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交 通 部 運 輸 研 究 所

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<tr><td colspan="6">摘要：
本研究考慮深水重力波斜行在一穩定三維強剪流上的情況。此流在水平方向緩慢變化，在垂直方向和一線性分佈稍有偏離，故應用一種和攝動法並無不同，但可顯著簡化公式以處理複雜情況的方法，可發展出描述波浪變化的 WKBJ 解。此解並與二維波作用守恆方程式相比，後者乃由 Jonsson, Brink-Kjær & Thomas (1978)的一維方程式自然延伸，因此考慮流中所含的旋度，但忽略在目前的情況下所可能產生的旋性擾動速度。此一比較以及一些數值計算顯示，除非對流的分佈狀況做某些限制，由 Jonsson et al.所定義的波作用在目前的三維流中將無法守恆。

當施加這些限制後，Jonsson et al.的考慮波流運動在一垂直剖面上的積分性質之緩慢變化的方法亦可應用在三維情況，其結果和二維波作用守恆方程式以及用目前的方法所導出的方程式一致。此一分析不僅代表 Jonsson et al.的結果之一有意義的延伸，且可某種程度地解釋為何 Jonsson et al.的方法以及波作用守恆方程式兩者在應用上有所限制，以及為何波作用密度的定義無法將在一般情況下所可能產生的旋性擾動速度包含在內。而在一個較非一般的情況，即使旋性擾動速度確可省略，只要上述限制沒有全部被滿足，Jonsson et al.的方法仍可能無法應用，且由他們所定義的波作用可能不守恆。然而，在此一情況，或在更一般的情況，應用目前的方法所導出的緩變方程式可繼續適用。</td></tr>
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ABSTRACT:

This paper considers deep-water gravity waves propagating obliquely on a steady three-dimensional, strongly sheared current. The current varies slowly in the horizontal directions and deviates slightly from a linear profile in the vertical direction, so that a WKBJ description of wave modulations can be developed by using an approach which is not separate from the perturbation scheme but can however simplify the formulation significantly to take up the rather complicated situation. The resulting modulation equation is compared with the two-dimensional action conservation equation, which represents a natural extension of the one-dimensional one derived by Jonsson, Brink-Kjær & Thomas (1978) and therefore takes the vorticity of the current into account but ignores the rotational perturbation velocity that may occur in the present situation. This comparison and the numerical computations indicate that unless certain restrictions are imposed on the distribution of the underlying current, the wave action defined by Jonsson *et al.* is not conserved in this three-dimensional flow.

When these restrictions are imposed, the approach by Jonsson *et al.*, which considers the slow modulations of the integral properties of the combined wave and current motion across a fixed vertical section, can also be applied in the three-dimensional case and results in an equation consistent with the two-dimensional action conservation equation and the equation derived by the present approach. This analysis, while representing a non-trivial extension of the result derived by Jonsson *et al.*, can also explain why the approach by Jonsson *et al.* and also the action conservation equation have a restricted application and why the definition of the wave action density cannot include the rotational perturbation velocity which may have the same order of magnitude as the irrotational one in a general situation. In a less general situation, even when the rotational perturbation velocity is negligible, as long as the restrictions mentioned above are not fully imposed, the approach by Jonsson *et al.* may still fail and the wave action defined without ambiguity may not be conserved. However, in this or in an even more general situation, the modulation equation derived by the present approach remains valid.

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1. Introduction

Previous studies of the wave-current interactions, well documented in the review articles by Peregrine (1976), Jonsson (1990) and Thomas & Klopman (1997), may be divided into two categories: the first is to study the interactions between the waves and the currents that are all horizontally uniform. In these studies, much attention has been given to the effects of the large amplitudes of waves and the strong shear of currents. Consequently, certain numerical calculations are often needed (see, for example, Simmen & Saffman 1985 and Teles da Silva & Peregrine 1988). However an analytical solution in terms of an infinite series in powers of a certain parameter, which characterizes the smallness of the deviation of the wave motion from the potential motion, was derived by Shrira (1993) for linear waves propagating obliquely on a steady, strongly sheared current. Since this series solution can be rapidly convergent in a practical situation, this solution, as pointed out by Shrira (1993), is useful to the study of the 'gradually varying problem', which is among the second category.

In the second category, the underlying current is allowed to vary slowly in horizontal directions due to perhaps a slowly varying bed. These variations will certainly result in the corresponding slow modulations of the wave amplitudes and wavelengths. Modern theories on this problem were begun by Longuet-Higgins & Stewart (1960, 1961), Whitham (1965), and Bretherton & Garrett (1968), in which the idea of radiation stress was introduced and the action conservation equation established for the case of an irrotational current. Although these theories can be applied to many practical situations (e.g. waves on the majority of tidal flows), there are situations (e.g. waves on a wind-drift current) in which a highly sheared current exists so that these theories may become invalid.

An extension of the theories from irrotational currents to rotational ones has successfully been made by Jonsson, Brink-Kjœr & Thomas (1978) in a two-dimensional analysis, in which new definitions and expressions for the radiation stress, the wave energy density, and the wave action density have been given to include the effects of the constant vorticity in a steady current. By consideration of the integral properties of the combined wave and current motion across a fixed vertical section, an analytical expression for the variation of the wave amplitude with distance was derived rigorously, which in the same paper has also been proved to be equivalent to the action conservation equation in terms of the newly defined wave action density.

The two-dimensional analysis made by Jonsson *et al.* (1978) results in the action conservation equation that is only one-dimensional and therefore has a limited application. To investigate the ray theory and the conservation of wave action in a three-dimensional wave-current field without the assumption of irrotationality, White (1999) applied a formal perturbation scheme to the boundary-value problem to obtain the WKBJ description of the modulations of linear waves. In this approach, the spatial scales of the current in the horizontal directions and in the vertical direction are assumed to be the same, meaning that the resulting solution is valid only if the current varies slowly not only in the horizontal directions but also in the vertical direction. Consequently, except a new equation for a spatially varying phase shift, the dispersion relation and the two-dimensional action conservation equation derived here for a weakly sheared current are not different from those for the case of an irrotational current. Therefore, the theory of White (1999) may not be considered as an extension of the theory of Jonsson *et al.* (1978) from a two-dimensional case to a three-dimensional one.

In this study, the variation of the current velocity with depth can be one order of magnitude faster than that in the horizontal directions, a situation which has also been considered by Jonsson *et al.* (1978) in a two-dimensional analysis instead of the present three-dimensional one. In this situation, if the variation of the current with depth is near linear and if only the first-order WKBJ solution is pursued, it is unnecessary to explicitly introduce the ordering parameters to scale the equations as did by White (1999) (see Shyu & Phillips 1990 and Shyu & Tung 1999), so that the difficulties with two length scales among the underlying current itself in this case can be avoided. On the other hand, according to Shrira (1993), if the deviation from a linear profile of the variation of the current velocity with depth is small, the series solution derived by Shrira (1993) will converge very rapidly, which renders a one-term WKBJ solution possible. Therefore, in §2, we shall temporarily neglect the slow variations in the horizontal directions and the slight deviation from a linear profile in the vertical direction of the underlying current to obtain an exact solution of the linear waves in this situation, which coincides with the zeroth-order term of the series solution derived by Shrira (1993) and will hereafter be referred to as the basic solution. This solution, if allowing its parameters to slowly vary, represents the first-order WKBJ solution of the slowly varying wave train, although the variations of these parameters, especially that of the wave amplitude, remain to be determined, for which the discussion in §2 can also provide important information.

After the basic solution being given in §2, the WKBJ solution for deep-water gravity waves propagating obliquely on a steady three-dimensional, strongly sheared current with non-uniform vorticity will be deduced in §3 by consideration of the effects of the slow variations in the horizontal directions and the slight deviation from a linear profile in the vertical direction of the underlying current on each term of the differential equations in the boundary-value problem. The results indeed take the same form as the basic solution given in §2, although their parameters are now slowly varying. The differential equation for determination of the modulation of the wave amplitude with distance has also been derived in §3, which completes the first-order WKBJ solution.

The resulting modulation equation is compared with the two-dimensional action conservation equation in §4 that represents a natural extension of the one-dimensional one derived by Jonsson *et al.* (1978) and therefore takes the vorticity of the current into account but ignores the rotational perturbation velocity which may have the same order of magnitude as the irrotational part of the wave motion in the present case. This comparison indicates that in three-dimensional flow, unless certain restrictions are imposed on the distribution of the underlying current, the wave action defined by Jonsson *et al.* without consideration of the rotational perturbation velocity is not conserved.

In order to see the reasons why the validity of the action conservation equation is limited to a certain range of situations and why the wave action density cannot be redefined to include the rotational perturbation velocity, in §5, the approach of Jonsson *et al.* (1978) will also be applied to three-dimensional flow. In this analysis, in order to obtain a useful equation for determination of the variation of the wave amplitude with distance, it is required to impose the same restrictions on the distribution of the underlying current as those imposed in §4 to validate the two-dimensional action conservation equation. The resulting equation is indeed identical with the reduced forms of the two-dimensional action conservation equation and the equation derived by the present approach in §3.

In a general situation without the restrictions imposed, the differences between the action conservation equation and the modulation equation derived by the present approach are significant and will be illustrated in §6 by numerical simulation, which can also provide numerical evidence in support of certain ideas in the present approach that remains valid in a general situation.

2. The basic solution

In this section we shall describe the exact solution of the linear deep-water gravity waves propagating obliquely on a steady current $\mathbf{U}\{U_x(z), U_y(z), 0\}$ uniform in the horizontal directions but strongly and linearly sheared (constant vorticity) in the vertical direction. This solution is closely related to the WKBJ solution, because when the velocity and the vorticity of the underlying current become slowly varying in the horizontal directions and in both the horizontal and vertical directions, respectively, the parameters in this basic solution will similarly vary slowly, resulting in the WKBJ solution which represents the first-order term of the asymptotic expansion of the solution for the 'gradually varying problem'. On the other hand, the basic solution defined here is also identical with the zeroth-order term of the series solution derived by Shrira (1993) for waves propagating on a horizontally homogeneous but vertically sheared current. In this series solution, the higher-order terms arises due to the variation of the vorticity in the vertical direction. Therefore, by neglecting the terms containing the derivatives of the vorticity in the theory of Shrira (1993), one may easily obtain the basic solution. However, to recapitulate the situation and to provide important information for the analysis in the ensuing section, the complete basic equation is derived here following the precedent of Shrira (1993).

Since in the present circumstances, considering the vorticity dynamics, especially the effects of rotation and extension or contraction of the vortex-lines, one may expect that the oscillatory wave motion is no longer irrotational, we start with the Euler equation for perturbations of velocity $\mathbf{u}\{u_x, u_y, u_z\}$ and pressure p linearized upon the flow $\mathbf{U}$

$$\left.\begin{aligned}
\frac{\partial u_x}{\partial t} + U_x\frac{\partial u_x}{\partial x} + U_y\frac{\partial u_x}{\partial y} + u_z\frac{\partial U_x}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial x} &= 0 \\
\frac{\partial u_y}{\partial t} + U_x\frac{\partial u_y}{\partial x} + U_y\frac{\partial u_y}{\partial y} + u_z\frac{\partial U_y}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial y} &= 0 \\
\frac{\partial u_z}{\partial t} + U_x\frac{\partial u_z}{\partial x} + U_y\frac{\partial u_z}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial z} + g &= 0 \\
\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} &= 0
\end{aligned}\right\} \tag{2.1}$$

where ρ is the density of the water and g the acceleration due to gravity. In (2.1), the choice of the $x-$ and $y-$ axes of the rectangular coordinates are at our disposal. On the other hand, since the underlying current $\mathbf{U}$ is uniform in the horizontal directions, the waves will not be refracted by the current. Therefore the $y-$axis can be chosen to be parallel to the wave crests so that all variables are independent of y and the above system of equations can be reduced to

$$\left.\begin{aligned} \frac{\partial u_x}{\partial t} + U_x\frac{\partial u_x}{\partial x} + u_z\Omega_y + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \\ \frac{\partial u_z}{\partial t} + U_x\frac{\partial u_z}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial z} + g = 0 \\ \frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} = 0 \end{aligned}\right\} \tag{2.2a}$$

and

$$\frac{\partial u_y}{\partial t} + U_x\frac{\partial u_y}{\partial x} - u_z\Omega_x = 0 \tag{2.2b}$$

where $\mathbf{\Omega}\{\Omega_x, \Omega_y, 0\}$ denotes the vorticity of the underlying current $\mathbf{U}$ with $\Omega_x = -\partial U_y/\partial z$ and $\Omega_y = \partial U_x/\partial z$ in the present situation. Notice that the variables u_y and U_y as well as the constant Ω_x are absent from (2.2a), meaning that if this situation also occurs to the free-surface boundary conditions, the solutions of u_x, u_z and p as well as the wave phase velocity will not be affected by the convection in the $y-$direction U_y and its shear Ω_x. Nevertheless, if $\Omega_x \neq 0$ and $u_z \neq 0$, according to (2.2b), the oscillatory velocity component u_y will occur, which is important for the development of the WKBJ description in the next section.

