(x) \approx A_0 \mathrm{Ai}(-r) - C_0 \mathrm{Ai}'(-r), \tag{5.5}$$

where $\mathrm{Ai}'(-r) = \{d\mathrm{Ai}(x)/dx\}_{x=-r}$ and

$$\left.\begin{aligned} \frac{2}{3}r^{\frac{3}{2}} &= -\int_0^x H^{\frac{1}{2}}\,dx, \\ A_0 &= \left(\frac{r}{H}\right)^{\frac{1}{4}} \cos\left(-\int_0^x \frac{1}{2}G/H^{\frac{1}{2}}\,dx\right), \\ C_0 &= r^{-\frac{1}{4}}H^{-\frac{1}{4}} \sin\left(-\int_0^x \frac{1}{2}G/H^{\frac{1}{2}}\,dx\right). \end{aligned}\right\} \tag{5.6}$$

For the sake of definiteness, we have taken $x = 0$ to be the caustic and assumed that $H > 0$ for $x < 0$, corresponding to a situation in which the reflected wave is shorter and its group velocity smaller than the incident wave (see figure 1 and recall that $C_g \equiv \partial\sigma/\partial k$).

The fitness of (5.5) and (5.6) can easily be verified by substituting them into (5.2), which results in

$$\frac{d^2 v}{dx^2} + v(H+G) = \frac{d^2 A_0}{dx^2}\mathrm{Ai}(-r) - \frac{d^2 C_0}{dx^2}\mathrm{Ai}'(-r) \tag{5.7}$$

as $\mathrm{Ai}''(-r)$ is eliminated in favour of $-r\mathrm{Ai}(-r)$. Since the coefficients A_0 and C_0 in (5.5) have been separated from the rapidly varying parts of the solution, the terms on the right-hand side of (5.7) are again negligible so that the differences between (5.2) and (5.7) are insignificant. Furthermore, if A_0 and C_0 are regular and therefore remain slowly varying at the caustic, the solution (5.5) (and (5.1)) is also regular here and will satisfy the equation (5.2) everywhere, including the caustic, within the present approximation. Therefore we shall demonstrate the regularity of A_0 and C_0 at the caustic in the following way.

In the vicinity of the caustic, from (4.1), (4.2) and the Taylor's theorem, we have

$$H = \frac{1}{4}(k_{x2} - k_{x1})^2 = N^2 = \psi_1 x + \psi_2 x^2 + \cdots,$$

where $\psi_1 = d\psi/dx|_{x=0}$ and $2\psi_2 = d^2\psi/dx^2|_{x=0}$. Thus

$$\left.\begin{aligned} -\int_0^x H^{\frac{1}{2}}\,dx &= (-\psi_1)^{\frac{1}{2}}(-x)^{\frac{3}{2}}\left[\frac{2}{3} + \frac{1}{5}\frac{\psi_2}{\psi_1}x + \cdots\right], \\ -\int_0^x \frac{1}{2}G/H^{\frac{1}{2}}\,dx &= G_0(-\psi_1)^{-\frac{1}{2}}(-x)^{\frac{1}{2}}\left[1 + \left(\frac{1}{3}\frac{G_1}{G_0} - \frac{1}{6}\frac{\psi_2}{\psi_1}\right)x + \cdots\right], \end{aligned}\right\} \tag{5.8}$$

where $G = G_0 + G_1 x + \cdots$. Substituting (5.8) in (5.6), we obtain

$$\left.\begin{aligned} r &= (-\psi_1)^{\frac{1}{3}}(-x)\left[1 + \frac{1}{5}\frac{\psi_2}{\psi_1}x + \cdots\right], \\ A_0 &= (-\psi_1)^{-\frac{1}{6}}\left[1 - \left(\frac{1}{5}\frac{\psi_2}{\psi_1} + \frac{1}{2}\frac{G_0^2}{\psi_1}\right)x + \cdots\right], \\ C_0 &= G_0(-\psi_1)^{-\frac{5}{6}}\left[1 - \left(\frac{7}{15}\frac{\psi_2}{\psi_1} - \frac{1}{3}\frac{G_1}{G_0}\right)x + \cdots\right], \end{aligned}\right\} \tag{5.9}$$

in this region. Since the expressions in (5.9) represent the Taylor-series expansions about $x = 0$ and their radii of convergence can be expected to be very large for a slowly varying current field, A_0, C_0 and r are indeed regular at the caustic. Therefore (5.1) together with (5.5) and (5.6) represents a uniformly valid asymptotic solution.

At points away from the caustic, $\mathrm{Ai}(-r)$ and $\mathrm{Ai}'(-r)$ can be replaced by their asymptotic approximations, which for r large and positive are

$$\mathrm{Ai}(-r) \approx \pi^{-\frac{1}{2}} r^{-\frac{1}{4}} \sin\left(\frac{2}{3} r^{\frac{3}{2}} + \frac{1}{4}\pi\right), \tag{5.10}$$

$$\mathrm{Ai}'(-r) \approx -\pi^{-\frac{1}{2}} r^{\frac{1}{4}} \cos\left(\frac{2}{3} r^{\frac{3}{2}} + \frac{1}{4}\pi\right). \tag{5.11}$$

Thus from (5.1), (5.5) and (5.6), we have

$$\begin{aligned} \eta \approx & H^{-\frac{1}{4}} \exp\left[\int_0^x \frac{1}{2}(-Q - iG/H^{\frac{1}{2}})\,dx\right] \exp i\left[\int_0^x k_{x1}\,dx + k_y y - n_0 t - \frac{1}{4}\pi\right] \\ & + H^{-\frac{1}{4}} \exp\left[\int_0^x \frac{1}{2}(-Q + iG/H^{\frac{1}{2}})\,dx\right] \exp i\left[\int_0^x k_{x2}\,dx + k_y y - n_0 t + \frac{1}{4}\pi\right] \end{aligned} \tag{5.12}$$

for $x \ll 0$. This solution represents the WKBJ approximation; it obviously fails at the caustic where $H = 0$, but can nevertheless indicate the existence of the incident and reflected waves, as well as show their relative amplitudes and phases (an irrelevant constant common factor was neglected from (5.12)). Consequently, we have the local amplitudes

$$a = \begin{cases} H^{-1/4} \exp\left[\int_0^x \frac{1}{2}(-Q - iG/H^{\frac{1}{2}})\,dx\right] & (k_1 \text{ component}); \\ H^{-1/4} \exp\left[\int_0^x \frac{1}{2}(-Q + iG/H^{\frac{1}{2}})\,dx\right] & (k_2 \text{ component}). \end{cases} \tag{5.13}$$

(notice that G is pure imaginary while H and Q are real), which have been proved to satisfy the action conservation principle at and away from the caustic.

6. Extensions to general cases

A difference between Smith's (1975) theory and the present theory is that in the latter, the expressions for the amplitudes of the incident and reflected waves take an explicit form while in the former, the variations of these quantities were demonstrated to fulfill the action conservation principle everywhere, including the caustic. The present expressions, especially those in (5.13), if also valid in a general situation, will later prove of great use in the improvement of an error magnification phenomenon occurred in the estimates of the amplitude of the reflected wave in terms of that of the incident wave in the vicinity of the caustic during a numerical computation, which are required for the ray solution to be continued after reflection at the caustic. To achieve this goal, we shall in this section demonstrate that even when the water is of intermediate depth and the underlying larger-scale irrotational flow is multidirectional and unsteady, the solutions of the wave field in the vicinity of the caustic still take the same forms as those derived in Shyu & Phillips (1990) and in the preceding section.

When the larger-scale flows are not unidirectional, the caustics are unlikely to be straight. Thus it is necessary to derive the solutions in a set of orthogonal curvilinear co-ordinates in which the x-axis is perpendicular to the caustic. Therefore we define all the lines x = constant are parallel curves and $x = 0$ corresponds to the caustic (see figure 2), so that the scale factor h_x in the x-direction is independent of the position. On the other hand, if at the caustic we set the scale factor in the y-direction $h_y = 1$, the variation of h_y in the x-direction has the simple relation

$$h_y = 1 - \frac{x}{R(y,t)} \tag{6.1}$$

where R is the radius of curvature of the caustic which is large compared with the wavelength if the underlying current is slowly varying. This co-ordinate system will certainly produce singularities of the differential equations at certain positions far away from the caustic, but since it is the present purpose to derive the solutions in the vicinity of the caustic, these singularities can be avoided in the present analysis.

In this curvilinear co-ordinate system, the WKBJ solution of each wave component can still be written as

$$\eta = a(x,y,t)e^{i\chi(x,y,t)}, \tag{6.2}$$

although the forms of the functions $a(x,y,t)$ and $\chi(x,y,t)$ in (6.2) are different from those in a rectangular co-ordinate system. From (6.2), the x and y components of the wavenumber and the observed frequency are

$$k_x = \frac{\partial\chi}{\partial x}, \quad k_y = \frac{1}{h_y}\frac{\partial\chi}{\partial y}, \quad n = -\frac{\partial\chi}{\partial t}, \tag{6.3}$$

respectively. In (6.2) and (6.3), the dependence of a, k_x, k_y and n on x, y and t are expected to be slow except that in the vicinity of the caustic, the variations of a and k_x with x will be rapid owing to the singularities at the caustic. Notice that the variations of h_y with x, y, t are also slow in view of (6.1).

From (6.2) and (6.3), equations (3.1) and (3.6) immediately follow with

$$\left.\begin{aligned} R_1 &= -i\frac{a_1'}{a_1}, \\ R_2 &= -i\frac{a_2'}{a_2}. \end{aligned}\right\} \tag{6.4}$$

The values of a_1'/a_1 and a_2'/a_2 cannot be determined without considerations of the Laplace equation and the kinematical and dynamical boundary conditions, which even in the vicinity of the caustic can result in the action conservation equation as demonstrated by Smith (1975) in exactly the same circumstance, who also derived the dispersion relation in this region which is again identical with that in the regions far from the caustic. Notice that although the WKBJ solution becomes invalid in the immediate vicinity of the caustic, the solution values of a_1, a_2, k_1 and k_2 in this region determined from the action conservation equation and the dispersion relation are still meaningful, because a combination of these parameters can represent the quantities in the uniformly valid solution (see (1.1) and (1.2)).

Next, following the same procedure as that in section 3, we again obtain (3.12) together with (3.13). Thus if we choose

$$\eta = v(x,y,t)\exp\left\{\frac{i}{2}(\chi_1+\chi_2) - \int_0^x \frac{Q}{2}\,dx\right\}, \tag{6.5}$$

which equivalent to (5.1) for a straight caustic, we can similarly achieve

$$v(x,y,t) = A_0\mathrm{Ai}(-r) - C_0\mathrm{Ai}'(-r) \tag{6.6}$$

with

$$\frac{2}{3}r^{\frac{3}{2}} = -\int_0^x H^{\frac{1}{2}}\,dx, \tag{6.7}$$

$$A_0 = \left(\frac{r}{H}\right)^{\frac{1}{4}}\cos\left(-\int_0^x \frac{1}{2}G/H^{\frac{1}{2}}\,dx\right), \quad C_0 = r^{-\frac{1}{4}}H^{-\frac{1}{4}}\sin\left(-\int_0^x \frac{1}{2}G/H^{\frac{1}{2}}\,dx\right), \tag{6.8}$$

in which

$$H = \frac{1}{4}(k_{x2}-k_{x1})^2, \tag{6.9}$$

$$G = P + \frac{i}{2}(k_{x1}+k_{x2})Q + \frac{i}{2}(k'_{x1}+k'_{x2}). \tag{6.10}$$

The new variable r now depends on x, y and t, but its variations with respect to y and t will be slow.

The adequacy of (6.5)–(6.10) as a uniformly valid solution for the case of a curved and/or moving caustic which occurred in a deep or intermediate-depth region, depends on whether the singularities at the caustic can be cancelled out from $\chi_1+\chi_2, Q, G$, etc., otherwise the above solution will become singular here and the coefficients A_0 and C_0 are no longer slowly varying in the vicinity of the caustic, which will decline the use of the approximation implied by (5.7). Therefore it is required in the following to demonstrate the regularity of (6.5)–(6.10) at the caustic.

Since even for a curved moving caustic in an intermediate-depth region, from the dispersion relation and the fact that $U_x + C_{gx} = 0$ at the caustic, one can always prove that k_{x1} and k_{x2} represent two branches of a double-valued function with the branch point at the caustic $x = 0$. Therefore the phase function of the incident wave can be written as

$$\chi_1 = [d_0 + d_1 x + d_2 x^2 + \cdots] + \frac{2}{3}\left[(-\psi_1)^{\frac{1}{3}}(-x) + O(x^2)\right]^{\frac{3}{2}} \tag{6.11}$$

(also see (9), Smith (1975)), where the coefficients of the Taylor series expansions about $x = 0$ in the two square brackets are functions of y and t, which except d_0 are slowly varying according to the discussion following (6.3). On substitution χ_1 into (6.3) we have

$$\left.\begin{aligned} k_{x1} &= [d_1 + 2d_2 x + O(x^2)] - [\psi_1 x + O(x^2)]^{\frac{1}{2}}, \\ h_y k_{y1} &= \left[\frac{\partial d_0}{\partial y} + \frac{\partial d_1}{\partial y} x + O(x^2)\right] - \frac{1}{3}\frac{\partial \psi_1}{\partial y}\left[(-\psi_1)^{-1}(-x)^3 + O(x^4)\right]^{\frac{1}{2}}. \end{aligned}\right\} \tag{6.12}$$

The other branches of (6.11) and (6.12) then provide respectively the phase and wave-number components of the reflected wave:

$$\chi_2 = [d_0 + d_1 x + d_2 x^2 + \cdots] - \frac{2}{3}\left[(-\psi_1)^{\frac{1}{3}}(-x) + O(x^2)\right]^{\frac{3}{2}}, \tag{6.13}$$

$$\left.\begin{aligned} k_{x2} &= [d_1 + 2d_2 x + O(x^2)] + [\psi_1 x + O(x^2)]^{\frac{1}{2}}, \\ h_y k_{y2} &= \left[\frac{\partial d_0}{\partial y} + \frac{\partial d_1}{\partial y} x + O(x^2)\right] + \frac{1}{3}\frac{\partial \psi_1}{\partial y}\left[(-\psi_1)^{-1}(-x)^3 + O(x^4)\right]^{\frac{1}{2}}. \end{aligned}\right\} \tag{6.14}$$

The above phase functions not only lead to the right forms of k_{x1} and k_{x2}, but also ensure that $\nabla \times \mathbf{k} = 0$ for both waves. However, the values of the series coefficients d_1, d_2, ψ_1, etc. can be determined only from the dispersion relation. We notice in passing that for a curved caustic, even though k_{y1} is unequal to k_{y2} when $x \neq 0$, their difference is much smaller than that between k_{x1} and k_{x2} and is proportional approximately to $(-x)^{3/2}$ (also with a smaller coefficient $(2/3)(-\psi_1)^{-1/2}\partial\psi_1/\partial y$) when the caustic is approached. A similar situation also occurs to the observed frequencies n_1 and n_2 of the incident and reflected waves for a moving caustic. These situations can benefit the numerical computations of the reflected wave significantly as will be seen in the next section. Also we emphasize that neglect of the higher powers of x in the Taylor series expansion of a slowly varying parameter is equivalent to neglect of its higher-order derivatives with respect to x, because the series coefficients are closely related to the derivatives of the same order.