The boundary conditions at the free surface $z = \eta(x, y, t)$ transformed on the plane $z = 0$ corresponding to the unperturbed free surface for linear waves can be written as

$$\frac{\partial \eta}{\partial t} + U_x\frac{\partial \eta}{\partial x} = u_z, \qquad p = \rho g\eta, \tag{2.3}$$

(see Shrira 1993) which are indeed free from u_y, U_y and Ω_x. Therefore, one can solve (2.2a) and (2.3) without consideration of u_y, after which u_y can be determined from (2.2b).

Differentiating the first and second equations in (2.2a) with respect to z and x respectively, combining the resulting equations into one to eliminate the pressure terms, and using the third of equations (2.2a), we obtain

$$\frac{\partial \omega_y}{\partial t} + U_x\frac{\partial \omega_y}{\partial x} = 0, \tag{2.4}$$

where $\omega_y \equiv \partial u_x/\partial z - \partial u_z/\partial x$ represents the vorticity component of the wave motion in the $y-$direction. Thus if initially $\omega_y = 0$ everywhere, from (2.4) it will remain so in an inviscid fluid. Therefore a two-dimensional velocity potential $\phi(x, z, t)$ can be defined such that

$$u_x = \frac{\partial \phi}{\partial x}, \qquad u_z = \frac{\partial \phi}{\partial z}.$$

The third of equations (2.2a) then requires

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial z^2}=0. \tag{2.5}$$

Thus in deep water we have

$$\phi = Ae^{kz}e^{\mathrm{i}(kx-n_0t)}, \tag{2.6}$$

where A is a constant, k the wavenumber and n_0 the observed frequency of a chosen Fourier component, and if the surface displacement

$$\eta = ae^{\mathrm{i}(kx-n_0t)}, \tag{2.7}$$

where the amplitude a is a constant, from the boundary conditions (2.3), we obtain

$$A=-\mathrm{i}\frac{\sigma}{k}a \tag{2.8}$$

and

$$p|_{z=0}=\rho g a e^{\mathrm{i}(kx-n_0t)},$$

where

$$\sigma \equiv n_0 - U_{xs}k \tag{2.9}$$

is the intrinsic frequency relative to frame of reference in which the mean surface velocity equals zero, and $U_{xs} \equiv U_x|_{z=0}$. Substituting all these results into the first and second equations of (2.2a), we have respectively the dispersion relation

$$gk=\sigma^2+\sigma\Omega_y \tag{2.10}$$

and the pressure fluctuations

$$p=-\rho g z+\rho g a e^{kz}e^{\mathrm{i}(kx-n_0t)}-\rho\sigma\Omega_y a z e^{kz}e^{\mathrm{i}(kx-n_0t)}. \tag{2.11}$$

The last term in (2.11), arising from the fact that $U_x = U_{xs}+\Omega_y z$, cannot be found when the underlying current is irrotational, and is important for the analysis in §5. On the other hand, the dispersion relation (2.10) is identical with the zeroth-order term of the series solution derived by Shrira (1993), and also consistent with those presented by Thompson (1949), Biesel (1950) and Teles da Silva & Peregrine (1988) for waves in an intermediate-depth region.

Finally, from (2.2b) we have

$$u_y = \frac{\sigma \Omega_x}{n_0 - U_x k} a e^{kz} e^{\mathrm{i}(kx - n_0 t)}, \tag{2.12}$$

meaning that when the wave profiles propagate obliquely on a horizontally uniform shear flow, a transverse rotational perturbation velocity will occur, which can be as large as $\partial\phi/\partial x$ and $\partial\phi/\partial z$ if Ω_x has the same order of magnitude as σ. On the other hand, if Ω_y has the same order of magnitude as σ, the two terms on the right-hand side of (2.10) have the same order of magnitude too. Therefore, in the following discussion we assume that Ω_y, Ω_x and σ all have the same order of magnitude, representing a strongly sheared current.

Notice that since u_y varies with x and z, the vorticity components ω_x and ω_z of the oscillatory wave motion in the $x-$ and $z-$direction are non-zero. On the other hand, since U_x varies with depth, according to (2.12), the value of u_y will become infinity at a critical layer where $n_0 - U_x k = 0$. To avoid this singularity, the perturbations may become unstable (see, for example, Morland, Saffman & Yuen 1991; Shrira 1993; Miles 2001). This subject however goes beyond the scope of this work. Therefore the situation that the critical layers exist will be circumvented in this study.

Also we remark that although in the present case, the solutions of u_x, u_z and p as well as the dispersion relation (2.10) can be determined without consideration of the transverse velocity component u_y, the existence of the latter will affect the slow modulation of wave train when the underlying current becomes gradually varying.

3. The modulation theory

In this section, the underlying current considered in §2 is allowed to vary slowly in both the $x-$ and $y-$directions and its vorticity components Ω_x and Ω_y can gradually vary not only in the horizontal directions but also in the vertical direction. In this situation, since all these variations are slow in the sense that their length scales are large compared with the wavelength, the solution described in §2, when allowing its parameters to slowly vary, represents the first term of the asymptotic expansion of the exact solution for this 'gradually varying problem'. The modulations of these parameters will be derived in this section, which can complete the so-called WKBJ solution for this case.

When a wave train propagates on a horizontally non-uniform current, the magnitude and direction of the wavenumber vector $\mathbf{k}$ will both change with distance. However, even in this case, the $x-$axis of the rectangular coordinates can be chosen to be parallel to the local $\mathbf{k}$ at the position under consideration and the $y-$axis is therefore parallel to the local wave crest. On the other hand, when the underlying current is non-uniform in the horizontal directions, the mean water surface may not be horizontal, but according to Phillips (1981), Longuet-Higgins (1985, 1987) and Henyey *et al.* (1988), the effects of its slope and curvature on the wave motion are equivalent to those with a level mean surface and with the gravitational acceleration g being replaced by the effective gravitational acceleration g' defined by Phillips (1981). Therefore, by using g' instead of g and by using the coordinate system chosen above in which the xy plane is tangent to the mean water surface at the position under consideration, the solutions described in §2 can still have implications for derivation of the WKBJ solution.

Since the differentiation of the slowly varying parameters increases the order of magnitude by one each time, to derive the first-order WKBJ solution, the second-order derivatives of the slowly varying parameters and the products of any two first-order derivatives of these parameters can all be neglected in the following discussion. Similarly, since the underlying current velocity is slowly varying in the horizontal directions and its vorticity is slowly varying in all directions , this treatment can also be applied to the derivatives with respect to x or y of U_x, U_y, Ω_x and Ω_y, and to the derivatives with respect to z of Ω_x and Ω_y. In this way, there is no need to explicitly introduce an ordering parameter to formally expand each of the unknowns in powers of the latter in the following discussion (see Shyu & Phillips 1990 and Shyu & Tung 1999). Also, we emphasize that when the first-order derivatives of the slowly varying parameters and quantities are taken into account, the second term of the asymptotic expansion of each unknown should also be considered implicitly.

As a result, each quantity that has been solved in §2 will now possess extra terms, which though one order of magnitude smaller than that obtained in §2 and eventually negligible within the present approximation, must be included in the analysis. This situation also occurs in a formal perturbation scheme in which the second term of the asymptotic expansion is considered to derive the secular condition which leads to the modulation equation for the first term of the expansion. After this, the second term of the asymptotic expansion can be neglected in the WKBJ solution.

In addition to the quantities appearing in §2, some quantities that vanish in §2 will now become non-zero owing to the slow variations of the mean flow in each direction. These quantities will certainly be one order of magnitude smaller than their counterparts defined in §2 and therefore are distinguished from them by using the symbols with a hat. For example, from the continuity equation

$$\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{\partial \widehat{U}_z}{\partial z} = 0, \tag{3.1}$$

and from the situation that $\partial U_x/\partial x \neq 0$ and $\partial U_y/\partial y \neq 0$ it follows that the component of the steady flow in the $z-$direction $\widehat{U}_z$, though still vanishing locally at the mean water surface (because the xy plane is tangent to this surface at the position under consideration), has a small but non-zero value at the depth on the order of one wavelength, at which the steady flow has a direct influence on the surface waves. Also the non-uniformity of the steady flow in the horizontal directions implies that the vorticity component

$$\widehat{\Omega}_z \equiv \frac{\partial U_y}{\partial x} - \frac{\partial U_x}{\partial y} \neq 0$$

On the other hand, since the curvature of the mean water surface is usually very small, the quantities $(\partial \widehat{U}_z/\partial x)_{z=0}$ and $(\partial \widehat{U}_z/\partial y)_{z=0}$ remain negligible locally. Finally, in the present case, the perturbation vorticity component $\widehat{\omega}_y$, though small, also becomes non-zero. Therefore, in addition to $\partial\phi/\partial x$ and $\partial\phi/\partial z$, the rotational velocity components of the wave motion $\widehat{u}_x$ and $\widehat{u}_z$ also exist and are one order of magnitude smaller than $\partial\phi/\partial x$ and $\partial\phi/\partial z$ as well as u_y.

Notice that if $\widehat{u}_z$ does not vanish at the mean water surface, since its fast variation in the horizontal directions can locally be represented by the function $\exp[\mathrm{i}(kx - n_0 t)]$ and since its slow variation can be neglected within the present approximation, it is always possible to define an irrotational velocity field $\widehat{\phi}$ which takes the same form as (2.6) locally so that $\partial\widehat{\phi}/\partial z$ can have the same value as $\widehat{u}_z$ at each point on the mean water surface. Therefore, after subtracting $\partial\widehat{\phi}/\partial z$ from $\widehat{u}_z$, subtracting $\partial\widehat{\phi}/\partial x$ from $\widehat{u}_x$, and in the meantime, adding $\widehat{\phi}$ to ϕ, the new rotational perturbation velocity becomes horizontal at the mean water surface, and the new velocity potential still takes the same form as (2.6) except that the second term of the asymptotic expansion of A becomes different. Consequently, the boundary condition

$$\widehat{u}_z = 0 \qquad \text{at} \qquad z = 0 \tag{3.2}$$

can be applied to simplify the analysis significantly.

In order to describe both the fast and the slow variations, the total velocity potential can be written as

$$\phi = A(x, y) \exp\left[\int_0^z l(x, y, z)\, dz\right] e^{\mathrm{i}\chi(x,y,t)} \tag{3.3}$$

with

$$\mathbf{k} = \nabla_h \chi, \qquad n_0 = -\frac{\partial \chi}{\partial t}, \tag{3.4}$$

where $\nabla_h \equiv (\partial/\partial x, \partial/\partial y)$ represents the horizontal gradient operator, $\mathbf{k}\{k_x, k_y\}$ the wavenumber vector, and $l(x, y, z)$ a slowly varying function of position. Since in the present coordinate system, $k_y = 0$ at the position under consideration, and from the relation (3.9) given below, $l|_{z=0} \approx k$, the expression (3.3) together with (3.4) is indeed identical with (2.6) locally if the higher-order terms in the asymptotic expansions of A, $\mathbf{k}$, and l are neglected and their slow variations are ignored.

From the first of equations (3.4) it follows immediately that

$$\frac{\partial k_y}{\partial x} = \frac{\partial k_x}{\partial y}. \tag{3.5}$$

Also, from (3.4)

$$\frac{\partial \mathbf{k}}{\partial t} + \nabla_h n_0 = 0,$$

which is the kinematical conservation equation (Phillips 1977). Since in the present case the underlying current is steady, we have $\partial\mathbf{k}/\partial t = 0$ so that from the above equation n_0 is constant everywhere.

Substitution of (3.3) and (3.4) into the three-dimensional Laplace equation yields

$$-k_x^2 + \mathrm{i}\frac{\partial k_x}{\partial x} + 2\mathrm{i}k_x \frac{1}{A}\frac{\partial A}{\partial x} + \mathrm{i}\frac{\partial k_y}{\partial y} + l^2 + \frac{\partial l}{\partial z} = 0 \qquad \text{at} \qquad z = 0 \tag{3.6}$$

in which the terms $(1/A)(\partial^2 A/\partial x^2)$ and $(1/A)(\partial^2 A/\partial y^2)$ have been neglected and the fact that $k_y = 0$ locally has also been taken into account. Furthermore, since the variation of the wave motion in the $y-$direction at the position under consideration is slow, the second-order derivative $(\partial^2\phi/\partial y^2)_{z=0} = \mathrm{i}(\partial k_y/\partial y)Ae^{\mathrm{i}\chi} + (\partial^2 A/\partial y^2)e^{\mathrm{i}\chi} \approx \mathrm{i}(\partial k_y/\partial y)Ae^{\mathrm{i}\chi}$ should also be negligible here, meaning that at the position under consideration

$$\frac{\partial k_y}{\partial y} = 0 \tag{3.7}$$

within the present approximation. This important suggestion will later be justified analytically and numerically.

In (3.6), since both l and $\partial l/\partial z$ exist, one cannot express l in terms of other parameters and their derivatives without another equation. In Shyu & Tung (1999), the relation (see their (2.14))

$$\left.\frac{\partial l}{\partial z}\right|_{z=0} = -\mathrm{i}\frac{\partial k}{\partial x}$$

has been derived from the Laplace equation for the exactly two-dimensional case in which the wave crest is straight. Since this relation involves only the small quantities representing the derivatives of the slowly varying parameters, the small curvature of the wave crest occurred in the present case will impose a modification of this relation even smaller and therefore negligible. Thus, in the present case, we still have

$$\left.\frac{\partial l}{\partial z}\right|_{z=0} = -\mathrm{i}\frac{\partial k}{\partial x} = -\mathrm{i}\left(\frac{k_x}{k}\frac{\partial k_x}{\partial x} + \frac{k_y}{k}\frac{\partial k_y}{\partial x}\right) = -\mathrm{i}\frac{\partial k_x}{\partial x} \tag{3.8}$$

because $k = (k_x^2 + k_y^2)^{1/2}$ and $k_y = 0$ locally. Therefore, substituting (3.7) and (3.8) into (3.6), we obtain

$$l^2|_{z=0} = k_x^2 - 2\mathrm{i}k_x\frac{1}{A}\frac{\partial A}{\partial x}.$$

Squaring both sides of it and neglecting the term $(1/2k_x)(1/A)^2(\partial A/\partial x)^2$ and the even higher order terms, we finally have

$$l|_{z=0} = k - \mathrm{i}\frac{1}{A}\frac{\partial A}{\partial x}. \tag{3.9}$$

We next consider the kinematic free-surface condition, which in the present case can be written as

$$\frac{\partial \eta}{\partial t} + \left(\frac{\partial \phi}{\partial x} + \widehat{u}_x + U_x\right)\frac{\partial \eta}{\partial x} + \left(\frac{\partial \phi}{\partial y} + u_y + U_y\right)\frac{\partial \eta}{\partial y} = \frac{\partial \phi}{\partial z} + \widehat{u}_z + \widehat{U}_z \quad \text{at} \quad z = \eta.$$

After Taylor series expansions about $z = 0$, we have the linear wave approximation

$$\frac{\partial \eta}{\partial t} + U_x \frac{\partial \eta}{\partial x} + U_y \frac{\partial \eta}{\partial y} = \frac{\partial \phi}{\partial z} - \eta \left(\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} \right) \quad \text{at} \quad z = 0 \tag{3.10}$$

in view of (3.1) and (3.2). Therefore if the surface displacement is

$$\eta = a(x, y) e^{\mathrm{i}\chi(x,y,t)}, \tag{3.11}$$

substitution of (3.3), (3.4), (3.7), (3.9) and (3.11) into (3.10) yields

$$\frac{A}{a} = -\mathrm{i}\frac{\sigma}{k} + \frac{1}{ak}\left(U_x \frac{\partial a}{\partial x} + U_y \frac{\partial a}{\partial y} \right) + \frac{1}{k}\left(\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} \right) + \mathrm{i}\frac{1}{ak}\frac{\partial A}{\partial x}. \tag{3.12}$$

From (3.12), neglecting the smaller terms containing the derivatives of the slowly varying functions, we have

$$\frac{A}{a} \approx -\mathrm{i}\frac{\sigma}{k}. \tag{3.13}$$

Its differentiation with respect to x and y yields

$$\frac{\partial A}{\partial x} = -\mathrm{i}\frac{\sigma}{k}\frac{\partial a}{\partial x} - \mathrm{i}\frac{a}{k}\frac{\partial \sigma}{\partial x} + \mathrm{i}a\frac{\sigma}{k^2}\frac{\partial k_x}{\partial x} \tag{3.14}$$

and

$$\frac{\partial A}{\partial y} = -\mathrm{i}\frac{\sigma}{k}\frac{\partial a}{\partial y} - \mathrm{i}\frac{a}{k}\frac{\partial \sigma}{\partial y} + \mathrm{i}a\frac{\sigma}{k^2}\frac{\partial k_x}{\partial y}. \tag{3.15}$$

(Recall that $\partial k/\partial x = \partial k_x/\partial x$, $\partial k/\partial y = \partial k_x/\partial y$.) Therefore, by substituting (3.14) into (3.12) for $\partial A/\partial x$, one can express A in terms of other quantities and their derivatives within the present approximation. Note that without consideration of the second term of the asymptotic expansion of each quantity, (3.13) is again identical with (2.8), though the parameters A, a, σ and k are now slowly varying.

Finally, the dynamical free-surface condition is imposed by the requirement that the pressure in the water at the free surface is equal to the atmospheric pressure which is assumed to be constant here. Therefore, if at the free surface, the component of the equation of motion in the $s-$direction (see figure 1), which is tangent to the instantaneous free surface and perpendicular to the local wave crest, is under consideration, the pressure gradient in this equation will vanish. The rest of the terms, though originally involving the components of the quantities in the $s-$, $y-$ and $n-$directions, can all be transformed into the terms containing the components in the $x-$, $y-$ and $z-$directions. This can be done because in figure 1

$$\cos\alpha \approx 1, \qquad \sin\alpha \approx \frac{\partial\eta}{\partial x}$$

for linear waves. The resulting equation can then be expressed as Taylor series expansions about the mean water surface $z=0$ so that after neglecting the higher-order terms of the asymptotic expansions $\widehat{\Omega}_z\partial\phi/\partial y$, $\widehat{u}_x\partial U_x/\partial x$, $(\partial U_y/\partial x)\partial\phi/\partial y$ and $U_y\partial\widehat{u}_x/\partial y$, and neglecting the nonlinear terms of the oscillatory wave motion, we have

$$\left\{g\frac{\partial\overline{\zeta}}{\partial x}+U_x\frac{\partial U_x}{\partial x}+U_y\frac{\partial U_x}{\partial y}\right\}+\left\{g'\frac{\partial\eta}{\partial x}+\frac{\partial^2\phi}{\partial x\partial t}+U_x\frac{\partial^2\phi}{\partial x^2}+\frac{\partial U_x}{\partial x}\frac{\partial\phi}{\partial x}+U_x\frac{\partial\widehat{u}_x}{\partial x}+U_y\frac{\partial^2\phi}{\partial x\partial y}\right.$$
$$+U_x\frac{\partial U_x}{\partial z}\frac{\partial\eta}{\partial x}+\frac{\partial U_x}{\partial x}\frac{\partial U_x}{\partial z}\eta+U_x\frac{\partial^2 U_x}{\partial x\partial z}\eta+\frac{\partial\widehat{u}_x}{\partial t}+\frac{\partial U_x}{\partial z}\frac{\partial\eta}{\partial t}+\frac{\partial U_x}{\partial y}\frac{\partial U_y}{\partial z}\eta+U_y\frac{\partial^2 U_x}{\partial y\partial z}\eta$$
$$\left.+U_y\frac{\partial U_x}{\partial z}\frac{\partial\eta}{\partial y}+\frac{\partial U_x}{\partial y}u_y\right\}=0 \qquad \text{at} \qquad z=0, \tag{3.16}$$

where $\overline{\zeta}$ is the height of the mean water surface (see figure 1).

Notice that in deriving (3.16), the fact that the velocity component in the $n-$direction at the instantaneous free surface equals $\partial\eta/\partial t$ for linear waves has been utilized. Furthermore, the situation that η represents the surface displacement of the waves in the $z-$direction (see figure 1) results in the replacement of g by g' in the second braces in (3.16), which in the present case is defined as $g\cos\theta$. Finally we remark that since the curvature of the mean water surface is usually very small, the derivative of g' with respect to x has been neglected in (3.16).

The above equation can also be deduced directly from the component of the equation of motion in the $x-$direction evaluated directly at the mean water surface (so that no Taylor series expansion about $z=0$ is required) if the validity of the second of equations (2.3) is assumed here. However the present derivation involves no such assumption so that it is preferred here and can serve as a proof of the validity of the second of equations (2.3) in the present circumstance.

In (3.16), the terms in the first braces are time-independent while the expression in the second braces represents a linear combination of the time-harmonic terms. Therefore the latter itself should vanish, resulting in

$$g'\frac{\partial\eta}{\partial x}+\frac{\partial^2\phi}{\partial x\partial t}+U_x\frac{\partial^2\phi}{\partial x^2}+\frac{\partial U_x}{\partial x}\frac{\partial\phi}{\partial x}+U_y\frac{\partial^2\phi}{\partial x\partial y}+U_x\frac{\partial U_x}{\partial z}\frac{\partial\eta}{\partial x}+\frac{\partial U_x}{\partial z}\frac{\partial\eta}{\partial t}+U_y\frac{\partial U_x}{\partial z}\frac{\partial\eta}{\partial y}$$
$$+\left(\frac{\partial U_x}{\partial x}\frac{\partial U_x}{\partial z}+U_x\frac{\partial^2 U_x}{\partial x\partial z}+\frac{\partial U_x}{\partial y}\frac{\partial U_y}{\partial z}+U_y\frac{\partial^2 U_x}{\partial y\partial z}\right)\eta=R \qquad \text{at} \qquad z=0, \tag{3.17}$$

where

$$R \equiv -\frac{\partial \widehat{u}_x}{\partial t} - U_x \frac{\partial \widehat{u}_x}{\partial x} - u_y \frac{\partial U_x}{\partial y} \tag{3.18}$$

also evaluated at $z = 0$.