From (6.11) and (6.13) it is immediately clear that $\chi_1 + \chi_2$ in (6.5) is regular at the caustic. On the other hand, from (6.12) and (6.14), we have in the vicinity of caustic

$$H = \frac{1}{4}(k_{x2} - k_{x1})^2 = \psi_1 x + O(x^2). \tag{6.15}$$

Consequently

$$-\int_0^x H^{\frac{1}{2}}\,dx = \frac{2}{3}(-\psi_1)^{\frac{1}{2}}(-x)^{\frac{3}{2}}[1 + O(x)] \tag{6.16}$$

and

$$-\int_0^x \frac{1}{2}G/H^{\frac{1}{2}}\,dx = G_0(-\psi_1)^{-\frac{1}{2}}(-x)^{\frac{1}{2}}[1 + O(x)] \tag{6.17}$$

if G is regular at the caustic and $G_0 \equiv G(x = 0)$. Substitution of (6.16) into (6.7) results in

$$r = (-\psi_1)^{\frac{1}{3}}(-x)[1 + O(x)]. \tag{6.18}$$

Again the radius of convergence of the power series in the square brackets in (6.18) can be expected to be large compared with the wavelength as long as the underlying current is slowly varying. Therefore (6.18) indicates that r is regular at the caustic. Furthermore, on substituting (6.15), (6.17) and (6.18) into (6.8), we have

$$\left.\begin{aligned} A_0 &= (-\psi_1)^{-\frac{1}{6}}[1 + O(x)], \\ C_0 &= G_0(-\psi_1)^{-\frac{5}{6}}[1 + O(x)]. \end{aligned}\right\} \tag{6.19}$$

in the vicinity of the caustic, so that A_0 and C_0 are also regular (and therefore slowly varying) at the caustic. All of these results enable us to conclude that the singularities at the caustic have been cancelled out from the solution (6.5)–(6.10) provided that G and Q are also regular here, which will be demonstrated as follows.

Since the parameters G and Q also appear in the WKBJ solution, their regularity can be proved through a consideration of the action conservation equation, the fulfillment of which by the WKBJ solution even in the vicinity of the caustic was demonstrated by Smith (1975). Thus, substituting (5.10) and (5.11) into (6.6), using (6.7) and (6.9), and also taking (6.3) into consideration, we obtain the WKBJ solution

$$\begin{aligned}\eta =&C(y,t)H^{-\frac{1}{4}}\exp\left[\int_0^x \frac{1}{2}(-Q-iG/H^{\frac{1}{2}})\,dx\right]\exp i\left(\chi_1-\frac{1}{4}\pi\right)\\&+C(y,t)H^{-\frac{1}{4}}\exp\left[\int_0^x \frac{1}{2}(-Q+iG/H^{\frac{1}{2}})\,dx\right]\exp i\left(\chi_2+\frac{1}{4}\pi\right),\end{aligned}\tag{6.20}$$

where the common factor $C(y,t)$ is independent of x. From (6.20), the local amplitudes of the incident and reflected waves are

$$\left.\begin{aligned}a_1&=CH^{-\frac{1}{4}}\exp\left[\int_0^x \frac{1}{2}(-Q-iG/H^{\frac{1}{2}})\,dx\right],\\a_2&=CH^{-\frac{1}{4}}\exp\left[\int_0^x \frac{1}{2}(-Q+iG/H^{\frac{1}{2}})\,dx\right].\end{aligned}\right\}\tag{6.21}$$

Therefore substitution of (6.15) for H and differentiation yield

$$\left.\begin{aligned}\frac{a_1'}{a_1}&=\frac{1}{2}(-Q-iG/H^{\frac{1}{2}})-\frac{1}{2}\frac{k_{x2}'-k_{x1}'}{k_{x2}-k_{x1}},\\\frac{a_2'}{a_2}&=\frac{1}{2}(-Q+iG/H^{\frac{1}{2}})-\frac{1}{2}\frac{k_{x2}'-k_{x1}'}{k_{x2}-k_{x1}}.\end{aligned}\right\}\tag{6.22a, b}$$

The above expressions for a_1'/a_1 and a_2'/a_2 are exactly identical with those in (6.4) if the expression (3.13) and the definition (6.10) for G are substituted into (6.4) for R_1 and R_2. This and the fact that (6.20) represents a rigorous asymptotic approximation of the solution (6.5)–(6.10) have put even more confidence in the assumption that if the singularities at the caustic are completely cancelled out from (3.12), this equation and the solution (6.5)–(6.10) will remain valid uniformly in a region containing the caustic.

The variations of a_1 and a_2 have been proved by Smith (1975) to satisfy the action conservation equation in the vicinity of the caustic. Therefore, the regularity of the parameters Q, G in (6.22a,b) at the caustic can be demonstrated through a consideration of the action conservation equation. First, from (6.12) and (6.14) it immediately follows that

$$\frac{1}{2}\frac{k'_{x2}-k'_{x1}}{k_{x2}-k_{x1}} = \frac{1}{4x}[1+O(x)] \tag{6.23}$$

near the caustic. Next, from the action conservation equation in the curvilinear co-ordinates

$$\frac{\partial}{\partial t}\left(\frac{E_1}{\sigma_1}\right) + \frac{\partial}{\partial x}\left[(U_x + C_{gx1})\frac{E_1}{\sigma_1}\right] + \frac{1}{h_y}\frac{\partial h_y}{\partial x}\left[(U_x + C_{gx1})\frac{E_1}{\sigma_1}\right] + \frac{1}{h_y}\frac{\partial}{\partial y}\left[(U_y + C_{gy1})\frac{E_1}{\sigma_1}\right] = 0, \tag{6.24}$$

in which the wave action density of the incident wave

$$\frac{E_1}{\sigma_1} = \frac{1}{2}\rho g \frac{a_1^2}{\sigma_1},$$

where E_1 represents its energy density and ρ the density of water. Therefore, substitution and expansion yield

$$\begin{aligned}\frac{a_1'}{a_1} = &-\frac{\sigma_1}{2(U_x + C_{gx1})}\frac{\partial}{\partial x}\left(\frac{U_x + C_{gx1}}{\sigma_1}\right) - \frac{\sigma_1}{2a_1^2(U_x + C_{gx1})}\frac{\partial}{\partial t}\left(\frac{a_1^2}{\sigma_1}\right)\\ &-\frac{1}{1-x/R}\frac{\sigma_1}{2a_1^2(U_x + C_{gx1})}\frac{\partial}{\partial y}\left[(U_y + C_{gy1})\frac{a_1^2}{\sigma_1}\right] + \frac{1}{2R}\frac{1}{1-x/R}\end{aligned} \tag{6.25}$$

in virtue of (6.1).

The relation between σ_1 and k_1 differs according to whether the water is deep or of moderate depth, but in any cases, by using (6.12) and the dispersion relation, one can always obtain the form of the expansion

$$\frac{U_x + C_{gx1}}{\sigma_1} = \sqrt{\psi_1 x}[\alpha_0 + O(x)] + [e_1 x + O(x^2)] \tag{6.26}$$

in the vicinity of the caustic, where α_0 and e_1 are the leading coefficients of the two Taylor series in the square brackets. The absence of e_0 from the second series is simply due to the fact that $U_x + C_{gx1} = 0$ at the caustic. Thus from (6.26), the first term on the right-hand side of (6.25)

$$-\frac{\sigma_1}{2(U_x + C_{gx1})}\frac{\partial}{\partial x}\left(\frac{U_x + C_{gx1}}{\sigma_1}\right) = -\frac{1}{4x}[1+O(x)] + \frac{1}{\sqrt{\psi_1 x}}[\beta_0 + O(x)] \tag{6.27}$$

near the caustic. Since in the vicinity of the caustic, variations of wave properties perpendicular to the caustic are large compared to variations along the caustic or with time, (6.27) represents the major contribution to a_1'/a_1 in (6.25) in this region. Therefore, from a comparison between (6.27) and (6.23) it is immediately clear that the term $H^{-\frac{1}{4}}$ in a_1 in the present solution (6.21), which leads to the last terms in (6.22a,b) and therefore (6.23), is indeed consistent with the predication by the action conservation principle as far as their first approximations are concerned. However, the values of G and Q cannot be obtained without further evaluation of the second and third terms on the right-hand side of (6.25).

Since the first approximation $a_1 \approx CH^{-\frac{1}{4}}$ has been justified, from (6.15) we have

$$a_1^2 \approx \frac{C^2}{\sqrt{\psi_{1x}}} \tag{6.28}$$

near the caustic. Also, from the dispersion relation, it is not difficult to see that the series expansions of σ_1 and $U_y + C_{gy1}$ will possess the same form as that of k_{x1} in (6.12). Thus, using these series and (6.28) as well as (6.26), and recalling that C and ψ_1 are slow functions of y and t, we obtain

$$-\frac{\sigma_1}{2a_1^2(U_x + C_{gx1})}\frac{\partial}{\partial t}\left(\frac{a_1^2}{\sigma_1}\right) \approx \frac{\xi_0}{\sqrt{\psi_{1x}}}, \tag{6.29}$$

$$-\frac{1}{1 - x/R}\frac{\sigma_1}{2a_1^2(U_x + C_{gx1})}\frac{\partial}{\partial y}\left[(U_y + C_{gy1})\frac{a_1^2}{\sigma_1}\right] \approx \frac{\zeta_0}{\sqrt{\psi_{1x}}}, \tag{6.30}$$

in which the Taylor expansion

$$\left(1 - \frac{x}{R}\right)^{-1} = 1 + \frac{x}{R} + \left(\frac{x}{R}\right)^2 + \cdots$$

is also used, producing the higher-order terms than that on the right-hand side of (6.30). Consequently, from (6.27), (6.29) and (6.30) we have

$$-iG_0 = 2(\beta_0 + \xi_0 + \zeta_0), \tag{6.31}$$

if (6.22a) and (6.25) are equal.

After the first term of the expansion for G was found, the next order terms in (6.29) and (6.30) can be pursued by substituting $G \approx G_0$ into (6.21), which will result in the terms of zeroth power of x in (6.29) and (6.30). These terms and the corresponding term in (6.27) as well as the term $1/2R$ arising from the last term of (6.25), excluding those attributed to the series in (6.23), can be identified with $Q_0 \equiv Q(x = 0)$ in (6.22a). This procedure can be continued to determine subsequent terms in the expansions for G and Q, and the results show that these expansions indeed take the form of a power series with center at $x = 0$. Although there is no way to estimate the radii of convergence of these two series in a general situation, since the variation of the underlying current is slow, it is not unreasonable to anticipate that these series will be uniformly convergent in a large (compared with the wavelength) area centering at the caustic. Therefore we conclude that G, Q (and P) are regular at the caustic. This conclusion can also be drawn from a consideration of the action conservation equation for the reflected wave, because in this case, following the same procedure, one can obtain the same results except that the signs of the terms containing $\sqrt{\psi_1 x}$ in (6.27)–(6.30) become opposite, which also occurs to (6.22b) compared with (6.22a), so that the parameters G and Q in (6.22b) have the same values as those in (6.22a) and are indeed regular at the caustic.

In summary, by investigating the power-series expansion about the caustic and therefore the regularity at the caustic of each parameter in the equation and solutions, we have demonstrated that even for a curved moving caustic and for waves in an intermediate-depth region, the uniform asymptotic and the WKBJ solutions in the vicinity of the caustic take the same forms as those derived in Shyu & Phillips (1990) and in the preceding section. Since the existence of the series expansions is proved from the dispersion relation and the action conservation equation which themselves were deduced by Smith (1975) from the Laplace equation and the kinematical and dynamical boundary conditions, the present solutions are not independent of the dynamics. These solutions have provided explicit expressions for the amplitudes, although the Class 2 terms G and Q in them can be determined only numerically. This explicitness will in the next section prove of great use to a practical numerical computation of the reflection phenomenon.

7. An application to numerical computations

In this section, we shall conduct numerical simulations in two cases: a straight caustic and a curved caustic. Since it will later become clear that estimates of the reflected wave from the incident wave in the vicinity of the caustic at each instant involves only the instantaneous values of various variables and their derivatives with respect to x, and since the difference between n_1 and n_2 near a moving caustic is as small as that between k_{y1} and k_{y2} near a curved caustic (see the discussion following (6.14)), any conclusions drawn from the simulation of a curved caustic about the application of the present theory may have implications for the case of a moving caustic. To eliminate other complication without loss of generality, we also assume that all waves are in deep water.

Since we have in the earlier sections derived the analytical solutions, including the expressions of the Class 2 terms, in the case when a straight caustic is caused by a deep-water gravity wave propagating obliquely upon a steady unidirectional current, the results of the present numerical computations for this case can be compared with the analytical solutions to show the accuracy of the numerical schemes applied in both cases of a straight and a curved caustics. To achieve this purpose, even for a straight caustic, we deliberately take the directions (denoted by x' and y') of the computational grid not along $\mathbf{U}$ so that no simplifications which may originally suitable to this special case will be made and the extension of the numerical model to the case of a curved caustic is straightforward. Also we note that although the analysis in the preceding section was made in a curvilinear co-ordinate system, without a prior knowledge of the location of the caustic, the differential equations can be solved numerically only on a rectangular grid for the incident wave, after which and after the caustic was determined numerically, the components of each vector relative to the curvilinear co-ordinate system defined in section 6 can be calculated from those referred to (x', y'), that will be utilized to determine the reflected wave in the vicinity of the caustic.

Determination of the incident wave and the caustic

Since in the present simulations the underlying currents are steady, the action conservation equation can be reduced to

$$\frac{\partial}{\partial x'}\left[(U_{x'}+C_{gx'})\frac{a^2}{\sigma}\right]+\frac{\partial}{\partial y'}\left[(U_{y'}+C_{gy'})\frac{a^2}{\sigma}\right]=0, \tag{7.1}$$

and the apparent frequency n remains constant (denoted by n_0 again) everywhere so that the wave-number components $k_{x'}$ and $k_{y'}$ can be determined entirely from the irrotationality

$$\frac{\partial k_{y'}}{\partial x'}-\frac{\partial k_{x'}}{\partial y'}=0 \tag{7.2}$$

and the dispersion relation

$$n_0=\left[g\left(k_{x'}^2+k_{y'}^2\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}+U_{x'}k_{x'}+U_{y'}k_{y'}. \tag{7.3}$$

The partial differential equations (7.1) and (7.2) are solved by using a finite difference scheme. Since it is not the present purpose to develop an efficient model, an explicit difference equation of first-order accuracy is used to approximate (7.1) and (7.2). As this solution scheme marches towards caustic (in the y'-direction, say), the derivatives with respect to y' should certainly be replaced by the forward difference. On the other hand, the derivatives with respect to x' are replaced by the forward difference or the backward difference depending on whether $U_{x'}+C_{gx'}$ is negative or positive. This choice is important to the stability of the present difference scheme, and according to the Courant-Friedrichs-Lewy (C.F.L.) condition, this scheme will be stable only if the ratio of the grid spacing $\Delta y'$ to $\Delta x'$ is smaller than the ratio of $|U_{y'}+C_{gy'}|$ to $|U_{x'}+C_{gx'}|$. Therefore, to ensure that the numerical solutions are accurate even in the vicinity of the caustic, the grid spacings $\Delta x'=20$ cm, $\Delta y'=0.0625$ cm are chosen for both simulations of a straight and a curved caustics.

When the difference formula for (7.2) is solved at each mesh point, we first obtain the solution of $k_{x'}$ at this point. Then the numerical solution of $k_{y'}$ at the same point is calculated from (7.3) by using the Secant Method. Consequently, all the quantities related to the kinematics of the incident wave, including the characteristic velocity $\mathbf{U} + \mathbf{C_g}$, can be determined, after which the action conservation equation (7.1) will be solved for the amplitude a at this point.

The above computations can be continued until at a certain point,the subroutine for the Secant Method fails to return a reliable and real root of (7.3) which signifies the occurrence of the blockage phenomenon at this point. If this occurs to point A in figure 3, the solution values at the points on the same row but on the right side of A can still be pursued. However, since point A is excluded from the integration domain, the solution values at point D on the next row cannot be computed with the present difference scheme as $U_{x'} + C_{gx'} > 0$ at this point. This difficulty can be overcome by using the forward difference instead of the backward difference to approximate $\partial k_{y'}/\partial x'$ in (7.2) at point D and at the points above it and in the same column. This enables us to continue the calculations of $\mathbf{k}$ (but not a) beyond the row containing A until the Secant Method fails again or $U_x + C_{gx}$ became negative at another point, E say, which always occurs in the column next to point A. Consequently, the line AE in figure 3 can approximate the true caustic satisfactorily if $\Delta y'$ is sufficiently small. Note that $U_x + C_{gx}$ here represents the component of $\mathbf{U} + \mathbf{C_g}$ in the direction perpendicular to the estimate caustic.

After the caustic at point E was decided, the numerical solution of a at point F can be calculated reliably by using the backward-difference scheme for the derivative with respect to x' in (7.1), and using the solution values at points B and C. This difference equation has a larger value of $\Delta y'$ but can however fulfill the C.F.L. condition for stability even in the immediate vicinity of the caustic, because the characteristic velocity $\mathbf{U} + \mathbf{C_g}$ of the incident wave at points B and F always have a component towards the caustic (see figure 3). The numerical solution of a at the rest of the points on the same row as point F can be calculated without difficulty by using the solution values at the points on the same row as points B and C. Therefore we now have all the informations required for a repetition of the above procedure to determine the next position of the caustic and calculate the solution values of the incident wave on that row.

The above strategy for estimates of the location of the caustic can be justified directly by a comparison between its numerical and analytical solutions in the case of a straight caustic (an indirect justification for the case of a curved caustic will also be given later). In this case, the conditions of the incoming wave prescribed on the boundaries AB and BC in figure 4 are determined from the requirements that $k_y = -0.94$ rad/m, $n_0 = 3.62$ rad/s, the component of the action flux in the x-direction is equal to 1 everywhere, and the velocity distribution of the underlying current is

$$U_x = -0.6882 - 0.0077x \ (\text{m/s}), \qquad U_y = 0.$$

In this situation, one may easily prove that $k_x = 4.9$ rad/m and $C_{gx} = 0.6882$ m/s at $x = 0$, so that the true caustic coincides with the y–axis. However, in the simulation, to make the calculation more realistic, the fact that k_y = constant everywhere has not been invoked in the computations of $\mathbf{k}$ in the integration domain, so that equations (7.2) and (7.3) have been solved simultaneously, and by using the above strategy, the caustic has been located, which as shown in figure 4 coincides with the true caustic extremely well.

On the other hand, to simulate the calculations for a curved caustic, we assume that the streamlines of the underlying larger-scale current are circles and the magnitude of the velocity at each point

$$|U| = -4.0\frac{3\pi/2 - \theta}{\pi} \cdot \frac{100}{r} \ (\text{m/s}),$$

where (r, θ) represents the polar coordinates of this point (see figure 5). This velocity distribution has zero vorticity everywhere except at the point $r = 0$, which represents a singular point but will be excluded from the integration domain because of the wave blockage phenomenon. Another feature of this distribution is that when r is very large, $|U|$ becomes vanishingly small. Therefore a uniform deep-water wavetrain with frequency $n_0 = 1.7$ rad/s propagating in a single direction can be prescribed on the boundaries AB and BC in figure 5 which are very far from the origin. From these boundary conditions and by using the numerical scheme, the variation of the wave-number of the incident wave in the integration domain was solved until a blockage point was first met. After this, the calculations were restricted within a much smaller area specified in figure 6, in which the blockage point A had been located at the previous stage and the numerical solutions of the wave-number components at each point on line AB had also been estimated. Thus, to calculate the wave-number as well as the amplitude of the incident wave in this small area, it is only necessary to prescribe the value of a_1 at each point on line AB in figure 6. In consideration of (6.21) and (6.15), the particular boundary condition of a_1 chosen here is

$$a_1 = (-\hat{x})^{-\frac{1}{4}}$$

where $-\hat{x}$ represents the distance from each point on AB to the dashed straight line in figure 6 which approximates the estimate caustic. Note that since a slowly modulated incoming wave is allowed by the theories, the above arrangement is convenient for development and tests of the present algorithm.

The amplitude and wave-number of the incident wave, including the location of the caustic, in the integration domain in figure 6 can therefore be estimated by using the above schemes and strategy; the resulting caustic as shown in figure 6 is indeed curved. Also we remark that in the present simulation of a curved caustic, the wave-number components k_x, k_y and the component U_x of the current velocity in the x-direction, including its derivative with respect to x, in the region near the caustic and the dotted line in figure 6 are all approximately equal to those in the simulation of a straight caustic. The extra convection in the y-direction by the current in the simulation of a curved caustic has caused a discrepancy between the observed frequencies in these two simulations. Therefore the non-dimensional modulation rates $|1/U_x k_x||\partial U_x/\partial x|$ in both cases in these regions have pretty much the same value and are both equal to 0.2% approximately. Hence any errors arising from asymptotic expansions might have the same order of magnitude in these two simulations, which in the case of a straight caustic can be estimated from a comparison between the numerical and analytic solutions.

Determination of the reflected wave

After the incident wave field and the position of the caustic were determined, we proceed to estimate the reflected wave in the vicinity of the caustic using Smith's (1975) theory and the present theory. The results can serve as the boundary conditions for calculations of the reflected wave in the regions away from the caustic.

Since the difference between k_{y1} and k_{y2} is very small near a curved caustic (see the note following (6.14)) and is zero in the case of a straight caustic, we let $k_{y2} = k_{y1}$ at each point in the vicinity of the caustic and then calculate the value of k_{x2} as another root of equation (7.3) which coalesces with k_{x1} at the caustic. The accuracy of these estimates can more or less be seen from a calculation of vorticity of the resulting $\mathbf{k}_2$ as shown in figure 7. (In this figure and in the following figures, the results presented are along the dotted lines in figures 4 and 6 for a straight and curved caustics respectively, and also the values of $\sqrt{\psi_1 x}$ estimated from (6.15) by neglecting the higher powers of x are chosen as the abscissas of these figures.) The results in figure 7 indicate that the irrotationality is approximately fulfilled by the estimates of $\mathbf{k}_2$ in the case of a curved caustic and this fulfillment is even more satisfactory in the case of a straight caustic as might be expected.

Next, we shall calculate a_2 in terms of a_1 in the vicinity of the caustic by using Smith's (1975) theory. According to Smith (1975), the flux of wave action normal to the caustic carried by the incident and by the reflected waves are equal and opposite at the caustic, so that we have

$$\left[(U_x + C_{gx1})\frac{a_1^2}{\sigma_1}\right]_{x=0} = -\left[(U_x + C_{gx2})\frac{a_2^2}{\sigma_2}\right]_{x=0} \tag{7.5}$$

Also integration with respect to x of (6.24) and of the corresponding equation for the reflected wave and substitution of (6.1) yield

$$\left.\begin{aligned}\left(1-\frac{x}{R}\right)\left[(U_x + C_{gx1})\frac{a_1^2}{\sigma_1}\right] - \left[(U_x + C_{gx1})\frac{a_1^2}{\sigma_1}\right]_{x=0} &= \int_0^x F_1(x,y,t)\,dx,\\ \left(1-\frac{x}{R}\right)\left[(U_x + C_{gx2})\frac{a_2^2}{\sigma_2}\right] - \left[(U_x + C_{gx2})\frac{a_2^2}{\sigma_2}\right]_{x=0} &= \int_0^x F_2(x,y,t)\,dx,\end{aligned}\right\} \tag{7.6}$$

where

$$\left.\begin{aligned}F_1 &= -\left(1-\frac{x}{R}\right)\frac{\partial}{\partial t}\left(\frac{a_1^2}{\sigma_1}\right) - \frac{\partial}{\partial y}\left[(U_y + C_{gy1})\frac{a_1^2}{\sigma_1}\right],\\ F_2 &= -\left(1-\frac{x}{R}\right)\frac{\partial}{\partial t}\left(\frac{a_2^2}{\sigma_2}\right) - \frac{\partial}{\partial y}\left[(U_y + C_{gy2})\frac{a_2^2}{\sigma_2}\right].\end{aligned}\right\} \tag{7.7}$$

Since F_1 and F_2 represent the Class 2 terms, it is only required to calculate their first approximations, which are

$$F_1 \approx F_2 \approx \frac{\tau_0}{\sqrt{\psi_1 x}} \tag{7.8}$$

in view of (6.28)–(6.30) and due to the fact that each of a, σ and $U_y + C_{gy}$ for the incident wave has the same one-term approximation as that for the reflected wave in the vicinity of the caustic. Thus, substituting (7.5) and (7.8) into (7.6) and carrying out the integration and expansion, we obtain

$$\left[(U_x + C_{gx2})\frac{a_2^2}{\sigma_2}\right] \approx -\left[(U_x + C_{gx1})\frac{a_1^2}{\sigma_1}\right] + 4\frac{\tau_0}{\psi_1}\sqrt{\psi_1 x} \tag{7.9}$$

at each point in the vicinity of the caustic. In (7.9), the terms containing the radius of curvature R of the caustic are of higher powers of x than the last term so that they were neglected consistently. The relations (7.7)–(7.9) together indicate that the ratio a_2/a_1 at each grid point near the caustic can in theory be estimated from the solution values of a_1, k_1 and k_2 obtained earlier. Also we emphasize that since the coefficients of the higher powers of x in the series expansion of a slowly varying parameter are proportional to its higher-order derivatives and therefore are very small, the approximation (7.9) can be very accurate even for a moderate value of x. A similar situation will also occur in the application of present theory.

When the present theory is under consideration, from (6.21) and (6.17) it immediately follows that

$$\frac{a_2}{a_1} = \exp\left[\int_0^x iG/H^{\frac{1}{2}}\,dx\right] \approx \exp\left[-2iG_0(-\psi_1)^{-\frac{1}{2}}(-x)^{\frac{1}{2}}\right]. \tag{7.10}$$

Since

$$\exp\left[-2iG_0(-\psi_1)^{-\frac{1}{2}}(-x)^{\frac{1}{2}}\right] \approx 1 - 2iG_0(-\psi_1)^{-\frac{1}{2}}(-x)^{\frac{1}{2}}, \tag{7.11}$$

it is clear that the parameter G_0 (or strictly speaking, $-iG_0(-\psi_1)^{-1/2}$) is closely related to the difference between the amplitudes of the incident and reflected waves in the vicinity of the caustic. From (6.22a,b) and (6.15), we also have

$$\frac{a_2'}{a_2} - \frac{a_1'}{a_1} = iG/H^{\frac{1}{2}} \approx iG_0/\sqrt{\psi_1 x} \tag{7.12}$$

so that the value of $-iG_0(-\psi_1)^{-1/2}$ can be estimated from the solutions of a_1'/a_1 and a_2'/a_2, which themselves can be calculated respectively from (6.25) and from the corresponding equation for a_2'/a_2 as follows.

The first terms on the right-hand side of equation (6.25) and on that of the corresponding equation for a_2'/a_2 can be computed solely from the numerical solutions of $\mathbf{k}_1$ and $\mathbf{k}_2$ respectively. On the other hand, the second and third terms represent the Class 2 terms and their one-term approximations are proportional to $1/\sqrt{\psi_{1x}}$ according to (6.29) and (6.30). Therefore, each of these terms in a_1'/a_1 and in a_2'/a_2 should be equal in magnitude and opposite in sign within the present approximation. Hence, even without solving (7.1) for a_2, the approximation of a_2'/a_2 at each point near the caustic can still be estimated in theory. Consequently, the value of $-iG_0(-\psi_1)^{-1/2}$ in (7.12) and eventually the values of a_2/a_1 in (7.10) in the vicinity of the caustic can be calculated. Notice that the last terms in (6.25) and in the corresponding equation for a_2'/a_2 are equal to each other and therefore are cancelled out from (7.12).

Figure 8 shows the estimates of a_2/a_1 using Smith's (1975) theory and the present theory in the simulation of a straight caustic, which coincide with each other very well. However, to make sure that no common errors (e.g., the discretization errors) have occurred to both estimators, in figure 8 we also calculate a_2/a_1 directly from the expression

$$\frac{a_2}{a_1} = \left[-\frac{(U_x + C_{gx1})\sigma_2}{(U_x + C_{gx2})\sigma_1}\right]^{\frac{1}{2}}$$

in which the values of $\sigma_1, \sigma_2, C_{gx1}$ and C_{gx2} at each point are determined simply by substitution of $k_y = -0.94$ rad/m and $n_0 = 3.62$ rad/s into the dispersion relation (2.2). The results in figure 8 indicate that the present numerical schemes with sufficiently small grid spacings are indeed very accurate (the deviation of the curve in figure 8 from a straight line near its left end is due to the errors of $\sqrt{\psi_{1x}}$ instead of a_2/a_1).

In figure 9 we also compare the numerical solutions of $-iG_0(-\psi_1)^{-1/2}$ and Q_0 with their analytical solutions, in which the numerical solution of Q_0 at each point in the vicinity of the caustic was calculated by using the relation

$$Q_0 \approx -2\frac{a_1'}{a_1} - i\frac{G_0}{\sqrt{\psi_{1x}}} - \frac{k_{x2}' - k_{x1}'}{k_{x2} - k_{x1}} \tag{7.14}$$

in view of (6.22a) and (6.15). The results in figure 9 are indeed very satisfactory except that in the immediate vicinity of the caustic, the numerical schemes have produced significant errors owing to the singularities of a_1, $\mathbf{k_1}$ and $\mathbf{k_2}$ at the caustic. This comparison has provided a valuable check on the expressions of the Class 2 terms, including (4.6) for ψ_1, which are available only for a straight caustic.

In the simulation of a curved caustic, this comparison is not feasible because of the absence of these expressions. However, when (7.9) is utilized to estimate a_2/a_1 in this case, it is found that the values of $(a_2/a_1)^2$ at all points in the vicinity of the caustic have become negative, which is impossible, meaning that significant errors have occurred to the estimates. On the other hand, when the present theory is applied, especially when (7.12) is invoked, the values of $-iG_0(-\psi_1)^{-1/2}$ also become negative and more importantly, they are far from being constant (see figure 10 and note that the ordinate now represents $iG_0(-\psi_1)^{-1/2}$ instead of $-iG_0(-\psi_1)^{-1/2}$). Therefore it is evident that in the simulation of a curved caustic, if the relation (7.12) is invoked, large errors have also occurred in the application of the present theory.