In (3.17), the terms on the right-hand side represented by R all contain the rotational perturbation velocity component $\widehat{u}_x$ or u_y. The sum of these terms will later be related to those involving only the steady flow and the irrotational part of the wave motion so that even without knowing $\widehat{u}_x$, the modulations of the wave train and the dispersion relation can still be determined. To achieve this purpose, we consider the component of the vorticity equation in the y – direction evaluated at the mean water surface for the entire flow. After neglecting the higher-order terms of the asymptotic expansions $\widehat{u}_x \partial \Omega_y / \partial x$, $(\partial \Omega_y / \partial y) \partial \phi / \partial y$, $\widehat{\omega}_y \partial U_y / \partial y$, $\Omega_y \partial^2 \phi / \partial y^2$, $(\partial U_y / \partial z) \partial \widehat{u}_x / \partial y$ and $\widehat{\Omega}_z \partial^2 \phi / \partial z \partial y$, and neglecting the nonlinear terms of the oscillatory wave motion, we have

$$\left\{ U_x \frac{\partial^2 U_x}{\partial x \partial z} + U_y \frac{\partial^2 U_x}{\partial y \partial z} + \frac{\partial U_x}{\partial y}\frac{\partial U_y}{\partial z} - \frac{\partial U_y}{\partial y}\frac{\partial U_x}{\partial z} \right\} + \left\{ \frac{\partial}{\partial z}\left(\frac{\partial \widehat{u}_x}{\partial t} + U_x \frac{\partial \widehat{u}_x}{\partial x} + u_y \frac{\partial U_x}{\partial y} \right) + \frac{\partial U_x}{\partial z}\frac{\partial \widehat{u}_z}{\partial z} \right.$$
$$\left. + \frac{\partial^2 U_x}{\partial x \partial z}\frac{\partial \phi}{\partial x} + \frac{\partial^2 U_x}{\partial z^2}\frac{\partial \phi}{\partial z} + \frac{\partial U_y}{\partial z}\frac{\partial^2 \phi}{\partial x \partial y} \right\} = 0 \qquad \text{at} \qquad z = 0 \tag{3.19}$$

in view of (3.2) and the fact that

$$\frac{\partial \widehat{u}_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial \widehat{u}_z}{\partial z} = 0.$$

Notice that although $\widehat{u}_z|_{z=0} = 0$, $(\partial \widehat{u}_z / \partial z)_{z=0}$ is in general non-zero.

In (3.19), the time-independent terms and the time-harmonic terms have again been separated so that we have for the steady flow

$$U_x \frac{\partial^2 U_x}{\partial x \partial z} + U_y \frac{\partial^2 U_x}{\partial y \partial z} + \frac{\partial U_x}{\partial y}\frac{\partial U_y}{\partial z} - \frac{\partial U_y}{\partial y}\frac{\partial U_x}{\partial z} = 0 \qquad \text{at} \qquad z = 0, \tag{3.20}$$

and for the oscillatory wave motion

$$\frac{\partial}{\partial z}\left(-\frac{\partial \widehat{u}_x}{\partial t} - U_x \frac{\partial \widehat{u}_x}{\partial x} - u_y \frac{\partial U_x}{\partial y} \right) = \frac{\partial U_x}{\partial z}\frac{\partial \widehat{u}_z}{\partial z} + \frac{\partial^2 U_x}{\partial x \partial z}\frac{\partial \phi}{\partial x} + \frac{\partial^2 U_x}{\partial z^2}\frac{\partial \phi}{\partial z} + \frac{\partial U_y}{\partial z}\frac{\partial^2 \phi}{\partial x \partial y} \qquad \text{at} \quad z = 0. \tag{3.21}$$

The expression in the parentheses in (3.21) is identical to that represented by R in (3.18). From (3.3), (3.5), (3.9) and the situation that $k_y = 0$ locally, it is not difficult to prove that the solution

$$R = \frac{\partial U_x}{\partial z}\widehat{u}_z + \frac{\partial^2 U_x}{\partial x \partial z}(\mathrm{i}\phi) + \frac{\partial^2 U_x}{\partial z^2}\phi + \frac{\partial U_y}{\partial z}\left(\mathrm{i}\frac{\partial \phi}{\partial y} \right) \qquad \text{at} \qquad z = 0 \tag{3.22}$$

can satisfy (3.21) within the present approximation. On the other hand, the uniqueness of this solution can be substantiated as follows.

First, the differentiation of $(\partial U_x/\partial z)\partial\phi/\partial z$ (instead of $(\partial^2 U_x/\partial z^2)\phi$ which appears in (3.22)) with respect to z will result in not only the term $(\partial^2 U_x/\partial z^2)\partial\phi/\partial z$ occurring in (3.21) but also the term $(\partial U_x/\partial z)\partial^2\phi/\partial z^2$ which is absent from (3.21). The latter term is even one order of magnitude larger than the former term. Next, the derivatives with respect to z of the terms $(\partial U_x/\partial x)\partial\phi/\partial x$ and $U_y\partial^2\phi/\partial x\partial y$, instead of $(\partial^2 U_x/\partial x\partial z)(\mathrm{i}\phi)$ and $(\partial U_y/\partial z)(\mathrm{i}\partial\phi/\partial y)$ chosen in (3.22), also contain each an extra term that is not negligible even if the steady flow $\mathbf{U}$ becomes irrotational so that $\widehat{u}_x$, u_y and $\widehat{u}_z$ in (3.21) vanish, which is certainly impossible. Finally, the choice of $(\partial U_x/\partial z)\widehat{u}_z$ rather than $U_x\partial\widehat{u}_z/\partial z$ in (3.21) is because the differentiation of the latter with respect to z also produces two terms which have the same order of magnitude but only one of them really occurs in (3.21), while the differentiation of the term $(\partial U_x/\partial z)\widehat{u}_z$ with respect to z results in $(\partial U_x/\partial z)\partial\widehat{u}_z/\partial z$ and $(\partial^2 U_x/\partial z^2)\widehat{u}_z$; the latter is indeed negligible or even vanishes at $z = 0$ according to (3.2). More important, the latter term can also be found in the original equation that leads to (3.19). This and the fact that the fast variations in the $z-$direction of the last three terms in both (3.21) and (3.22) are simply specified by the function e^{kz} can put even more confidence in the solution (3.22).

Since $\widehat{u}_z|_{z=0} = 0$, the relation (3.22) reduces to

$$R = \frac{\partial^2 U_x}{\partial x\partial z}(\mathrm{i}\phi) + \frac{\partial^2 U_x}{\partial z^2}\phi + \frac{\partial U_y}{\partial z}\left(\mathrm{i}\frac{\partial\phi}{\partial y}\right) \qquad \text{at} \qquad z = 0 \tag{3.23}$$

of which the right-hand side is indeed devoid of the rotational perturbation velocity.

Notice that the differentiation of the right-hand side of (3.18) with respect to y will increase the order of magnitude of each term by one so that the differentiation of the right-hand side of (3.23) with respect to y, which results in the term $(\partial U_y/\partial z)\mathrm{i}(\partial^2\phi/\partial y^2)_{z=0}$ among others, should also be negligible. This justifies (3.7) analytically as $(\partial^2\phi/\partial y^2)_{z=0} \approx \mathrm{i}(\partial k_y/\partial y)Ae^{\mathrm{i}\chi}$. The numerical evidence of (3.7) can also be found in §6.

The equation (3.17) together with (3.23) involves only η and ϕ as the unknowns. Therefore, substituting (3.3) and (3.11) into (3.17), using (3.12) and (3.14) to eliminate A in favour of a, neglecting the terms containing the second-order derivatives of the slowly varying functions and the products of any two first-order derivatives of these functions, and then crossing out the common factor, we obtain

$$\left\{\left(g' + 2\sigma U_x + U_x\frac{\partial U_x}{\partial z}\right)\frac{1}{a}\frac{\partial a}{\partial x} + \left(2\sigma U_y + U_y\frac{\partial U_x}{\partial z}\right)\frac{1}{a}\frac{\partial a}{\partial y} + U_x\frac{\partial \sigma}{\partial x} + U_y\frac{\partial \sigma}{\partial y} + 2\sigma\frac{\partial U_x}{\partial x} + \sigma\frac{\partial U_y}{\partial y}\right.$$

$$\left. + \frac{\partial U_x}{\partial x}\frac{\partial U_x}{\partial z} + \frac{\partial U_x}{\partial y}\frac{\partial U_y}{\partial z} + U_x\frac{\partial^2 U_x}{\partial x\partial z} + U_y\frac{\partial^2 U_x}{\partial y\partial z} - \frac{\sigma}{k}\frac{\partial^2 U_x}{\partial x\partial z} - \left(\frac{\sigma}{k}\frac{1}{a}\frac{\partial a}{\partial y} + \frac{1}{k}\frac{\partial \sigma}{\partial y} - \frac{\sigma}{k^2}\frac{\partial k_x}{\partial y}\right)\frac{\partial U_y}{\partial z}\right\}$$

$$+ \mathrm{i}\left\{g'k - \sigma^2 - \sigma\frac{\partial U_x}{\partial z} + \frac{\sigma}{k}\frac{\partial^2 U_x}{\partial z^2}\right\} = 0 \qquad \text{at} \qquad z = 0. \tag{3.24}$$

Since without loss of generality, the amplitude $a(x, y)$ in (3.11) can be defined as a real function, from the imaginary and real parts of (3.24) and by making use of (3.20), we finally have the dispersion relation

$$g'k = \sigma^2 + \sigma\Omega_{ys} - \frac{\sigma}{k}\frac{\partial^2 U_x}{\partial z^2}\bigg|_{z=0} \tag{3.25}$$

and the equation specifying the slow modulation of the amplitude a

$$(g' + 2\sigma U_{xs} + \Omega_{ys}U_{xs})\frac{1}{a}\frac{\partial a}{\partial x} + (2\sigma + \Omega_{ys})U_{ys}\frac{1}{a}\frac{\partial a}{\partial y} + U_{xs}\frac{\partial \sigma}{\partial x} + U_{ys}\frac{\partial \sigma}{\partial y} + 2\sigma\frac{\partial U_{xs}}{\partial x} + \sigma\frac{\partial U_{ys}}{\partial y}$$

$$+ \left(\frac{\partial U_{xs}}{\partial x} + \frac{\partial U_{ys}}{\partial y}\right)\Omega_{ys} - \frac{\sigma}{k}\frac{\partial^2 U_x}{\partial x\partial z}\bigg|_{z=0} + \left(\frac{\sigma}{k}\frac{1}{a}\frac{\partial a}{\partial y} + \frac{1}{k}\frac{\partial \sigma}{\partial y} - \frac{\sigma}{k^2}\frac{\partial k_x}{\partial y}\right)\Omega_{xs} = 0 \tag{3.26}$$

where U_{xs}, U_{ys}, Ω_{xs} and Ω_{ys} denote the values of U_x, U_y, Ω_x and Ω_y at the mean water surface respectively, so that

$$\Omega_{xs} \equiv \frac{\partial U_z}{\partial y}\bigg|_{z=0} - \frac{\partial U_y}{\partial z}\bigg|_{z=0} \approx -\frac{\partial U_y}{\partial z}\bigg|_{z=0}, \qquad \Omega_{ys} \equiv \frac{\partial U_x}{\partial z}\bigg|_{z=0} - \frac{\partial U_z}{\partial x}\bigg|_{z=0} \approx \frac{\partial U_x}{\partial z}\bigg|_{z=0}.$$

Notice that since Ω_{ys} represents specifically the component of vorticity perpendicular to the local $\mathbf{k}$ and the latter may vary in the x – direction, the quantities $\partial\Omega_{ys}/\partial x$ and $(\partial^2 U_x/\partial x\partial z)_{z=0}$ are generally not equal to each other; their difference will be evaluated in (4.7).

Also we emphasize that if the higher-order term $-(\sigma/k)(\partial^2 U_x/\partial z^2)_{z=0}$ and those inherent in g' are neglected, the dispersion relation (3.25) is again identical with (2.10), though the quantities k, σ, and Ω_{ys} are now slowly varying. Since these higher-order terms, when substituting in (3.26), become negligible within the present approximation, and since the slow variation of $\mathbf{k}$ and σ can be determined from that of the underlying current by using

(2.9), (2.10), (3.5) and the fact that $n_0 = \text{constant}$, the last term in (3.25) can be discarded and g' can be replaced by g in the WKBJ solution. Therefore, we have

$$gk = \sigma^2 + \sigma\Omega_{ys}. \tag{3.27}$$

This approximation will cause a spatially varying phase shift suggested by White (1999) which as pointed out by White is insignificant if one is only interested in modulations of the waves over large spatial scales, i.e. scales much larger than a wavelength.

Using (3.27) together with (2.9) and (3.5), and considering that $n_0 = \text{constant}$ everywhere, all quantities in (3.26) except the amplitude a become known so that the modulation of a can be determined numerically from (3.26), after which the modulations of A and the amplitudes of p and u_y can also be determined by using (3.13), (2.11) and (2.12) respectively. All of these represent the first-order WKBJ solution of the waves propagating obliquely on a steady three-dimensional, horizontally slowly varying and vertically strongly sheared current with non-uniform but slowly varying vorticity.

Incidentally we note that if the component of the equation of motion in the $y-$direction evaluated at the instantaneous free surface is considered, following the same approach which leads to (3.24) and using the component of the vorticity equation in the $z-$direction evaluated at $z = 0$, one can obtain an equation similar to (3.24). This equation, upon using the result

$$U_x \frac{\partial^2 U_y}{\partial x \partial z} + U_y \frac{\partial^2 U_y}{\partial y \partial z} + \frac{\partial U_y}{\partial x}\frac{\partial U_x}{\partial z} - \frac{\partial U_x}{\partial x}\frac{\partial U_y}{\partial z} = 0 \qquad \text{at} \qquad z = 0 \tag{3.28}$$

derived from the time-independent terms of the component of the vorticity equation in the $x-$direction, can be cancelled out completely, meaning that the dynamical free-surface condition can indeed be satisfied by the present solution.

4. The action conservation equation

When the underlying large-scale current is irrotational, the modulation of the wave amplitude is determined by the action conservation principle established by Bretherton & Garrett (1968), in which the wave action density is defined as the wave energy density divided by the intrinsic frequency. This principle has later been proved by Jonsson *et al.* (1978) to remain valid for two-dimensional gravity waves propagating on a steady two-dimensional, horizontally slowly varying current which can nevertheless vary rapidly but linearly with depth. In their theory, a different definition of the wave energy and the intrinsic frequency has been made, which coincides with that of Bretherton & Garrett only if the vorticity in the current vanishes. However, as pointed out by a referee of the paper, the result of Jonsson *et al.* (1978) can also be written in terms of the wave energy defined as the sum of the potential and kinetic energies calculated respectively from the perturbation surface displacement and the perturbation velocity, and in terms of the intrinsic frequency defined by (2.9). This wave energy, as found by the same referee, is $(1/4)\rho a^2(2g - \Omega_{ys}\sigma/k)$ so that in this two-dimensional problem, the action conservation principle can be written as

$$\frac{d}{dx}\left[(U_{xs} + C_g)\frac{1}{4}\rho a^2\left(2g - \Omega_{ys}\frac{\sigma}{k}\right)/\sigma\right] = 0, \tag{4.1}$$

where $C_g \equiv \partial\sigma/\partial k$ represents the group velocity. (The difference between U_{xs} and the 'formal surface velocity' defined by Jonsson *et al.* (1978) is proportional to a^2 and therefore is negligible in the above equation.)

Since the dispersion relation (3.27) for the three-dimensional case does not involve any quantities that will vanish in the two-dimensional case, this relation can be substituted into (4.1), resulting in

$$(g + 2\sigma U_{xs} + \Omega_{ys}U_{xs})\frac{1}{a}\frac{\partial a}{\partial x} + U_{xs}\frac{\partial\sigma}{\partial x} + 2\sigma\frac{\partial U_{xs}}{\partial x} + \frac{\partial U_{xs}}{\partial x}\Omega_{ys} = 0, \tag{4.2}$$

because in the two-dimensional flow of inviscid fluid, $(\partial^2 U_x/\partial x\partial z)_{z=0} = 0$ in view of (3.20), a situation which has also been assumed in Jonsson *et al.* (1978). Therefore the comparison between (3.26) and (4.2) indicates that for the case of $U_y = 0$ and $\partial/\partial y = 0$, the result (3.26) coincides with the action conservation equation. (Recall that within the present approximation, g' can be replaced by g in (3.26).)

When $U_y \neq 0$ and $\Omega_{xs} \neq 0$, but the rotational perturbation velocity u_y occurring in this situation is ignored in the definition of the wave action density, the action conservation equation in the present coordinate system becomes

$$\frac{\partial}{\partial x}\left[(U_{xs}+C_{gx})\frac{1}{4}\rho a^2\left(2g-\Omega_{ys}\frac{\sigma}{k}\right)/\sigma\right]+\frac{\partial}{\partial y}\left[(U_{ys}+C_{gy})\frac{1}{4}\rho a^2\left(2g-\Omega_{ys}\frac{\sigma}{k}\right)/\sigma\right]=0 \tag{4.3}$$

representing a natural extension of (4.1). In this situation, from the kinematical conservation equation,

$$\frac{\partial n_0}{\partial x}=\frac{\partial}{\partial x}(\sigma+U_{xs}k_x+U_{ys}k_y)=\frac{\partial\sigma}{\partial x}+U_{xs}\frac{\partial k_x}{\partial x}+k\frac{\partial U_{xs}}{\partial x}+U_{ys}\frac{\partial k_x}{\partial y}=0 \tag{4.4}$$

$$\frac{\partial n_0}{\partial y}=\frac{\partial}{\partial y}(\sigma+U_{xs}k_x+U_{ys}k_y)=\frac{\partial\sigma}{\partial y}+U_{xs}\frac{\partial k_x}{\partial y}+k\frac{\partial U_{xs}}{\partial y}=0 \tag{4.5}$$

in view of (3.5) and the fact that $k_y=0$ and $\partial k_y/\partial y=0$ locally. Therefore, by using (3.20), (3.27) and (4.4), the expansion of (4.3) yields

$$\begin{aligned}&(g+2\sigma U_{xs}+\Omega_{ys}U_{xs})\frac{1}{a}\frac{\partial a}{\partial x}+(2\sigma+\Omega_{ys})U_{ys}\frac{1}{a}\frac{\partial a}{\partial y}+U_{xs}\frac{\partial\sigma}{\partial x}+U_{ys}\frac{\partial\sigma}{\partial y}+2\sigma\frac{\partial U_{xs}}{\partial x}+\sigma\frac{\partial U_{ys}}{\partial y}\\&+\left(\frac{\partial U_{xs}}{\partial x}+\frac{\partial U_{ys}}{\partial y}\right)\Omega_{ys}-\frac{1}{2}\frac{\sigma}{k}\left.\frac{\partial^2 U_x}{\partial x\partial z}\right|_{z=0}+\frac{1}{2}\left(\frac{\partial U_{xs}}{\partial y}-\frac{U_{xs}}{k}\frac{\partial k_x}{\partial y}+\frac{\sigma}{k^2}\frac{\partial k_x}{\partial y}\right)\Omega_{xs}=0,\end{aligned} \tag{4.6}$$

in which the terms $(1/2)(\sigma^2/k^2)\partial k_y/\partial y$ and $(1/2)(\sigma/k^2)(\partial U_x/\partial z)\partial k_y/\partial y$ have been neglected in view of (3.7). Notice that since in (4.3) (and (3.26)) the quantity Ω_{ys} represents the component of vorticity tangential to the mean water surface and perpendicular to the local $\mathbf{k}$, we have

$$\begin{aligned}\frac{\partial\Omega_{ys}}{\partial x}&=\frac{\partial}{\partial x}\left[\frac{\partial}{\partial z}\left(\frac{k_x}{k}U_x+\frac{k_y}{k}U_y\right)\right]_{z=0}=\frac{\partial}{\partial x}\left[\frac{k_x}{k}\left.\frac{\partial U_x}{\partial z}\right|_{z=0}+\frac{k_y}{k}\left.\frac{\partial U_y}{\partial z}\right|_{z=0}\right]\\&=\left.\frac{\partial^2 U_x}{\partial x\partial z}\right|_{z=0}+\frac{1}{k}\frac{\partial k_y}{\partial x}\left.\frac{\partial U_y}{\partial z}\right|_{z=0}\\&=\left.\frac{\partial^2 U_x}{\partial x\partial z}\right|_{z=0}-\frac{1}{k}\frac{\partial k_x}{\partial y}\Omega_{xs}\end{aligned} \tag{4.7}$$

at the position under consideration, meaning that the quantity $\partial\Omega_{ys}/\partial x$ arising from the expansion of (4.3) is not equal to $(\partial^2 U_x/\partial x\partial z)_{z=0}$. The relation (4.7), including that $k_y=0$ and $\partial k_y/\partial y=0$, which are all valid only locally, has been utilized in deriving (4.6) so that in this form, (4.6) is valid only locally too and can therefore be compared with (3.26).

The comparison between (3.26) and (4.6) indicates that in a general case in which Ω_{xs} and $(\partial^2 U_x/\partial x\partial z)_{z=0}$ are not small compared with σ and $\partial\sigma/\partial x$ respectively, the wave action defined by Jonsson *et al.* (1978), which does not take the rotational perturbation velocity u_y into account here, is not conserved. Even if $\Omega_{xs} \approx 0$ (so that $u_y \approx 0$ according to (2.12)) but $(\partial^2 U_x/\partial x\partial z)_{z=0}$ has the same order of magnitude as $\partial\sigma/\partial x$, the wave action defined by Jonsson *et al.* still cannot be conserved. This situation is consistent with the theory of Jonsson *et al.* (1978) in which the wave action has been proved to be conserved under the assumption that Ω_y is constant in a two-dimensional flow.

If both Ω_{xs} and $(\partial^2 U_x/\partial x\partial z)_{z=0}$ vanish but the remaining terms in (3.26) and (4.6) are non-zero, meaning that the flow is three-dimensional, equations (3.26) and (4.6) coincide with each other exactly. Thus the action conservation equation derived by Jonsson *et al.* remains valid in a more general situation than that considered in Jonsson *et al.* (1978). Therefore it is interesting to see whether this conclusion can also be drawn from an analysis using their approach, which may also explained why the condition that Ω_{xs} and $(\partial^2 U_x/\partial x\partial z)_{z=0}$ are both small compared with σ and $\partial\sigma/\partial x$, respectively, is crucial for the validity of the action conservation equation.

5. Application of Jonsson, Brink-Kjœr & Thomas' approach

The approach applied by Jonsson *et al.* (1978) is based on the integral properties of the combined wave and current motion across a fixed vertical section. This approach was first applied by Longuet-Higgins & Stewart (1960) in their derivation of the radiation stress tensor and was applied by Phillips (1977) to derive the expressions for the conservation of mass, momentum and energy when a wave train propagates obliquely on a variable irrotational current. The special arrangement made by Jonsson *et al.* (1978) is suitable for a rotational current. However, to make this approach successful, it is required that any vertical integral from the bottom to the free surface involved in the analysis can be evaluated in terms of simple functions so that their derivatives with respect to x and y can avoid the special functions, like the exponential-integral function, which cannot be found in (4.6). This requirement cannot be fulfilled when the rotational perturbation velocity u_y given by (2.12), in which the denominator varies rapidly, is not negligible. This is also the reason why the wave action density cannot be redefined to include u_y if the latter is not negligible. Therefore, the condition that $\Omega_x = 0$ everywhere (but $U_y \neq 0$) is imposed here first, which ensures that $u_y = 0$ according to (2.12). This situation may actually occur when a highly sheared flow with non-uniform vorticity is generated by a wind stress acting on the boundary of an irrotational tidal flow and in the direction perpendicular to this flow.