To identify the sources of these errors, we temporarily neglect the second and third terms on the right-hand side of (6.25) and neglect the last term in (7.9) correspondingly (by assuming that $\partial[(U_y + C_{gy})a^2/\sigma]/\partial y = 0$ and $\partial(a_1^2/\sigma_1)/\partial t = 0$). Therefore, if the resulting quantities are designated by a bar, we have

$$\left(\frac{\overline{a_2}}{a_1}\right)_{SM} \approx \left[-\frac{(U_x + C_{gx1})/\sigma_1}{(U_x + C_{gx2})/\sigma_2}\right]^{\frac{1}{2}} \tag{7.15}$$

from Smith's (1975) theory, and

$$\left(\frac{\overline{a_2}}{a_1}\right)_{ST} \approx \exp\left[-2i\overline{G_0}(-\psi_1)^{-\frac{1}{2}}(-x)^{\frac{1}{2}}\right] \tag{7.16}$$

from the present theory, in which

$$i\overline{G_0}/\sqrt{\psi_{1x}} \approx \overline{\frac{a_2'}{a_2}} - \overline{\frac{a_1'}{a_1}}, \tag{7.17}$$

where

$$\left.\begin{aligned} \overline{\frac{a_1'}{a_1}} &= -\frac{\sigma_1}{2(U_x + C_{gx1})}\frac{\partial}{\partial x}\left(\frac{U_x + C_{gx1}}{\sigma_1}\right), \\ \overline{\frac{a_2'}{a_2}} &= -\frac{\sigma_2}{2(U_x + C_{gx2})}\frac{\partial}{\partial x}\left(\frac{U_x + C_{gx2}}{\sigma_2}\right). \end{aligned}\right\} \tag{7.18}$$

The results in figure 11 show a large discrepancy between $(\overline{a_2/a_1})_{SM}$ and $(\overline{a_2/a_1})_{ST}$ in the simulation of a curved caustic, contrary to the prediction by the theories. However, if the exponent in (7.16) is determined from (7.17) multiplied by $2\overline{x}$ instead of $2x$, where

$$2\overline{x} \equiv \frac{k_{x2} - k_{x1}}{k_{x2}' - k_{x1}'} \tag{7.19}$$

and therefore approximates $2x$ in theory according to (6.23), then the consistency between $(\overline{a_2/a_1})_{SM}$ and the new estimates of $(\overline{a_2/a_1})_{ST}$ is improved significantly in figure 11, implying that the errors in $i\overline{G_0}/\sqrt{\psi_{1x}}$ and in $2\overline{x}$ have mostly been cancelled out from (7.16). This cancellation will become even clearer in the following discussion.

If the error in $\overline{a_2'/a_2} - \overline{a_1'/a_1}$ can be offset by that in $2\overline{x}$, this situation, though not exactly the same, may actually occur to each of $\overline{a_1'/a_1}$ and $\overline{a_2'/a_2}$. Therefore, by using the ratio $\overline{R} = \overline{x}/x$, the errors in $\overline{a_1'/a_1}$ and $\overline{a_2'/a_2}$ can be remedied. Figure 12 shows the results of $\overline{a_1'/a_1}/\overline{R}$ and $\overline{a_2'/a_2}/\overline{R}$ which are much closer to the dominant term $-1/4x$ (cf., (6.27)) than the original estimates. Thus large errors have occurred to the estimates of the first terms on the right-hand sides of (6.25) and of the corresponding equation for a_2'/a_2 which represent $\overline{a_1'/a_1}$ and $\overline{a_2'/a_2}$ respectively. Nevertheless the results in figure 13 indicate that the sums of the estimates of the first and third terms can be very accurate as they are not only close to the dominant term $-1/4x$ but also coincide very well with the values of a_1'/a_1 estimated directly from numerical differentiation of a_1 obtained earlier. Therefore the estimates of the third term also contain a large error which offsets that in the estimates of the first term, but can however lead to significant errors in the estimates of $-iG_0(-\psi_1)^{-1/2}$ in (7.12) (and $\tau_0(-\psi_1)^{-1/2}$ in (7.9)) if the values of a_2'/a_2 in the vicinity of the caustic are estimated using the method described above.

The above results lead us to believe that all these very large errors have originated from misalignment of the curvilinear co-ordinate lines. Since in the vicinity of the caustic, the values of $|U_x + C_{gx1}|$ and $|U_x + C_{gx2}|$ are small (recall that $U_x + C_{gx1} = U_x + C_{gx2} = 0$ at the caustic) compared with $|U_y + C_{gy1}|$ and $|U_y + C_{gy2}|$ in the simulation of a curved caustic (see figure 14), very slight misalignment of the x-axis can cause large relative errors in the estimates of $|U_x + C_{gx1}|$ and $|U_x + C_{gx2}|$ (and therefore in the estimates of $\overline{a_1'/a_1}$ and $\overline{a_2'/a_2}$ according to (7.18)). Furthermore, while the major term $-1/4x$ in $\overline{a_1'/a_1}$ and $\overline{a_2'/a_2}$ are offset in (7.17), these relative errors will be magnified even further in the estimates of $-i\overline{G_0}(-\psi_1)^{-1/2}$ using (7.17). On the other hand, since in the vicinity of the caustic, $\partial a_1/\partial y$ and $\partial a_2/\partial y$ are very small compared with $\partial a_1/\partial x$ and $\partial a_2/\partial x$, very slight misalignment of the co-ordinate lines can also produce large errors in the estimates of $\partial a_1/\partial y$ and $\partial a_2/\partial y$. These errors together with those in $|U_x + C_{gx1}|$ account for the disproportionately large percent changes in the third term on the right-hand side of (6.25) and in the last term in (7.9). However, for those quantities (for example, a_1'/a_1) which are insensitive to a slight rotations of the co-ordinate axes, their estimates, no matter which method is applied, will remain nearly unchanged under slight misalignment of the co-ordinate lines, that can explain why in figure 13 the sum of the estimates of the first and third terms on the right-hand side of (6.25) coincides very well with that obtained directly from numerical differentiation of a_1 which itself is the numerical solution of (7.1).

Since equation (7.1) was solved on a rectangular grid which is independent of the location of the caustic, and since $a_1' \equiv \partial a_1/\partial x \gg \partial a_1/\partial y$, slight misalignment of the co-ordinate lines has obviously no effect on the solution values of a_1 and has only a very small effect on the estimates of a_1'/a_1 obtained straightforwardly from numerical differentiation of a_1. Therefore the estimates of a_1'/a_1 in figure 13 should be very accurate, except that in the immediate vicinity of the caustic large discretization errors may occur due to the singularities at the caustic. Incidentally, the small differences between a_1'/a_1 and $-1/4x$ in figure 13 have provided evidence that the position of the curved caustic (from which the values of x were measured) had been located accurately.

From the above discussion it is clear that in a general situation, the error magnification phenomenon will render the relations (7.9) and (7.12) useless. Even the values of $(\overline{a_2/a_1})_{SM}-1$ cannot be estimated reliably from (7.15), because as $|U_x + C_{gx1}|/\sigma_1$ and $|U_x + C_{gx2}|/\sigma_2$ are small in the vicinity of the caustic, their difference, which divided by their mean value is responsible for $\overline{a_2/a_1}$ being unequal to unity, is even smaller so that the error magnification phenomenon also occurs to the estimates of $(\overline{a_2/a_1})_{SM}-1$ in (7.15). This can be substantiated by showing the relation between the estimates of $(\overline{a_2/a_1})_{SM}$ and $(-x)^{1/2}$ in figure 15 which is far from being linear (see figure 8 for a comparison) and therefore should be incorrect. The linear relationship between the estimates of $(\overline{a_2/a_1})_{SM}$ (or $(\overline{a_2/a_1})_{ST}$) and $(k_{x2} - k_{x1})/2$ ($\approx \sqrt{\psi_1 x}$ in theory) in figure 11, both of which containing significant errors, just shows another invariant under rotation of the axes by a small angle. The development of the analytical theories for all these phenomena would be extremely difficult if not impossible, therefore we here rest content with the discussion of the consistency between the results.

The error magnification phenomenon will certainly become less severe in the regions far away from the caustic, however in these regions the one-term approximations given in (7.8), (7.10), (7.12), etc. may become insufficient. On the other hand, although the approximation $a_2/a_1 \approx 1$ can be accurate enough in the immediate vicinity of the caustic, the estimates of a_1 themselves in this region may contain significant errors due to the fact that a_1 and $\mathbf{k}_1$ are singular at the caustic. Besides, when the caustic is curved and the reflected wave field is still solved on a rectangular grid for convenience, it is often required to determine the boundary conditions of the reflected wave at the grid points with diverse values of x. As a consequence, the approximation $a_2/a_1 \approx 1$ cannot be applied equally well on these points. Therefore another effort should be made to avoid the error magnification phenomenon.

Since both a_1'/a_1 and $1/4x$ can be estimated accurately and from (6.22a), (6.23) and (6.15) we have

$$-\frac{iG_0}{2\sqrt{\psi_1 x}} \approx \frac{a_1'}{a_1} + \frac{1}{4x}, \tag{7.20}$$

the approximation of $-iG_0(-\psi_1)^{-1/2}$ can therefore be estimated reliably from this relation. By substitution of this value into (7.10), we may finally obtain the approximation of a_2/a_1 at each point in the vicinity of the caustic.

In (7.20) the zeroth power of x has been neglected which in (7.12) is completely cancelled out. This cancellation also occurs to the first power of x in (7.9) (recall that (7.9) was derived by integration of the action equation with respect to x). Therefore it seems that the estimates of $-iG_0(-\psi_1)^{-1/2}$ by using (7.20) represent a lower-order approximation than those by using (7.12), but this is true only if the second and third terms on the right-hand side of (6.25) are vanishingly small, otherwise the estimates of a_2'/a_2 in (7.12) and $(4\tau_0/\psi_1)\sqrt{\psi_1 x}$ in (7.9) using the method described above will introduce the truncation errors of the zeroth and first powers of x respectively, because of the use of the one-term asymptotic approximations in this method. Hence, in a general situation, even without considerations of the error magnification phenomenon, it is still impossible to achieve the same accuracy as that shown in figures 8 and 9 for a straight caustic, although one may expect that the truncation errors in (7.20) will decrease if the modulation rates of the current field get progressively smaller. Nevertheless, even in the present simulation of a curved caustic, since the error magnification phenomenon has mostly been avoided in the application of (7.20), the resulting estimates of $-iG_0(-\psi_1)^{-1/2}$ in figure 16 approach to a constant far more satisfactorily than those in figure 10.

Since the true value of $-iG_0(-\psi_1)^{-1/2}$ in figure 16 is unknown, for a comparison between the numerical and analytical solutions, we also estimate this quantity by using (7.20) in the simulation of a straight caustic. The results in figure 17 indicate that the new estimates of $-iG_0(-\psi_1)^{-1/2}$, though less accurate than those in figure 9, can fit the analytical solution to within 25%. Therefore, if the true value of a_2/a_1 at a certain point is 1.25, then neglect of the exponent in (7.10) produces a relative error of 20% in a_2/a_1, but this figure can be reduced to about 5% by an application of the present theory.

After the approximation of a_2/a_1 (and therefore a_2) at each point in the vicinity of the caustic has been determined, these values and the values of $\mathbf{k}_2$ in the same region obtained earlier can serve as the boundary conditions for calculations of the reflected wave in the regions away from the caustic. This task is just routine and therefore requires no elaboration here.

8. Conclusions

When short deep-water gravity waves propagate obliquely upon a steady unidirectional irrotational current and are reflected by the latter, a second-order ordinary differential equation for the surface displacement of the short waves is derived from the Laplace equation and the kinematical and dynamical boundary conditions. This equation takes the same form as that derived by Shyu & Phillips (1990), although the expressions of the Class 2 terms in the coefficients of the present equation are much more complicated than those in Shyu & Phillips (1990). The regularity of this equation at the caustic is demonstrated and its uniform asymptotic solution and the corresponding WKBJ solution are subsequently derived. The satisfaction of the action conservation principle by this WKBJ solution at every point including the caustic has also been proved elsewhere.

Except the expressions of the Class 2 terms, Shyu & Phillips' (1990) solutions and the present solutions take the forms valid even for waves in an intermediate-depth region and near a curved moving caustic induced by an unsteady multidirectional irrotational current. This suggestion is verified in a curvilinear co-ordinate system from considerations of the dispersion relation and the action conservation equation which themselves have been deduced by Smith (1975) in the vicinity of the caustic in exactly the same situation. In this general situation, the Class 2 term in the solutions which is responsible for the amplitude of the reflected wave being unequal to that of the incident wave in the vicinity of the caustic, can be estimated in a numerical calculation. The algorithm for this estimation is developed and tested in the numerical simulations of a straight and a curved caustics, but its validity in the case of a moving caustic is also obvious. The results of these simulations indicate that for a curved caustic, while the errors due to misalignment of the co-ordinate lines are magnified very seriously in the previous estimates of the amplitude of the reflected wave in the vicinity of the caustic from a consideration of the action conservation principle, this situation can be improved significantly by using the present algorithm.

The cause of the error magnification phenomenon is that when the characteristic velocity component $U_y + C_{gy}$ in the direction along the caustic is significant, very slight misalignment of the co-ordinate lines will in the vicinity of the caustic produce large percent changes in $U_x + C_{gx1}$ and $U_x + C_{gx2}$ and in those related to the convergence of the action flux in the y-direction; the latter is constrained by the action conservation equation. Therefore, in this situation, any efforts to evaluate the difference between the amplitudes a_1 and a_2 of the incident and reflected waves in the vicinity of the caustic using the action conservation principle directly will inevitably fail. However, in the application of the present theory, since the forms of the expressions for a_1 and a_2 have been established, the term in these expressions which is responsible for a_2 being unequal to a_1 in the vicinity of the caustic can therefore be estimated straightforwardly from the numerical solutions of a_1 and $\partial a_1/\partial x$, which even in the vicinity of the caustic are insensitive to a slight rotation of the co-ordinate lines and therefore are devoid of the error magnification phenomenon.

Finally, we note that since the extension of the theory to the general case in section 6 is based on the dispersion relation and the action conservation equation and their properties, especially those which are essential to the analysis, are common in many situations, the conclusion in section 6 about the forms of the solutions might also be drawn for the capillary blockage phenomenon (Phillips 1981), for waves propagating on a rotational current with uniform vorticity, or for an even more general situation. However, to verify these conjectures rigorously, the validity of the dispersion relations and the action equations in the vicinity of the caustic in these situations should be demonstrated by extensions of Smith's (1975) theory. Also we notice that the dotted line in figure 1 implies that a different type (the curve in figure 1 changes from convex to concave near the point at which the dotted line is a tangent to this curve) of the gravity blockage phenomenon may occur in the regions with much weaker currents and smaller k_x. This situation, if it is true, would make it possible for double reflection and eventually a triple turning point of the gravity waves to occur, similar to those of the capillary/gravity waves which were suggested and solved by Trulsen & Mei (1993).

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Caption of Figure

Figure 1. Solutions of the dispersion relation (2.3) for given n_0. The dashed line represents the situation occurred at the caustic where the solution points A and B coalesce and therefore $k_{x1} = k_{x2}$. The dotted line represents another tangent of the curve, signifying the occurrence of the blockage and reflection phenomenon of a different type.

Figure 2. Definition sketch.

Figure 3. A diagram for illustration of the strategy to locate the caustic and compute the incident wave near the caustic.

Figure 4. The domain of integration and the true (y–axis) and predicated (dashed line) locations of the caustic in the simulation of a straight caustic. All test results described below are along the dotted line.

Figure 5. The domain of integration and the directions of the incoming wave and the current field in the simulation of a curved caustic.

Figure 6. A part of the integration domain in figure 5 in which the caustic (solid curve) is located and the reflected wave estimated. All test results described below are along the dotted line.

Figure 7. Dimensionless vorticity of the estimate wave-numbers of the reflected waves in the simulations of a straight caustic (circles) and a curved caustic (triangles).

Figure 8. Estimates of a_2/a_1 by using Smith's (1975) theory ($\odot$) and the present theory ($\times$) in the simulation of a straight caustic. The curve represents the true values calculated from (7.13).