The vertical integrals after being solved will be differentiated with respect to x and y to establish the equations for the conservation of mass, momentum and energy. Since each differentiation will increase the order of magnitude by one, these vertical integrals themselves can be evaluated without consideration of the slow variations of the quantities in the integrands. For example, although the slow variation of $\partial U_x/\partial z$ in the $z-$direction can significantly affect the vertical integrals of the quantities involving U_x if the water is very deep, this slow variation influences the surface waves only slightly according to (3.25)–(3.27). Therefore, for the sake of simplicity and without loss of the generality of the wave modulation theory, the slow variation of $\partial U_x/\partial z$ in the $z-$direction can be assumed vanishing here. This argument can also be applied to the variation of U_z in the $z-$direction which can occur due to the slow variation of the mean flow in the horizontal directions but can however be neglected in the vertical integrals by assuming that the value of $\partial U_x/\partial x + \partial U_y/\partial y$ is non-zero (but small) only in the region near the mean water surface in which the mean flow can have influence over the wave motion. Therefore, in the following discussion it is assumed that $U_z = 0$ as did by Longuet-Higgins & Stewart (1960), Phillips (1977) and Jonsson *et al.* (1978). Similarly, in this approach, it is unnecessary to consider the second term of the asymptotic expansion of each unknown, meaning that the components of the rotational perturbation velocity $\widehat{u}_x$ and $\widehat{u}_z$ can also be disregarded in the vertical integrals. Therefore, it suffices to substitute all the solutions in §2 except u_y into the vertical integrals for the local properties of the wave motion.

Following the precedent of Jonsson *et al.* (1978) (and Phillips (1977) for a three-dimensional analysis), we first define the radiation stress

$$S_{\alpha\beta} = \delta_{\alpha\beta}\overline{\int_{-h}^{\eta} p\,dz} - \delta_{\alpha\beta}\int_{-h}^{0}(-\rho g z)\,dz + \rho\overline{\int_{-h}^{\eta}\widetilde{u}_\alpha\widetilde{u}_\beta\,dz} - \rho\int_{-h}^{0}\widetilde{U}_\alpha\widetilde{U}_\beta\,dz, \quad (\alpha,\beta=1,2), \tag{5.1}$$

where a overbar denotes averaging over the (constant) observed period, $\delta_{\alpha\beta}$ is the unit tensor ($\delta_{\alpha\beta}=1$ if $\alpha=\beta$ and vanishes otherwise), h the local mean water depth,

$$\widetilde{u}_1 \equiv \widetilde{u}_x \equiv \frac{\partial\phi}{\partial x} + U_x, \quad \widetilde{u}_2 \equiv \widetilde{u}_y \equiv \frac{\partial\phi}{\partial y} + U_y = U_y \tag{5.2}$$

the total horizontal velocity components, and $\widetilde{U}_1 \equiv \widetilde{U}_x$, $\widetilde{U}_2 \equiv \widetilde{U}_y$ the x, y components of a 'formal current velocity'. The profiles of the latter are defined as

$$\widetilde{U}_x(z) = \widetilde{U}_{xs} + \Omega_y z, \qquad \widetilde{U}_y(z) = \widetilde{U}_{ys}, \tag{5.3}$$

where $\widetilde{U}_{xs}$, $\widetilde{U}_{ys}$, and Ω_y are independent of z. The relations between $\widetilde{U}_x$ and U_x and between $\widetilde{U}_y$ and U_y can be established from the requirement that

$$\int_{-h}^{0}\widetilde{U}_x(z)\,dz = \overline{\int_{-h}^{\eta}\widetilde{u}_x(z)\,dz}, \qquad \int_{-h}^{0}\widetilde{U}_y(z)\,dz = \overline{\int_{-h}^{\eta}\widetilde{u}_y(z)\,dz}. \tag{5.4}$$

Substituting (2.6), (2.8), (5.2) and (5.3) into (5.4) and recalling that

$$U_x = U_{xs} + \Omega_y z, \qquad U_y = U_{ys}, \tag{5.5}$$

we obtain

$$\widetilde{U}_{xs} = U_{xs} + \frac{2\sigma+\Omega_y}{4h}a^2, \qquad \widetilde{U}_{ys} = U_{ys} \tag{5.6}$$

correct to the second order in (ak). To achieve these results, the mean water depth h is assumed to be large compared with the wavelength so that the solutions (2.6) and (2.8) for deep-water waves can be valid.

In order to solve the first integral in (5.1) to the second order in (ak), the mean pressure distribution

$$\overline{p} = -\rho g z - \frac{\rho}{2}\sigma^2 a^2 e^{2kz}, \tag{5.7}$$

correct to the second order in (ak) and valid even for a vortical flow (see (3.2.17) in Phillips (1977)) is also required in addition to (2.11). By substituting all these results and the solutions given in §2 into (5.1), we obtain

$$\left.\begin{aligned} S_{11} &= \frac{\rho}{4}ga^2 + \frac{\rho}{4}\Omega_y(2\sigma+\Omega_y)a^2h \\ S_{22} &= \frac{\rho}{4k}\sigma\Omega_y a^2 \\ S_{12} &= S_{21} = 0 \end{aligned}\right\} \tag{5.8}$$

correct to the second order in (ak).

Next the total mean energy flux per unit area

$$F_\alpha = \overline{\int_{-h}^{\eta}\left[p+\rho g(z+b)+\frac{\rho}{2}(\widetilde{u}_1^2+\widetilde{u}_2^2+\widetilde{u}_3^2)\right]\widetilde{u}_\alpha\,dz}, \qquad (\alpha=1,2) \tag{5.9}$$

where

$$\widetilde{u}_3 \equiv \widetilde{u}_z \equiv \partial\phi/\partial z + U_z = \partial\phi/\partial z \tag{5.10}$$

and $b=b(x,y)$ represents the height of the mean water surface above a reference level (see figure 2) specified for determination of the potential energy. Notice that the term $\rho g b\widetilde{u}_\alpha$ must be included in (5.9) to take into account the situation that the mean water surface is not horizontal, because this term will result in the terms containing $\partial b/\partial x$ or $\partial b/\partial y$ in the final energy conservation equation, which like other terms in this equation, contain only one first-order derivative of the slowly varying quantities and therefore cannot be neglected.

By substitution and after some lengthy manipulations, we obtain

$$\begin{aligned} F_1 =& \frac{\rho}{4k}\sigma^2U_{xs}a^2 + \frac{\rho}{4k}g\sigma a^2 + \frac{\rho}{2}gU_{xs}a^2 - \frac{3}{8}\rho a^2\left(-2\sigma\Omega_yU_{xs}h + \frac{2}{3}\sigma\Omega_y^2h^2 - \Omega_y^2U_{xs}h + \frac{1}{3}\Omega_y^3h^2\right) \\ &+ \frac{\rho}{2}\int_{-h}^{0}\widetilde{U}_x^3\,dz + \frac{\rho}{2}\int_{-h}^{0}\widetilde{U}_y^2\widetilde{U}_x\,dz + \rho g b h\widetilde{U}_{mx} \end{aligned} \tag{5.11}$$

$$F_2 = \frac{\rho}{2}gU_{ys}a^2 + \frac{\rho}{4}a^2\left(\sigma\Omega_yU_{ys}h + \frac{1}{2}\Omega_y^2U_{ys}h\right) + \frac{\rho}{2}\int_{-h}^{0}\widetilde{U}_x^2\widetilde{U}_y\,dz + \frac{\rho}{2}\int_{-h}^{0}\widetilde{U}_y^3\,dz + \rho g b h\widetilde{U}_{my} \tag{5.12}$$

where $\widetilde{U}_{mx} \equiv (1/h)\overline{\int_{-h}^{\eta}\widetilde{u}_x(z)\,dz} = \widetilde{U}_{xs} - \Omega_yh/2$ and $\widetilde{U}_{my} \equiv (1/h)\overline{\int_{-h}^{\eta}\widetilde{u}_y(z)\,dz} = \widetilde{U}_{ys}$ (see (5.3) and (5.4)), the average-over-depth velocity.

On the other hand, the mean total momentum flux $M_{\alpha\beta}$ per unit area equals the sum of the first and third terms on the right-hand side of (5.1). Thus

$$M_{\alpha\beta} = S_{\alpha\beta} + \frac{\rho}{2}gh^2\delta_{\alpha\beta} + \rho\int_{-h}^{0}\widetilde{U}_\alpha\widetilde{U}_\beta\,dz, \qquad (\alpha,\beta=1,2). \tag{5.13}$$

The horizontal components of the mean total pressure force acting on the fluid at the bed per unit length in the $x-$ and $y-$ directions are

$$P_1 = \rho g h \frac{\partial D}{\partial x} \qquad \text{and} \qquad P_2 = \rho g h \frac{\partial D}{\partial y},$$

respectively; see figure 2. The equations $-\partial M_{\alpha 1}/\partial x - \partial M_{\alpha 2}/\partial y + P_\alpha = 0, (\alpha = 1, 2)$ of total momentum conservation therefore take the form

$$\frac{\partial S_{11}}{\partial x} + \frac{\partial S_{12}}{\partial y} + \frac{\partial}{\partial x}\left(\rho \int_{-h}^{0} \widetilde{U}_x \widetilde{U}_x \, dz\right) + \frac{\partial}{\partial y}\left(\rho \int_{-h}^{0} \widetilde{U}_x \widetilde{U}_y \, dz\right) + \rho g h \frac{\partial b}{\partial x} = 0, \tag{5.14}$$

$$\frac{\partial S_{21}}{\partial x} + \frac{\partial S_{22}}{\partial y} + \frac{\partial}{\partial x}\left(\rho \int_{-h}^{0} \widetilde{U}_x \widetilde{U}_y \, dz\right) + \frac{\partial}{\partial y}\left(\rho \int_{-h}^{0} \widetilde{U}_y \widetilde{U}_y \, dz\right) + \rho g h \frac{\partial b}{\partial y} = 0. \tag{5.15}$$

Also the equation expressing total energy conservation is simply

$$\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = 0, \tag{5.16}$$

where F_1 and F_2 are given by (5.11) and (5.12) respectively.

The equations (5.14)–(5.16) can be combined into one equation to eliminate the terms devoid of the wave amplitude a. To achieve this purpose, we multiply (5.14) by $\widetilde{U}_{mx}$ and (5.15) by $\widetilde{U}_{my}$, and then subtract the resulting equations from (5.16), so that the terms originated from the last terms in (5.11), (5.12), (5.14) and (5.15) can immediately be cancelled out in this operation, considering the mass conservation equation

$$\frac{\partial}{\partial x}(\widetilde{U}_{mx} h) + \frac{\partial}{\partial y}(\widetilde{U}_{my} h) = 0. \tag{5.17}$$

The integrals in (5.11), (5.12), (5.14) and (5.15) can also yield the terms free from a in this operation. However, by using (5.17) repeatedly and in consideration of (3.20), (5.3), (5.4) and (5.6), it can be proved that

$$\begin{aligned} &\frac{\partial}{\partial x}\left(\frac{1}{2}\int_{-h}^{0} \widetilde{U}_x^3 \, dz + \frac{1}{2}\int_{-h}^{0} \widetilde{U}_x \widetilde{U}_y^2 \, dz\right) + \frac{\partial}{\partial y}\left(\frac{1}{2}\int_{-h}^{0} \widetilde{U}_x^2 \widetilde{U}_y \, dz + \frac{1}{2}\int_{-h}^{0} \widetilde{U}_y^3 \, dz\right) - \widetilde{U}_{mx}\frac{\partial}{\partial x}\left(\int_{-h}^{0} \widetilde{U}_x^2 \, dz\right) \\ &- \widetilde{U}_{mx}\frac{\partial}{\partial y}\left(\int_{-h}^{0} \widetilde{U}_x \widetilde{U}_y \, dz\right) - \widetilde{U}_{my}\frac{\partial}{\partial x}\left(\int_{-h}^{0} \widetilde{U}_x \widetilde{U}_y \, dz\right) - \widetilde{U}_{my}\frac{\partial}{\partial y}\left(\int_{-h}^{0} \widetilde{U}_y^2 \, dz\right) \\ =&\frac{1}{12} h^3 \Omega_y \frac{\partial \Omega_y}{\partial x}\left(\frac{2\sigma + \Omega_y}{4h} a^2 - \frac{1}{2}\Omega_y h\right). \end{aligned} \tag{5.18}$$

Therefore, if $\partial\Omega_y/\partial x \neq 0$, the terms originated from the integrals in (5.11), (5.12), (5.14) and (5.15) cannot be cancelled out in this operation and will yield a term free from a which cannot be eliminated by other terms, meaning that in this situation the integral approach will fail for an apparent reason. Hence we here restrict our consideration further to the case that $\partial\Omega_y/\partial x = 0$, which corresponds to the second requirement mentioned at the end of §4 for the validity of the action conservation equation (since $\partial\Omega_y/\partial x = (\partial^2 U_x/\partial x\partial z)_{z=0}$ in the present situation when $\Omega_x = 0$ according to (4.7)).

The rest of the terms in the resulting equation from this operation, all containing the wave amplitude a, can also be divided into three groups. The first group is devoid of h while each term in the second and third groups contains respectively h and h^2 as the common factor. Since h can be very large compared with the wavelength and can be changed without affecting the wave motion and the current field in the region near the mean water surface, the terms in these three groups should be balanced separated, leading to three equations. However, if $\partial\Omega_y/\partial x = 0$, the terms in the third group are completely cancelled out, and in the meantime, the two equations from the first and second groups coincide with each other exactly and can be written as

$$(g + 2\sigma U_{xs} + \Omega_y U_{xs})\frac{1}{a}\frac{\partial a}{\partial x} + (2\sigma + \Omega_y)U_{ys}\frac{1}{a}\frac{\partial a}{\partial y} + U_{xs}\frac{\partial \sigma}{\partial x} + U_{ys}\frac{\partial \sigma}{\partial y} + 2\sigma\frac{\partial U_{xs}}{\partial x} + \sigma\frac{\partial U_{ys}}{\partial y} + \left(\frac{\partial U_{xs}}{\partial x} + \frac{\partial U_{ys}}{\partial y}\right)\Omega_y = 0. \tag{5.19}$$

Therefore the condition $\partial\Omega_y/\partial x = 0$ is indeed important for the application of the integral approach.

Since in the present analysis, $\Omega_y = \Omega_{ys}$, equation (5.19) is exactly identical with (4.6) (and (3.26) if g' is replaced by g in the latter) when $\Omega_{xs} = 0$ and $\partial\Omega_{ys}/\partial x = 0$. Therefore, by using the approach of Jonsson *et al.* (1978), we again prove that when $\Omega_{xs} = 0$ and $\partial\Omega_{ys}/\partial x = 0$, even if $U_y \neq 0$ and $\partial U_y/\partial y \neq 0$, the action conservation equation remains valid. However, in a more general situation in which $\Omega_{xs} = 0$ but $\partial\Omega_{ys}/\partial x \neq 0$, although the rotational perturbation velocity u_y still vanishes so that the wave action density can be defined without ambiguity and the vertical integrals can be evaluated in terms of simple functions, the approach of Jonsson *et al.* (1978) may still fail and the action conservation equation becomes invalid as shown in §4. However, in this situation and in an even more general situation in which the underlying current varies slowly in the horizontal directions and its vorticity components Ω_x and Ω_y are both large and vary slowly in all directions, equation

(3.26) remains valid. In this general situation, the difference between the predictions by (3.26) and by the action conservation equation (4.6) in which the wave action density is defined without consideration of u_y, will be illustrated numerically in the next section.

6. Numerical computations

Since the differential equations (3.26) and (4.6) are derived in the specifically oriented rectangular coordinates and since the forms of these equations are valid only locally, it is convenient to solve these equations by using a special stepwise numerical integration as illustrated in the following numerical simulation.

In this simulation, the velocity distribution of the underlying current is given as

$$\left.\begin{aligned} U'_x &= -2.0 - 0.002085x' - 0.005049y' + 1.5z' + 0.00075x'z' - 0.0013y'z' \ \ (\mathrm{m\,s^{-1}}) \\ U'_y &= 1.1547 - 0.00005x' - 0.004915y' + 0.866z' + 0.000433x'z' - 0.00075y'z' \ \ (\mathrm{m\,s^{-1}}) \end{aligned}\right\} \tag{6.1}$$

where x', y', z', unlike the coordinates x, y, z which move with the position under consideration, represent a fixed rectangular coordinate system as shown in figure 3, and U'_x, U'_y the velocity components in the $x'-$ and $y'-$directions. This distribution under a rotation of the coordinates by an angle $30°$ about the $z'-$direction also takes the form

$$\left.\begin{aligned} U''_x &= -1.1547 - 0.005x'' - 0.005y'' + 1.732z'' - 0.001732y''z'' \ \ (\mathrm{m\,s^{-1}}) \\ U''_y &= 2 - 0.002y'' \ \ (\mathrm{m\,s^{-1}}) \end{aligned}\right\} \tag{6.2}$$

in terms of the new coordinates x'', y'', z'' and the new components U''_x, U''_y in the $x''-$ and $y''-$directions. In this form, this distribution can easily be proved to satisfy the vorticity equations (3.20) and (3.28).

Notice that from (6.2), the quantities $\partial U''_y/\partial z''$ and $\partial^2 U''_x/\partial x''\partial z''$ both vanish. Therefore, if at a certain position the waves propagate in the $x''-$direction, the wave action can be conserved according to both (3.26) and (5.19) compared with (4.6). However, for waves propagating in other directions, equation (5.19) becomes invalid and the following computations will show that the wave action defined without consideration of the rotational perturbation velocity which has the same order of magnitude as the irrotational one in this situation is not conserved.

To conduct the computations, the boundary conditions are prescribed on $x' = 0$ on which the amplitude $a = $ constant $= 1$ m and the component of the wavenumber on the $y'-$direction $k'_y = $ constant $= 0$, meaning that at each point on $x' = 0$, the axes of the coordinates x, y, z applied in (3.26) and (4.6) are in the same direction as those of x', y', z', respectively. On the other hand, since $n_0 = $ constant everywhere, from (2.9) and (3.27) it is clear that the magnitude of the wavenumber $\mathbf{k}$ cannot remain constant as U_{xs} and Ω_{ys} vary on $x' = 0$ according to (6.1). However, since $n_0 = 0.5\ \mathrm{rad\,s^{-1}}$ is chosen here, these variations and that of k are slow near the origin of the coordinates x', y', z' at which the wavelength is about 23.5 m in view of (2.9) and (3.27).

On $x' = 0$, the value of k'_y at each point has been prescribed so that the value of the wavenumber component on the x' – direction k'_x at the same point can be calculated by using (2.9) and (3.27). Next, from the values of k'_x at points A and C in figure 3, one can estimate the value of $\partial k'_x/\partial y'$ at point B by using the approximation

$$\frac{\partial k'_x}{\partial y'}(B) \approx \frac{k'_x(A) - k'_x(C)}{2\Delta y'}.$$

Since $\partial k'_y/\partial x' = \partial k'_x/\partial y'$ and

$$\frac{\partial k'_y}{\partial x'}(B) \approx \frac{k'_y(E) - k'_y(B)}{\Delta x'} = \frac{k'_y(E)}{\Delta x'},$$

the value of $k'_y(E)$ can therefore be determined approximately from the values of $k'_x(A)$ and $k'_x(C)$, and is usually non-zero. Substituting the value of $k'_y(E)$ into (2.9) and (3.27), we also obtain the value of $k'_x(E)$.

When $k'_y(E)$ is non-zero, to directly apply (3.26) at point E, a new rectangular coordinate system is required so that we next proceed to determine the components $k_x(G)$ and $k_y(G)$ in the coordinates x, y, z in which $k_y(E) = 0$ (see figure 3). This can be done because from (3.7) we have $k_y(D) = 0$ and $k_y(F) = 0$ approximately, so that in this new coordinate system the situation at point E becomes the same as that at point B in the old coordinate system. Therefore, by using the same group of formulae and equations, one can estimate the values of $k_x(G)$ and $k_y(G)$. This procedure can be repeated to determine the variations of $\mathbf{k}$ along the line whose tangent is everywhere parallel to the local $\mathbf{k}$.

At each point on this line, after the values of $\mathbf{k}$ and therefore σ as well as their derivatives with respect to x and y are determined, all quantities involved in (3.26) except $(1/a)\partial a/\partial x$ and $(1/a)\partial a/\partial y$ become known. Therefore (3.26) can be integrated step by step along the same line for the solution values of a along this line. The difference between this procedure and that for the solution of $\mathbf{k}$ is that the solution values of a at the two neighboring points on the adjacent lines on both sides of the line under consideration will be utilized to estimate the value of $\partial a/\partial y$ at the point under consideration, which provides the last data for determination of the value of a at the next point on this line, using (3.26). Thus the values of a on each of the lines whose tangents are everywhere parallel to the local $\mathbf{k}$ depend on the values of a on other lines, contrary to the situation that the values of $\mathbf{k}$

on each of these lines are independent of those on other lines. This is because these lines do not coincide with the characteristic curves of equation (3.26), but on the other hand, the equations (2.9), (3.5) and (3.27) represent a degenerate hyperbolic system in which all directions are formally characteristic (see Whitham 1974, §5.1). Since the situation that the values of $\mathbf{k}$ on each line are independent of those on other lines has been realized in the present computations by using (3.7), equation (3.7) may therefore be consistent with the set of equations (2.9), (3.5) and (3.27).

When point E is under consideration, since the points D' and F' in figure 3 are not on the $y-$axis, the value of $\partial a/\partial y$ at point E can be estimated only by an iterative algorithm. This algorithm starts with an initial guess $(\partial a/\partial y)(E) = (a(D') - a(F'))/(2\Delta y')$ and then estimates $(\partial a/\partial x)(E)$ by using (3.26). These two quantities also satisfy the relation

$$\lim_{\Delta y' \to 0} \frac{a(D') - a(F')}{2\Delta y'} = \sin\theta \frac{\partial a}{\partial x}(E) + \cos\theta \frac{\partial a}{\partial y}(E), \tag{6.3}$$

where $\theta = \theta_1 = \theta_2$ in this case (see figure 3). Therefore, approximating the value of

$$\lim_{\Delta y' \to 0} \frac{a(D') - a(F')}{2\Delta y'}$$

by $(a(D') - a(F'))/2\Delta y'$ and substituting the first iterate of $(\partial a/\partial x)(E)$ into (6.3), we obtain a new approximation for $(\partial a/\partial y)(E)$ so that these computations can be repeated until the iterates converge.

These computations will become a little more complicated when the iterative algorithm is applied to the next point G and the points after, because at each of these points, its two neighboring points on the adjacent lines usually cannot be connected with each other by a straight line passing through the point under consideration, meaning that $\theta_1 \neq \theta_2$ now. Therefore, instead of (6.3), we consider

$$\lim_{d_1 \to 0} \frac{a(D') - a(E)}{d_1} = \sin\theta_1 \frac{\partial a}{\partial x}(E) + \cos\theta_1 \frac{\partial a}{\partial y}(E),$$

$$\lim_{d_2 \to 0} \frac{a(E) - a(F')}{d_2} = \sin\theta_2 \frac{\partial a}{\partial x}(E) + \cos\theta_2 \frac{\partial a}{\partial y}(E),$$

where d_1 and d_2 represent respectively the distances between D' and E and between E and F'. By using these two relations, one may obtain two approximations for $(\partial a/\partial y)(E)$ in each iteration, but their mean value can eventually be applied in the iterative algorithm, which may improve the stability of the numerical solution.

The solution values of a along a single line are shown in figure 4, in which we also compute the numerical solution of (4.6) by using exactly the same algorithm. The results indeed indicate that in a three-dimensional, strongly sheared current, the variation of the wave amplitude with distance is significantly different from that predicted by the action conservation equation (4.6).