Figure 9. Estimates of $-iG_0(-\psi_1)^{-1/2}$ (circles) and Q_0 (triangles) in the simulation of a straight caustic by using (7.12) and (7.14) respectively and comparisons with their analytical solutions (horizontal lines).

Figure 10. Estimates of $iG_0(-\psi_1)^{-1/2}$ (circles) and Q_0 (triangles) in the simulation of a curved caustic by using (7.12) and (7.14) respectively.

Figure 11. Estimates of $\overline{a_2/a_1}$ in the simulation of a curved caustic, $\Delta = (\overline{a_2/a_1})_{SM}$; $+ = (\overline{a_2/a_1})_{ST}$ (using x); $\odot = (\overline{a_2/a_1})_{ST}$ (using $\bar{x}$).

Figure 12. Estimates of $\overline{a_1'/a_1}$ and $\overline{a_2'/a_2}$ with and without using the error reducing strategy (e.r.s.) and comparisons with $-1/4x$, $\times = \overline{a_1'/a_1}$ (without using

e.r.s.); $\odot = \overline{a_1'/a_1}$ (using e.r.s.); $\diamond = \overline{a_2'/a_2}$ (without using e.r.s.); $\Delta = \overline{a_2'/a_2}$ (using e.r.s.); $+ = -1/4x$.

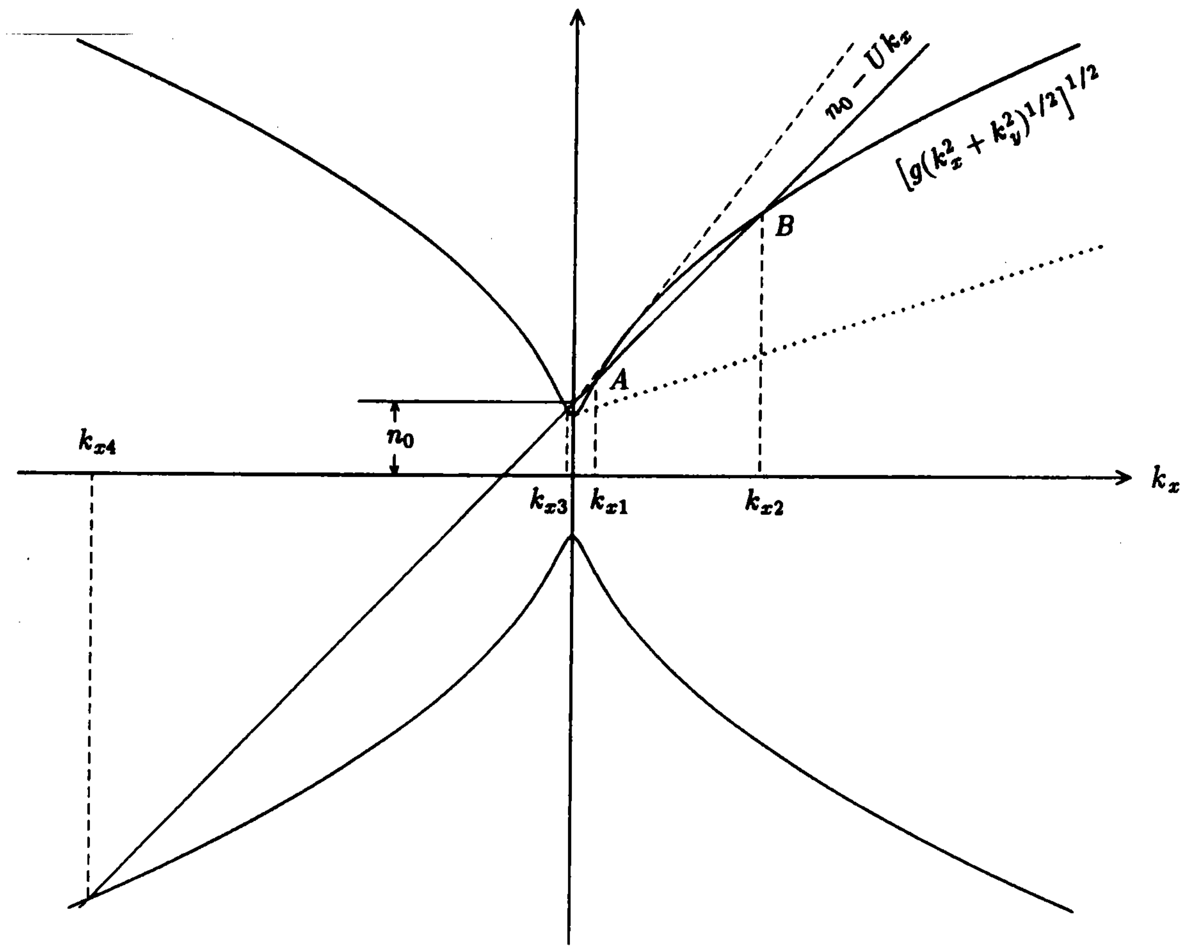
Figure 13. Comparisons between two kinds of estimates of a_1'/a_1 and between a_1'/a_1 and $-1/4x$ (+). The circles and the triangles are the estimates of a_1'/a_1 obtained respectively from numerical differentiation of a_1 and from summation of the estimates of $\overline{a_1'/a_1}$ ($\times$) and (6.30).

Figure 14. Values of $(U_x + C_{gx})/(U_y + C_{gy})$ of the incident ($\odot$) and reflected ($\times$) waves in the simulation of a curved caustic.

Figure 15. Same as figure 11 except that the values of $(-x)^{1/2}$ are chosen as the abscissa.

Figure 16. Estimates of $-iG_0(-\psi_1)^{-1/2}$ by using (7.20) in the simulation of a curved caustic.

Figure 17. Estimates of $-iG_0(-\psi_1)^{-1/2}$ by using (7.20) in the simulation of a straight caustic. The horizontal line represents the analytical solution.



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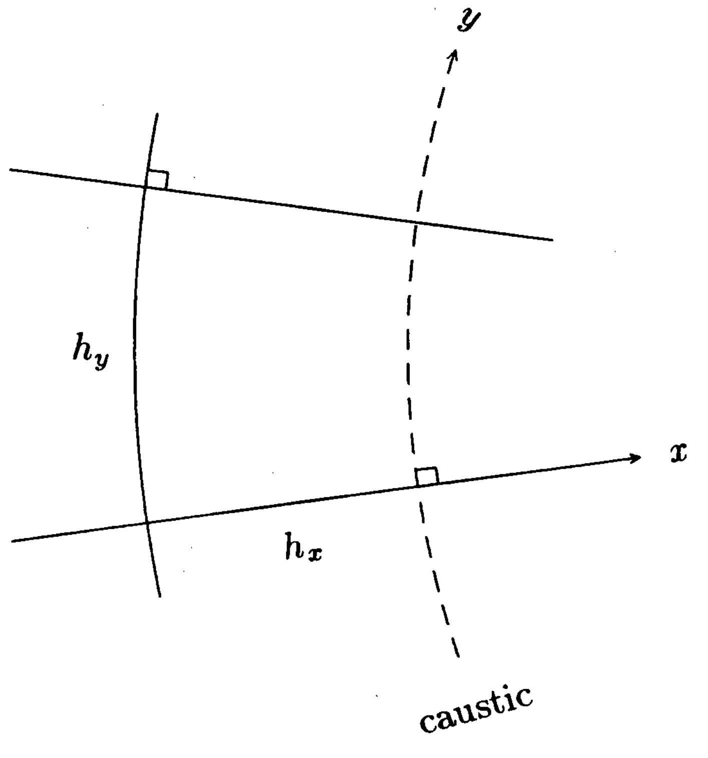

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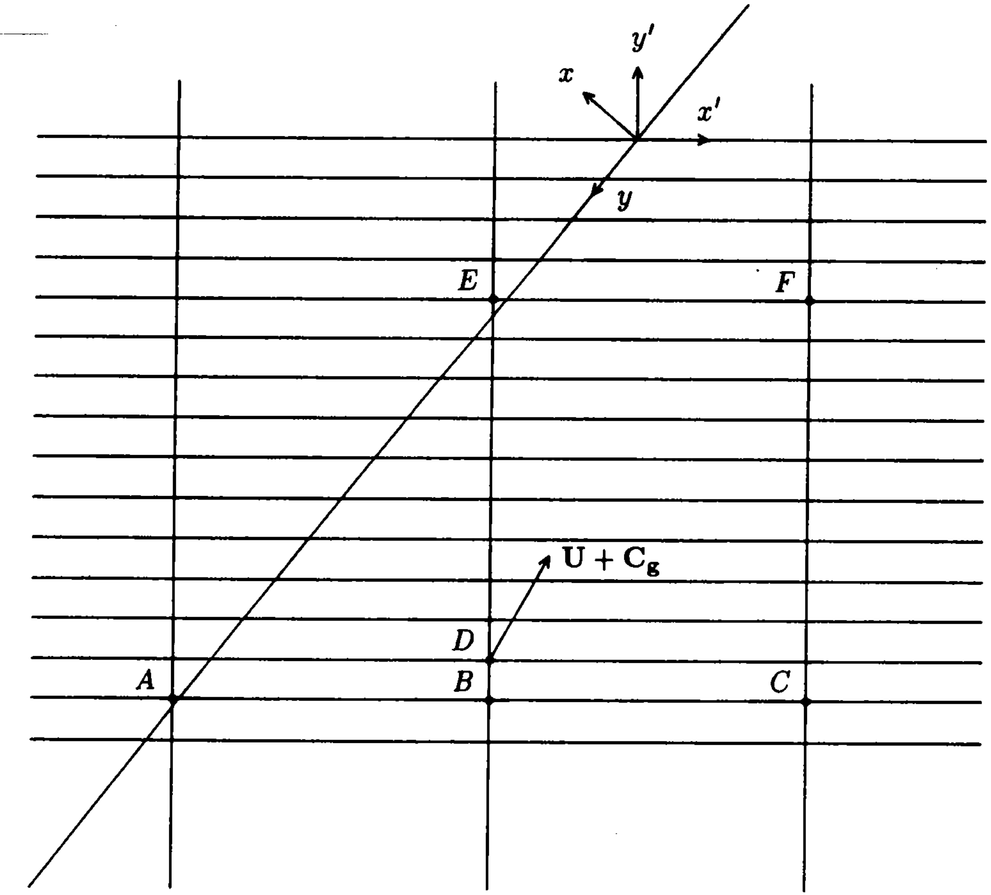

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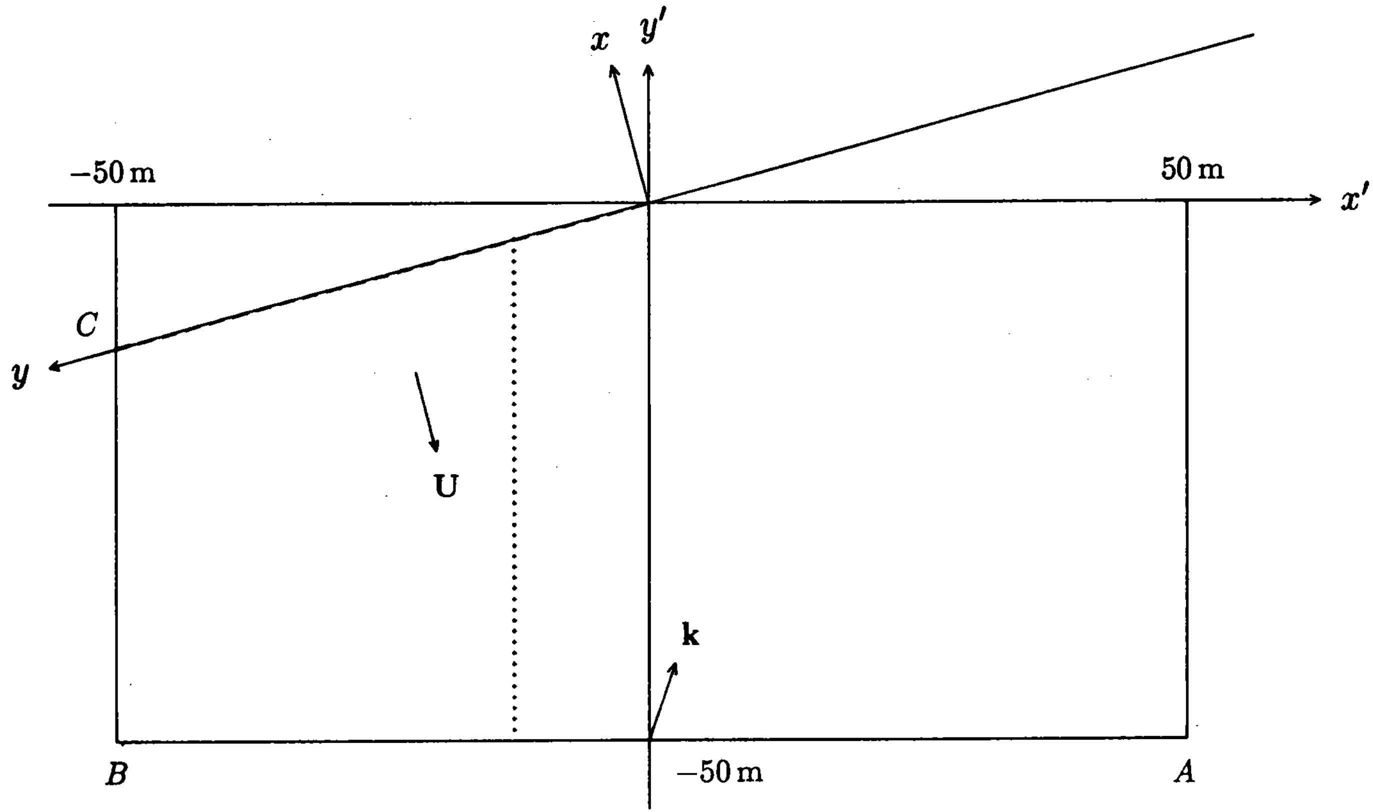

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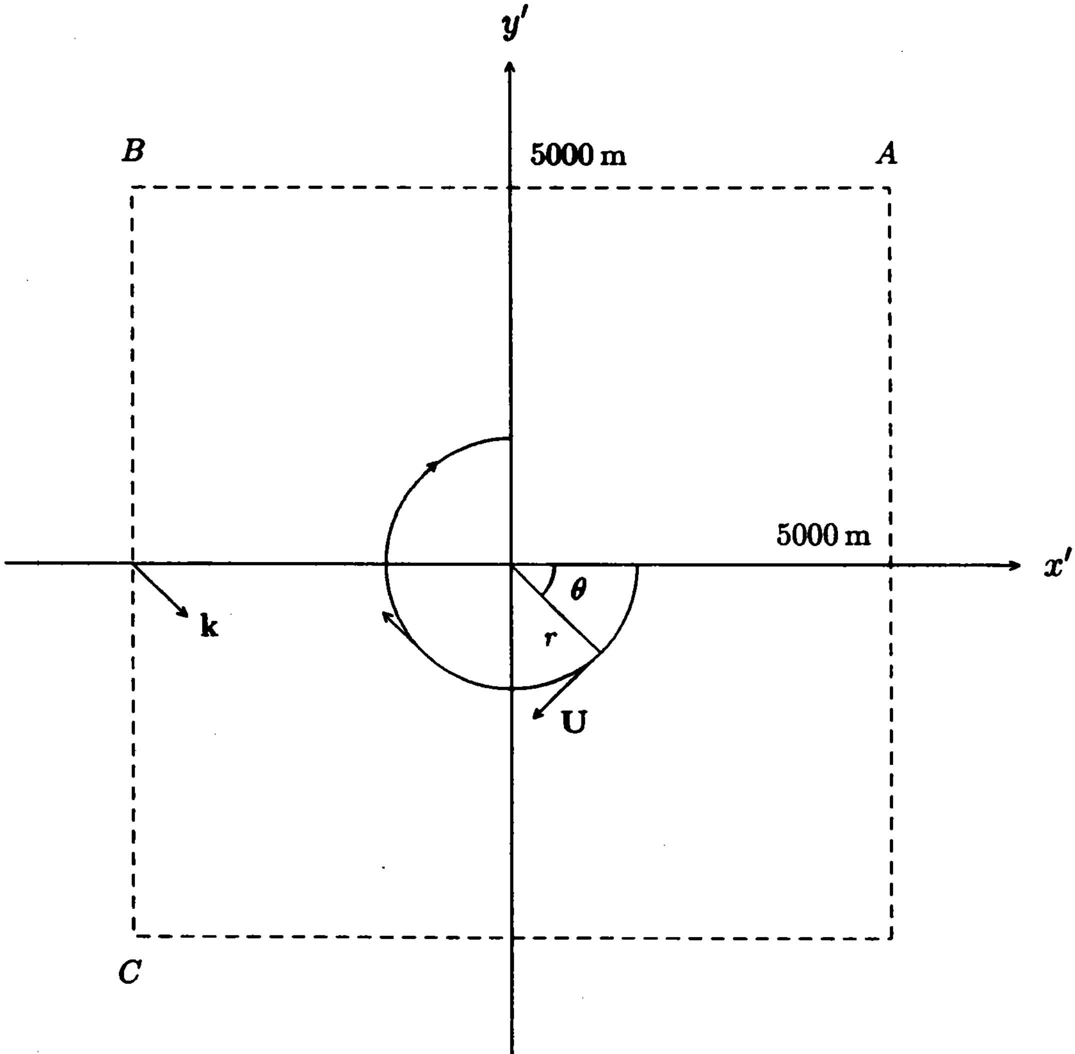

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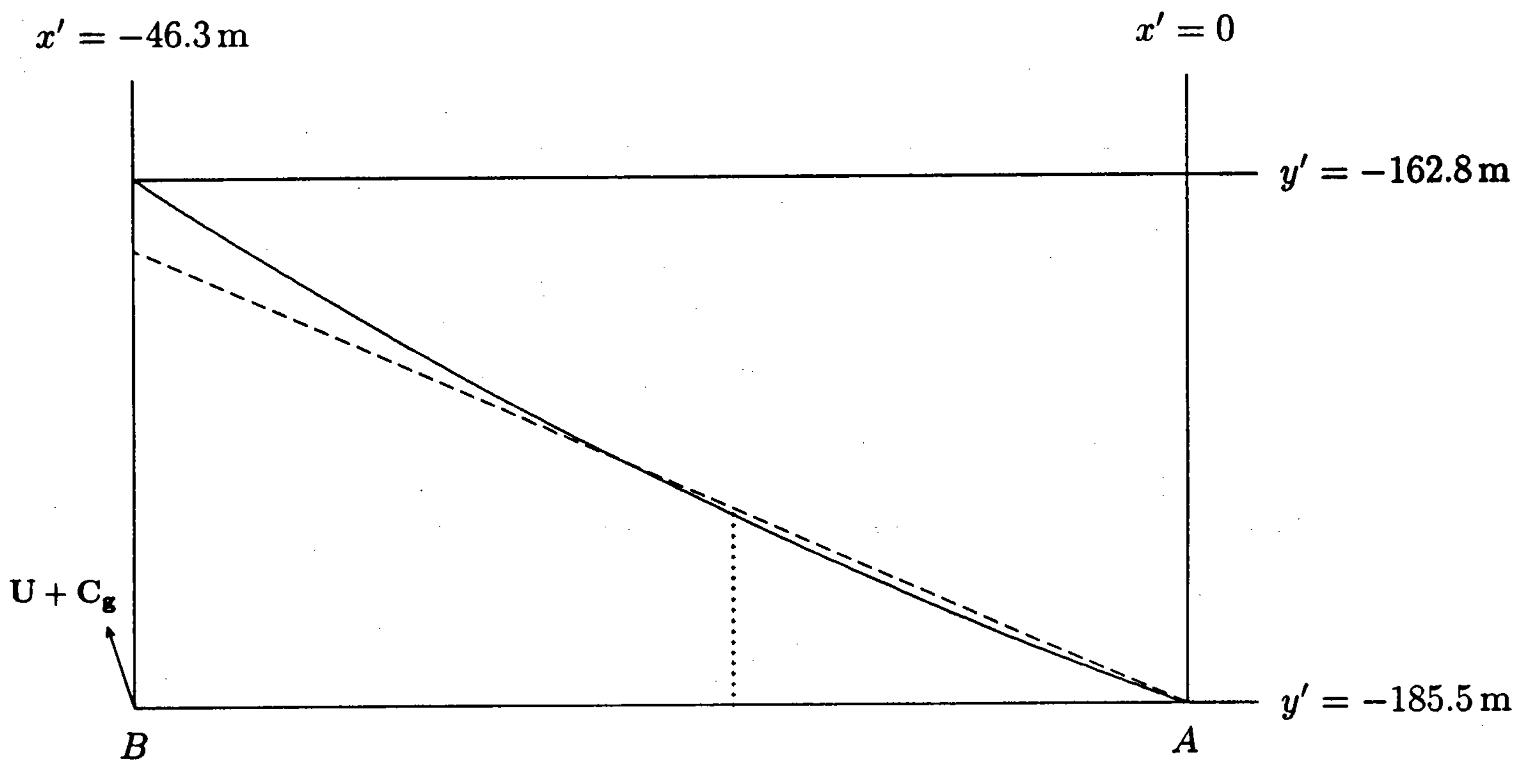

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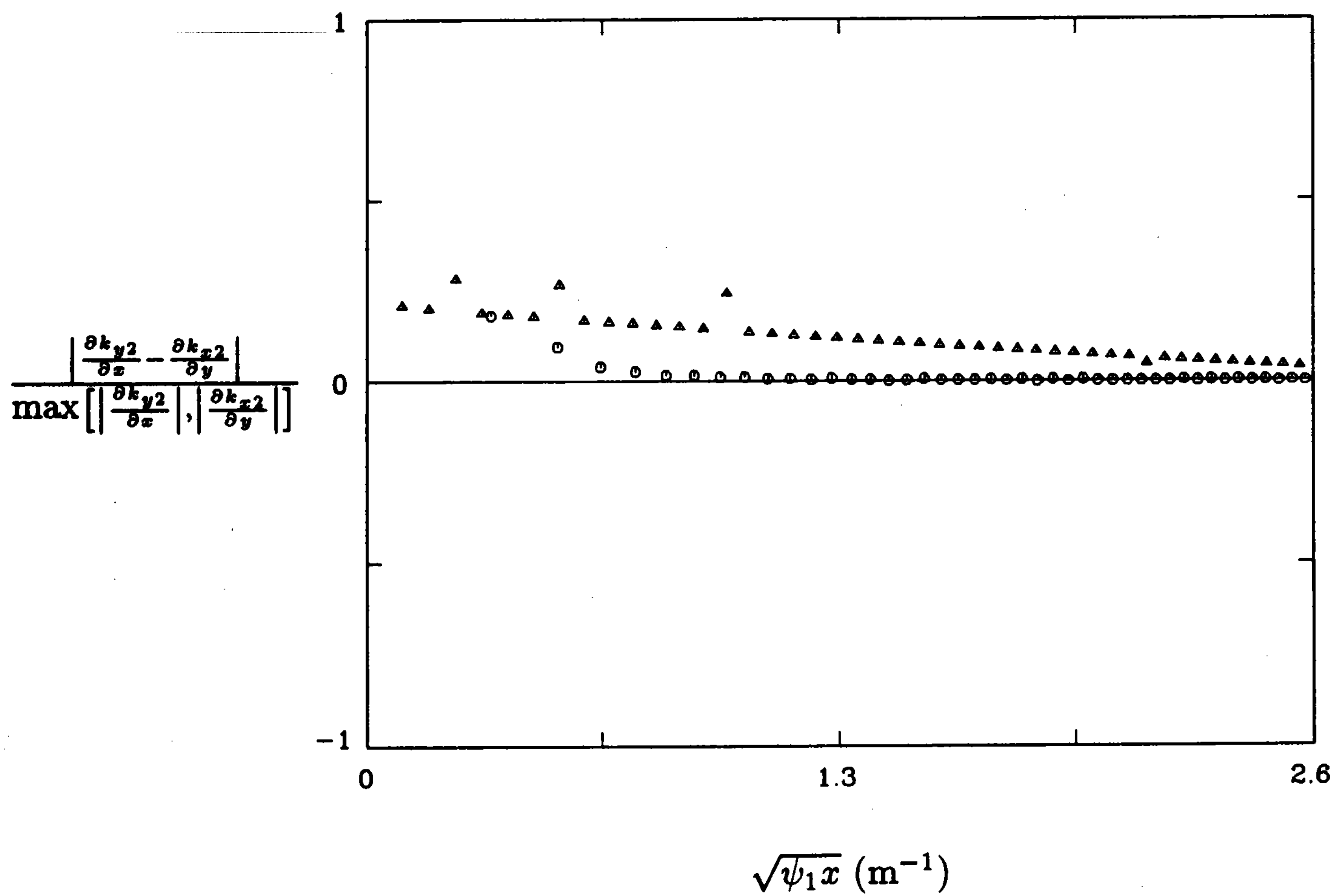
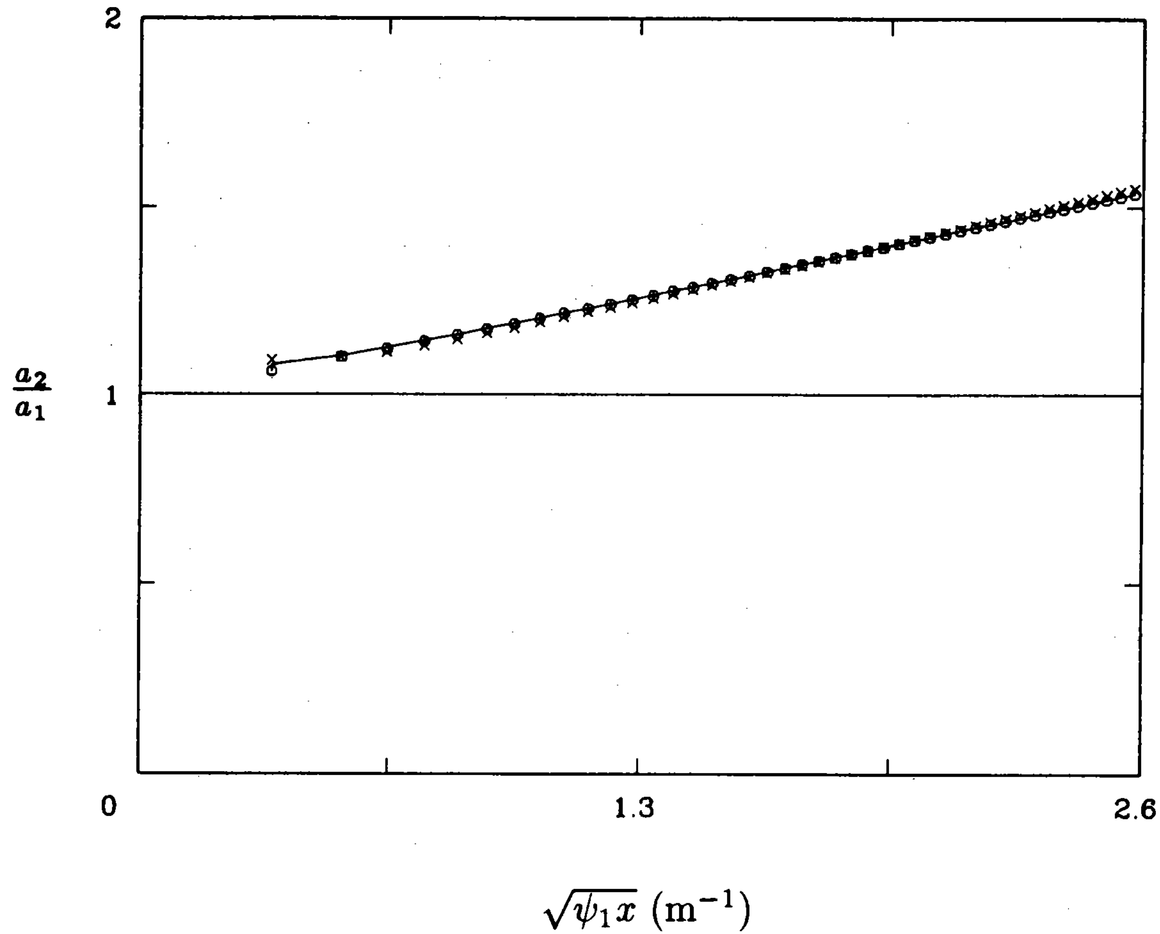


$\frac{\left|\frac{\partial k_{y2}}{\partial x} - \frac{\partial k_{x2}}{\partial y}\right|}{\max\left[\left|\frac{\partial k_{y2}}{\partial x}\right|, \left|\frac{\partial k_{x2}}{\partial y}\right|\right]}$
1
0
−1
0
1.3
2.6
$\sqrt{\psi_1 x}$ (m^{-1})

Figure 7.



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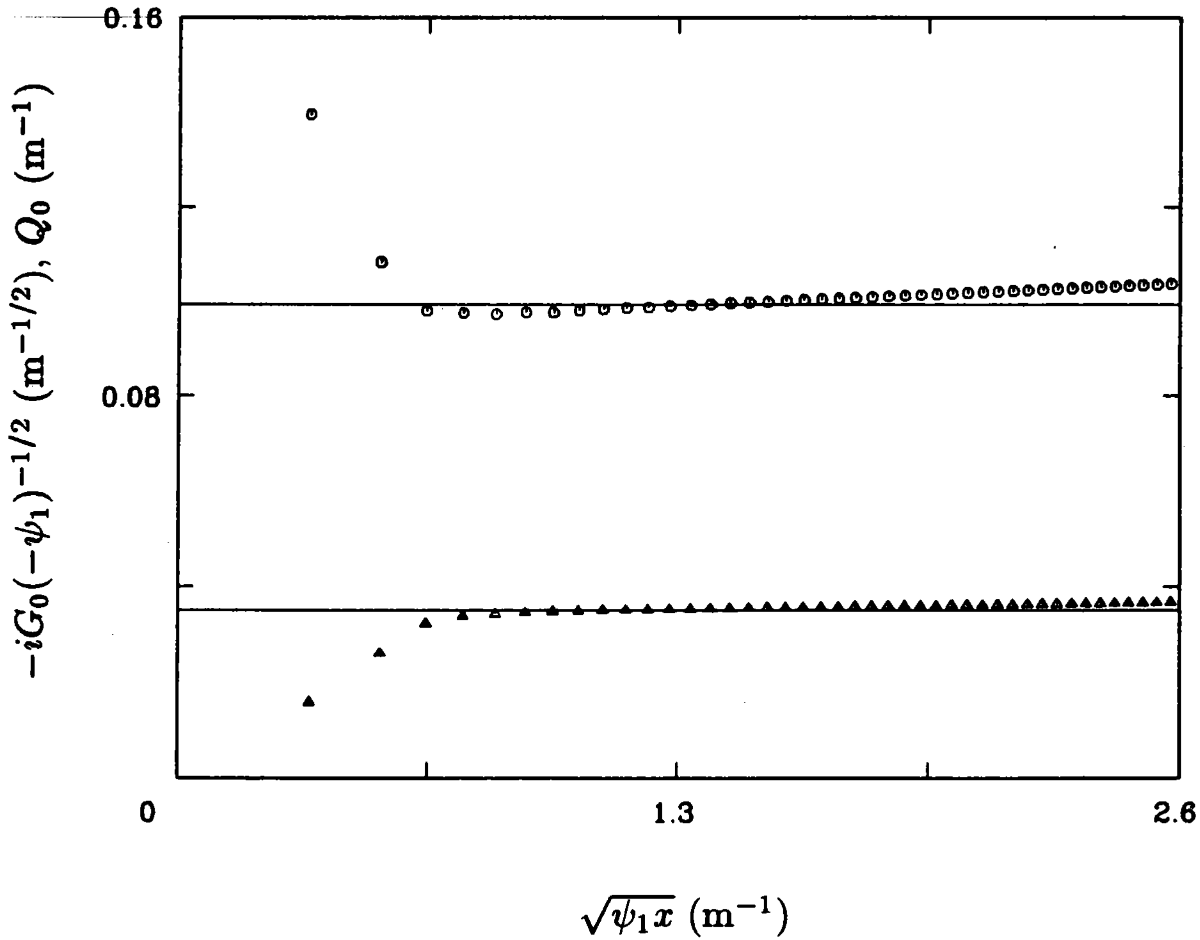

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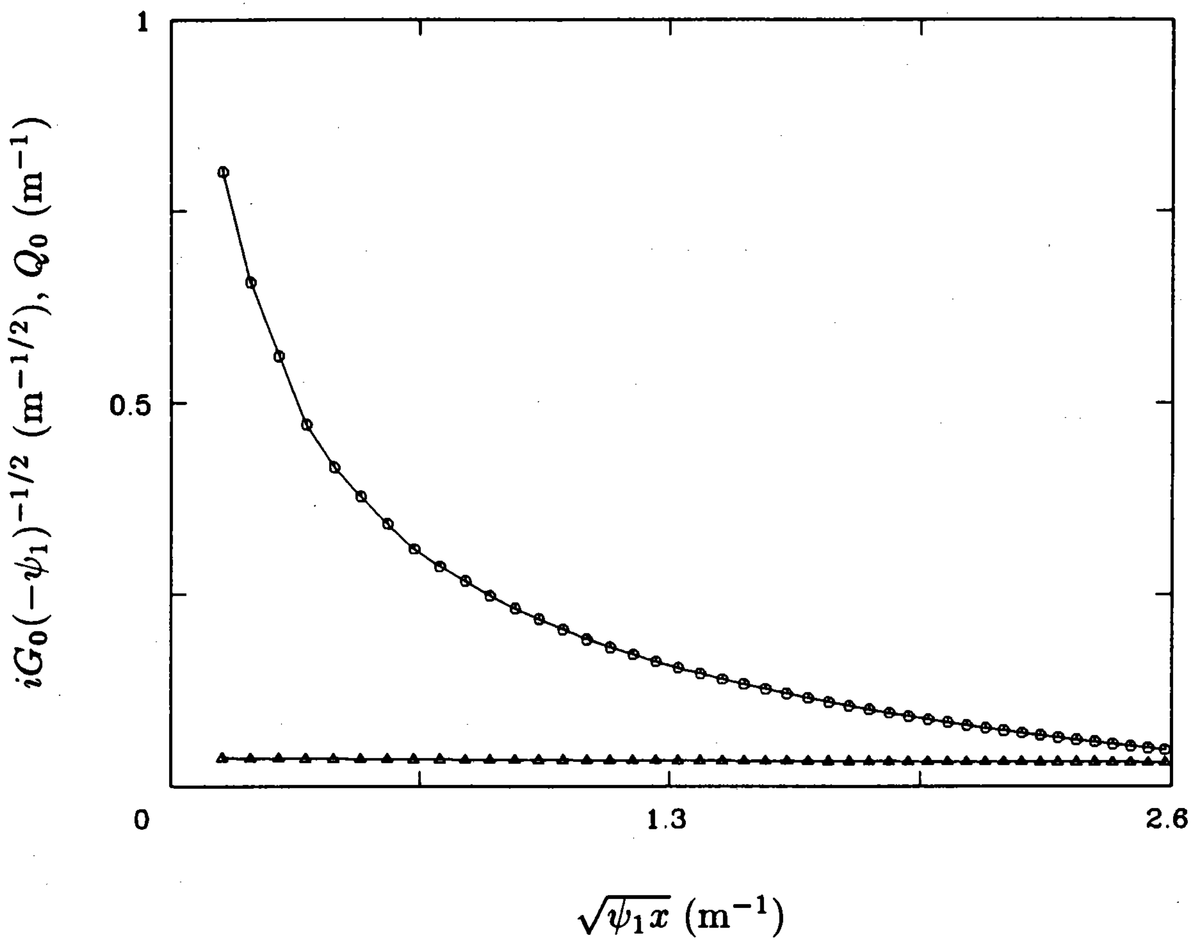

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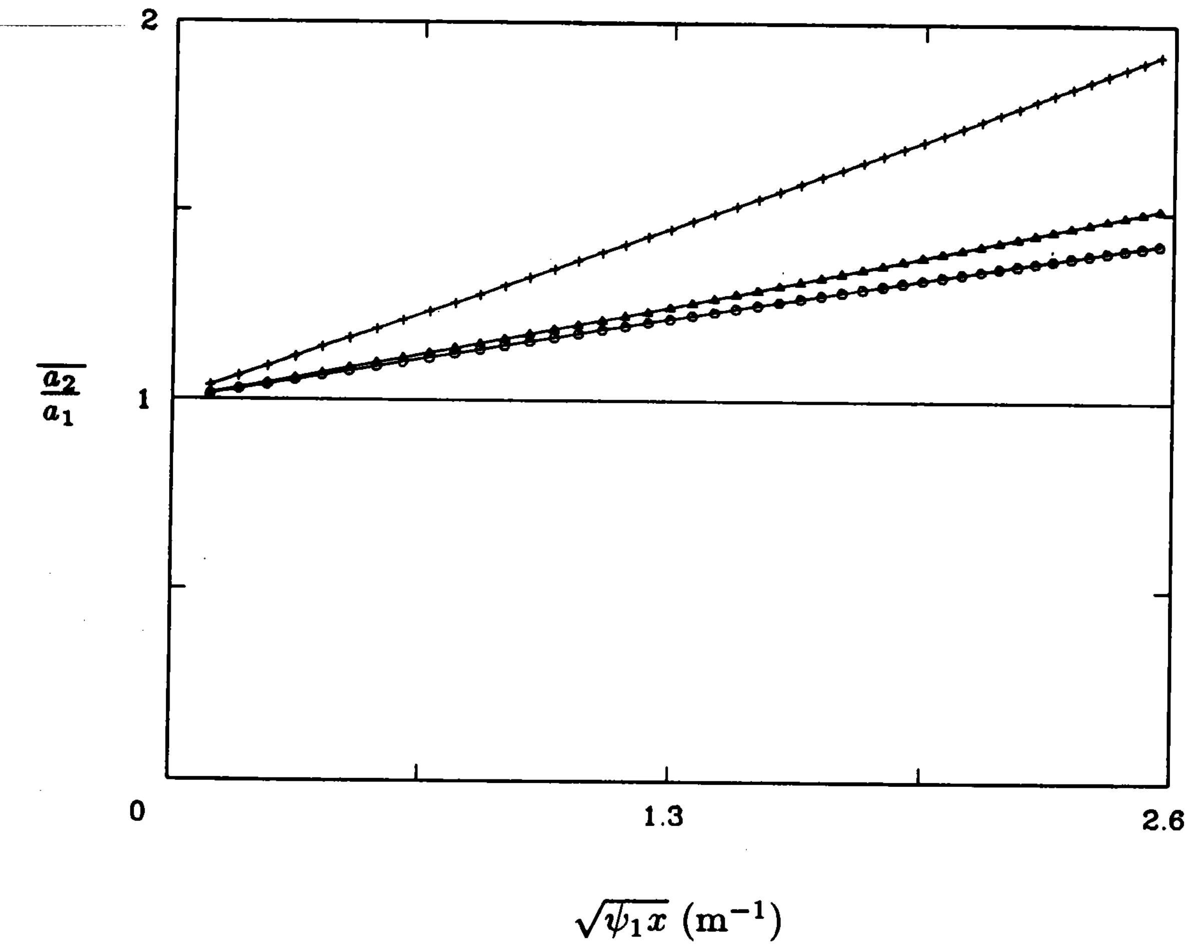

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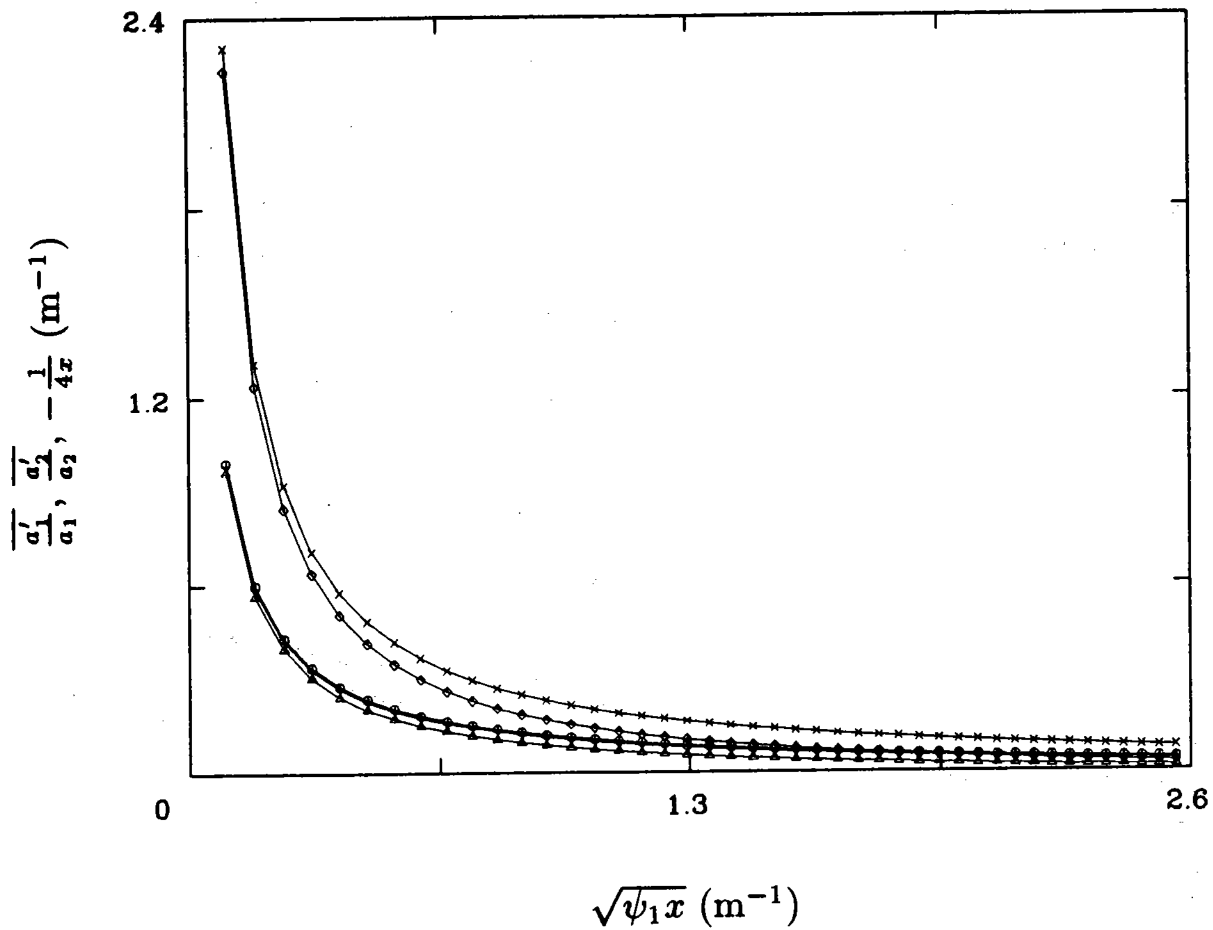

2.4
1.2
0
1.3
2.6
$\frac{\overline{a_1'}}{a_1}, \frac{\overline{a_2'}}{a_2}, -\frac{1}{4x}$ (m^{-1})
$\sqrt{\psi_1 x}$ (m^{-1})

Figure 12.

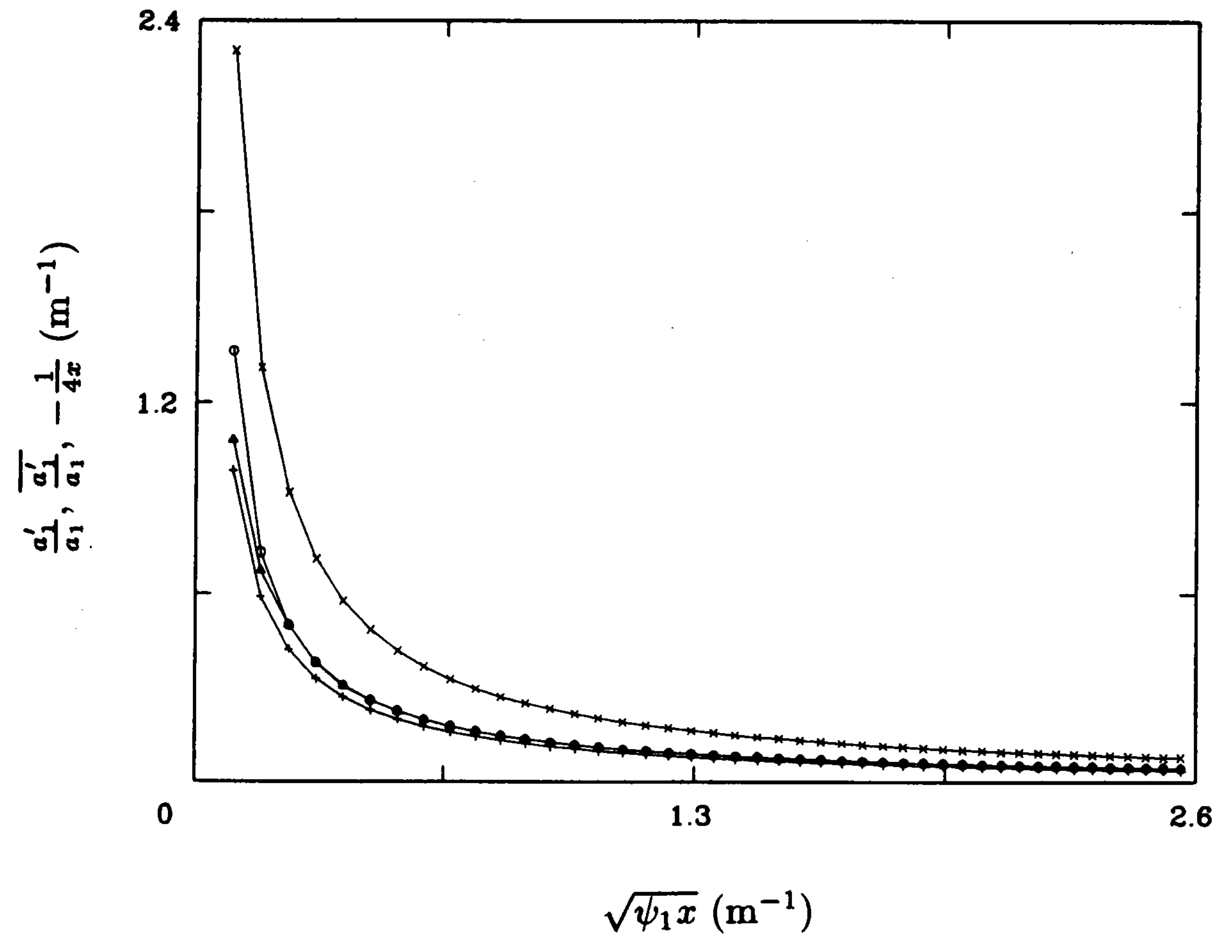
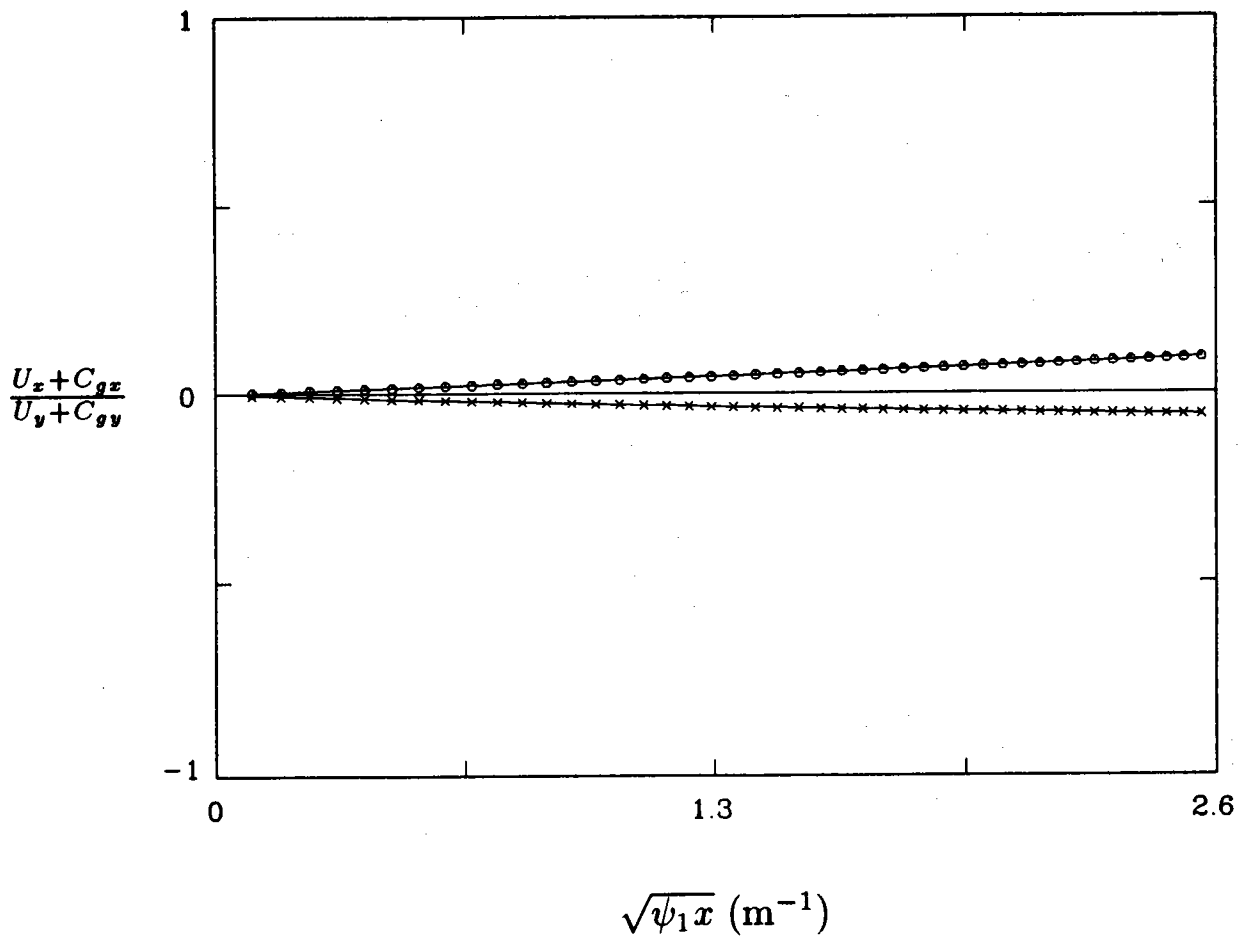

2.4
1.2
0
1.3
2.6
$\frac{a_1'}{a_1}$, $\overline{\frac{a_1'}{a_1}}$, $-\frac{1}{4x}$ (m^{-1})
$\sqrt{\psi_1 x}$ (m^{-1})

Figure 13.



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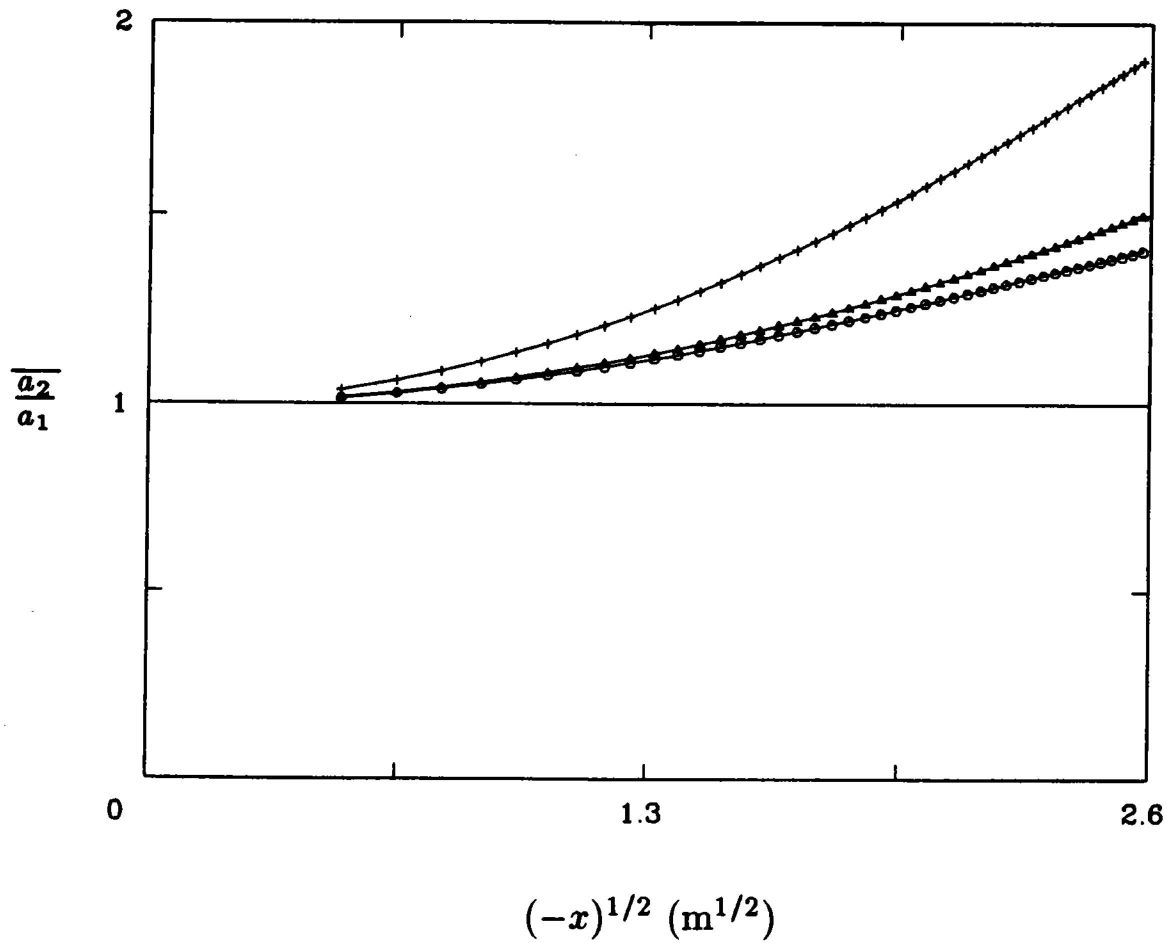

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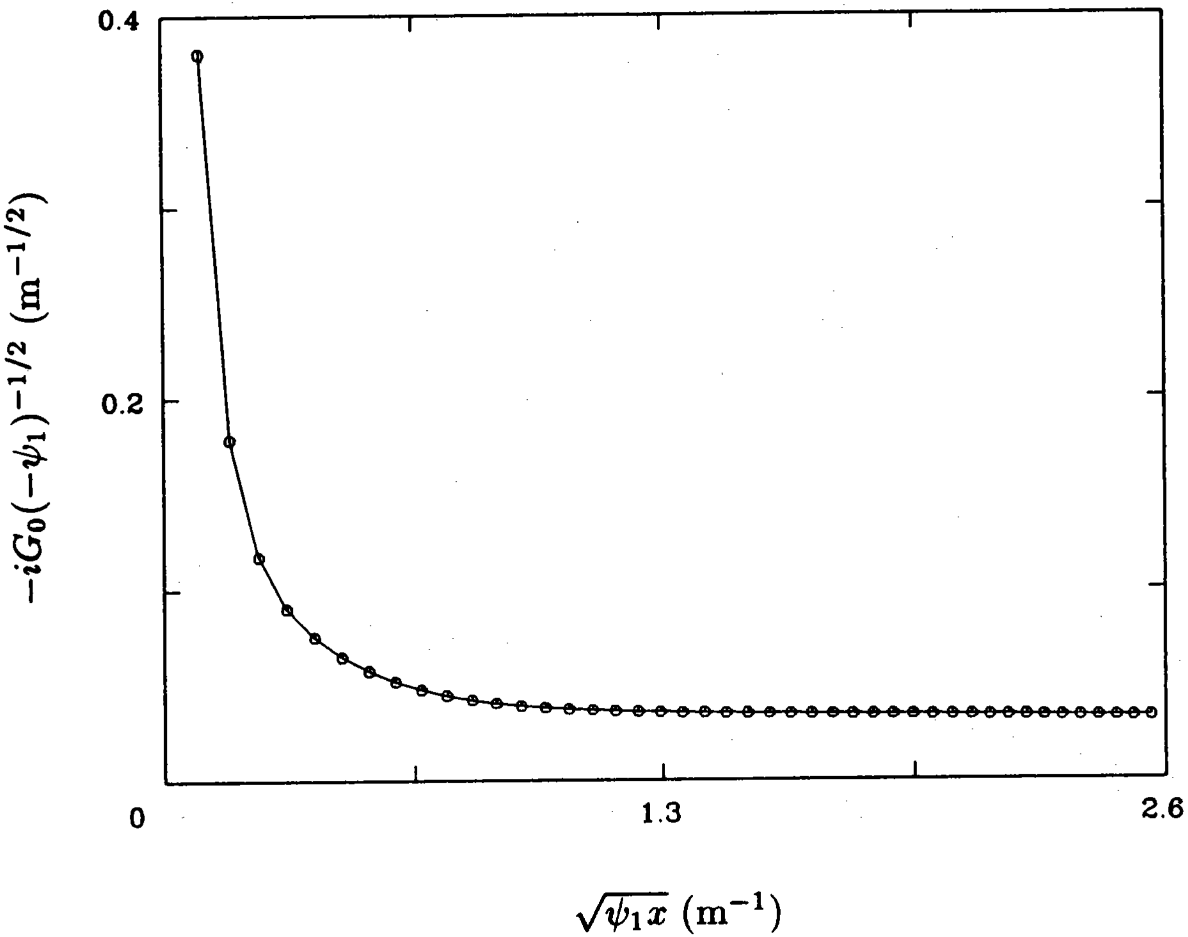

Figure 16.

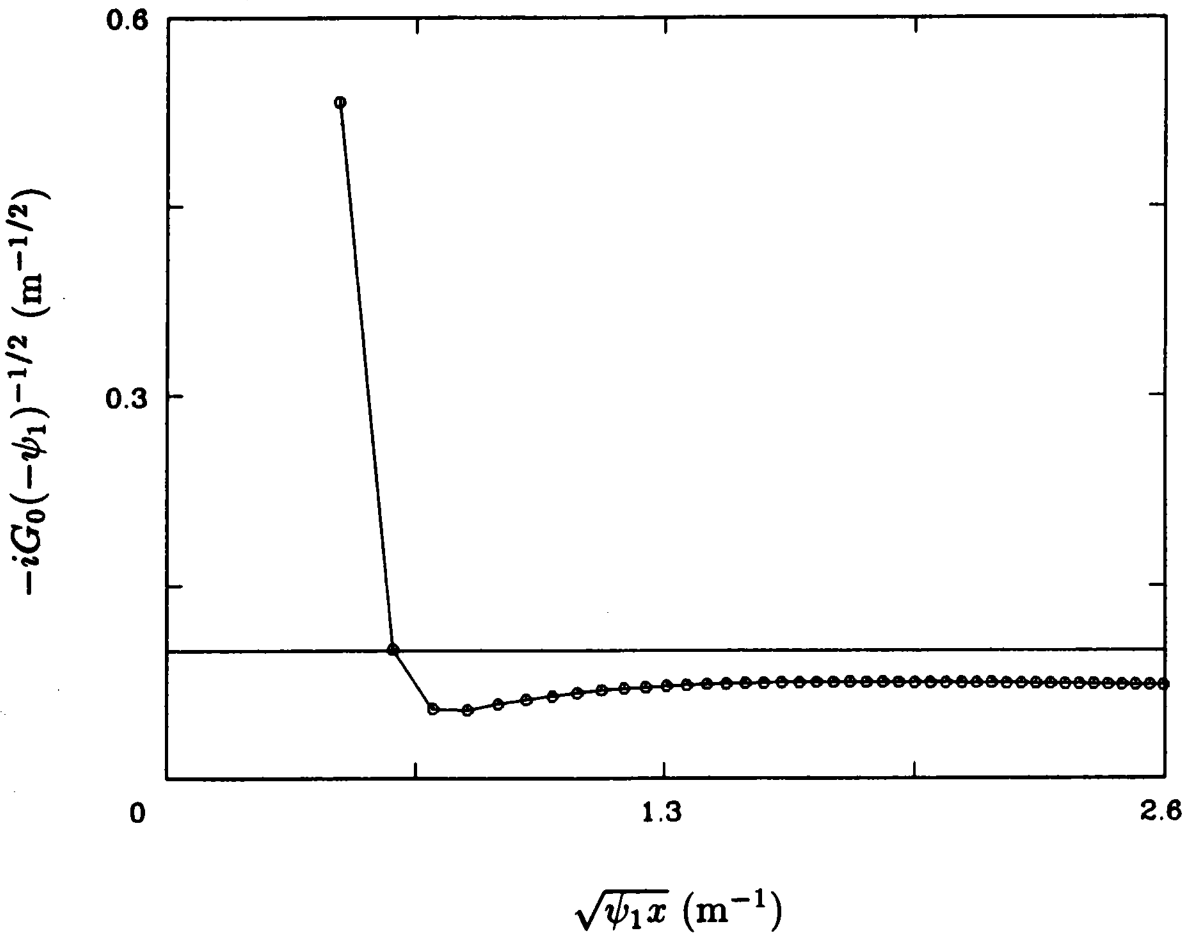

Figure 17.

台中港附近波浪預報模式設計：海流效應

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