In order to check the computer program and to justify (3.7) which has been utilized not only in the derivation of (3.26) but also in the above numerical computations, we also solve the action conservation equation (4.3) and equation (3.26) on a rectangular mesh in the coordinates x', y', z' in figure 3. In these computations, to determine $\mathbf{k}$ at the next point, the value of $\partial k'_x/\partial y'$ at the point under consideration can always be estimated from the values of k'_x at the neighboring points which have been determined at the previous step of the computations, so that it is unnecessary to make use of (3.7) here. Similarly, the second term in (4.3) can be estimated from the data determined previously, so that this equation can also be solved by ordinary stepwise numerical integration without using the iterative algorithm. On the other hand, each quantity in (3.26) which represents the component of a vector or tensor can be expressed in terms of the components of this vector or tensor in the coordinates x', y', z' through a transformation of axes at each point under consideration. Therefore (3.26) can also be solved numerically by using rectangular grids at the expense of introducing extra terms in the equation, which will complicate the equation significantly, but can however avoid using the iterative algorithm.

By using a rectangular mesh in the coordinates x', y', z', the values of a and $\mathbf{k}$ at the points marked with crosses in figure 5 have been calculated, from which the values of a and $\mathbf{k}$ at the points marked with circles in figure 5 can also be estimated by linear interpolation. These results as shown in figure 4 and 6 coincide very well with their counterparts obtained by stepwise integration along a curve whose tangent is everywhere parallel to $\mathbf{k}$. Therefore, the use of (3.7) in the numerical computations has been justified. Furthermore, since in deriving (4.6) from (4.3), two terms containing $\partial k_y/\partial y$ have been neglected, the situation that the solution values of (4.3) and (4.6) at each point coincide with each other closely can also justify the neglect of the terms containing $\partial k_y/\partial y$ in the analysis, although in the present case both (4.3) and (4.6) cannot really describe the variation of the amplitude.

7. Conclusions

By using an approach which is not separate from the traditional perturbation scheme but can deal with the complicated situation in which a deep-water gravity wave train propagates obliquely on a steady three-dimensional, strongly sheared current that varies slowly in the horizontal directions and deviates slightly from a linear profile in the vertical direction, the first-order WKBJ solution, including the modulation equation of the wave amplitude, has been derived rigorously. This modulation equation is in general inconsistent with the two-dimensional action conservation equation which represents a natural extension of the one-dimensional one derived by Jonsson *et al.* (1978) and therefore take the vorticity of the current into account but ignores the rotational perturbation velocity that may have the same order of magnitude as the irrotational part of the wave motion in the present situation. Thus, in a region with a strongly sheared current, the wave spectrum data estimated from the action conservation equation may sometimes be misleading.

When the combined wave and current motion becomes two-dimensional and the vorticity is constant, the modulation equation derived here reduces to the one-dimensional action conservation equation deduced by Jonsson *et al.* (1978). Even if the underlying rotational current is three-dimensional, as long as Ω_{xs} and $(\partial^2 U_x/\partial x \partial z)_{z=0}$ are small compared with σ and $\partial\sigma/\partial x$ respectively, the present result can still coincide with the reduced form of the two-dimensional action conservation equation in this case.

To explain why these two restrictions on the distribution of the underlying current are both required for the validity of the action conservation equation, the approach of Jonsson *et al.* (1978) considering the integral properties of the combined wave and current motion across a fixed vertical section has also been applied in the three-dimensional flow. From this analysis, it is immediately clear that if Ω_x has the same order of magnitude as σ so that the rotational perturbation velocity u_y has the same order of magnitude as $\partial\phi/\partial x$ and $\partial\phi/\partial z$, since the vertical integrals from the bottom to the free surface of the terms involving u_y cannot be evaluated in terms of simple functions, the wave action density cannot be redefined to include the contribution of u_y in the usual sense.

Similarly, in order that all the vertical integrals involved can be evaluated in terms of simple functions, the condition that Ω_x is small compared with σ is also required for the approach of Jonsson *et al.* (1978) being valid. In addition, to cancel the terms unrelated to the wave-current interaction to obtain a useful and consistent differential equation for determination of the variation of the amplitude, it is also required that $(\partial^2 U_x/\partial x \partial z)_{z=0}$ is small compared with $\partial\sigma/\partial x$. The resulting equation is indeed identical with the reduced forms of the two-dimensional action conservation equation (4.6) and the modulation equation (3.26) in this case.

Finally we emphasize that even when $\Omega_x = u_y = 0$ so that the wave action density can be defined without ambiguity, as long as $(\partial^2 U_x/\partial x \partial z)_{z=0}$ is not small compared with $\partial\sigma/\partial x$, the wave action is not conserved, meaning that the failure of the conservation of wave action cannot be attributed solely to the neglect of the rotational part of the wave motion in the definition of the wave action density, which will increase the difficulties for a physical interpretation of this failure and for establishment of a new conservation principle. However, equation (3.26) can be utilized to determine numerically the variation of the amplitude with distance in a general situation.

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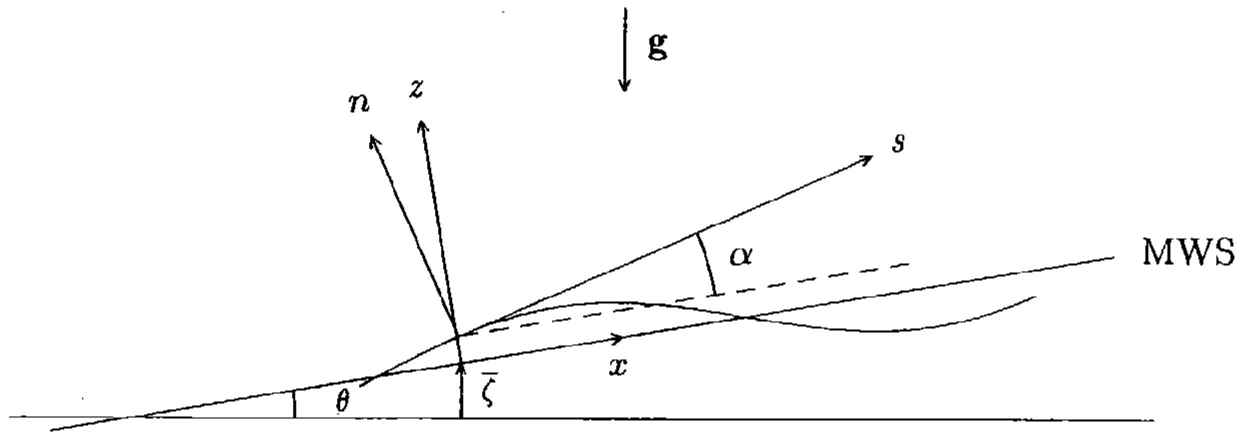

Fig. 1. Definition sketch.

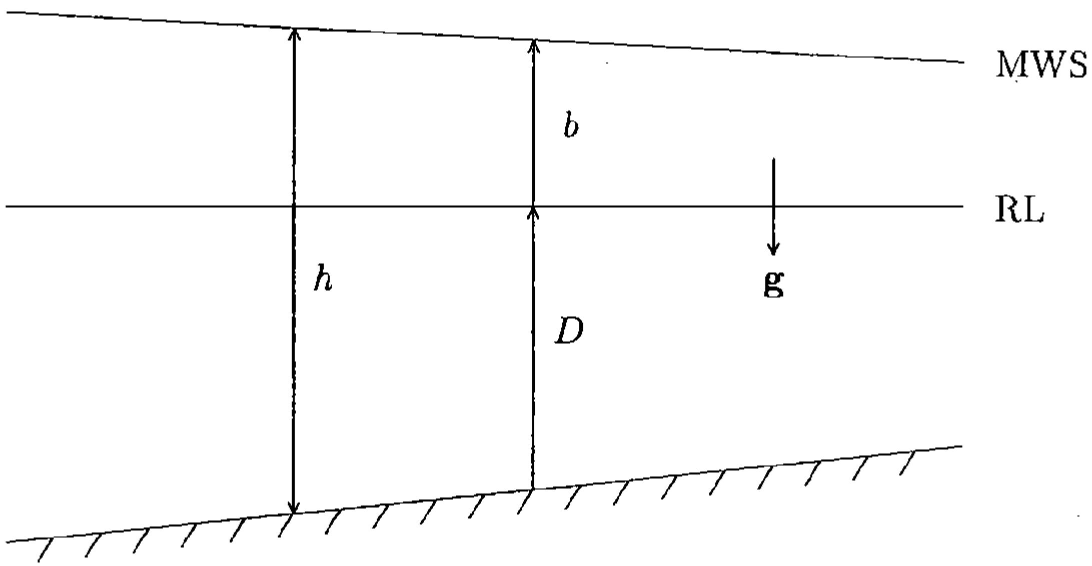

Fig. 2. Definition sketch.

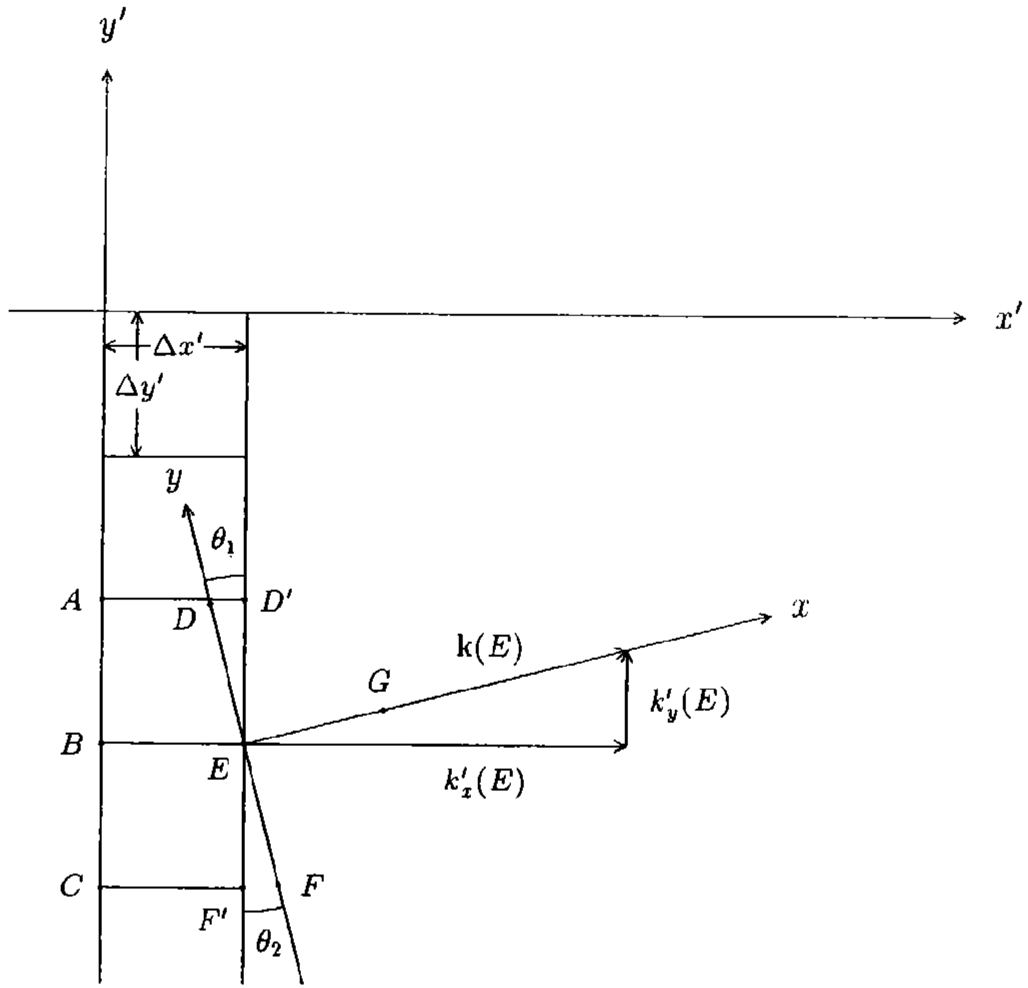

Fig. 3. A diagram for illustration of the stepwise integration along a direction parallel to $\mathbf{k}$.

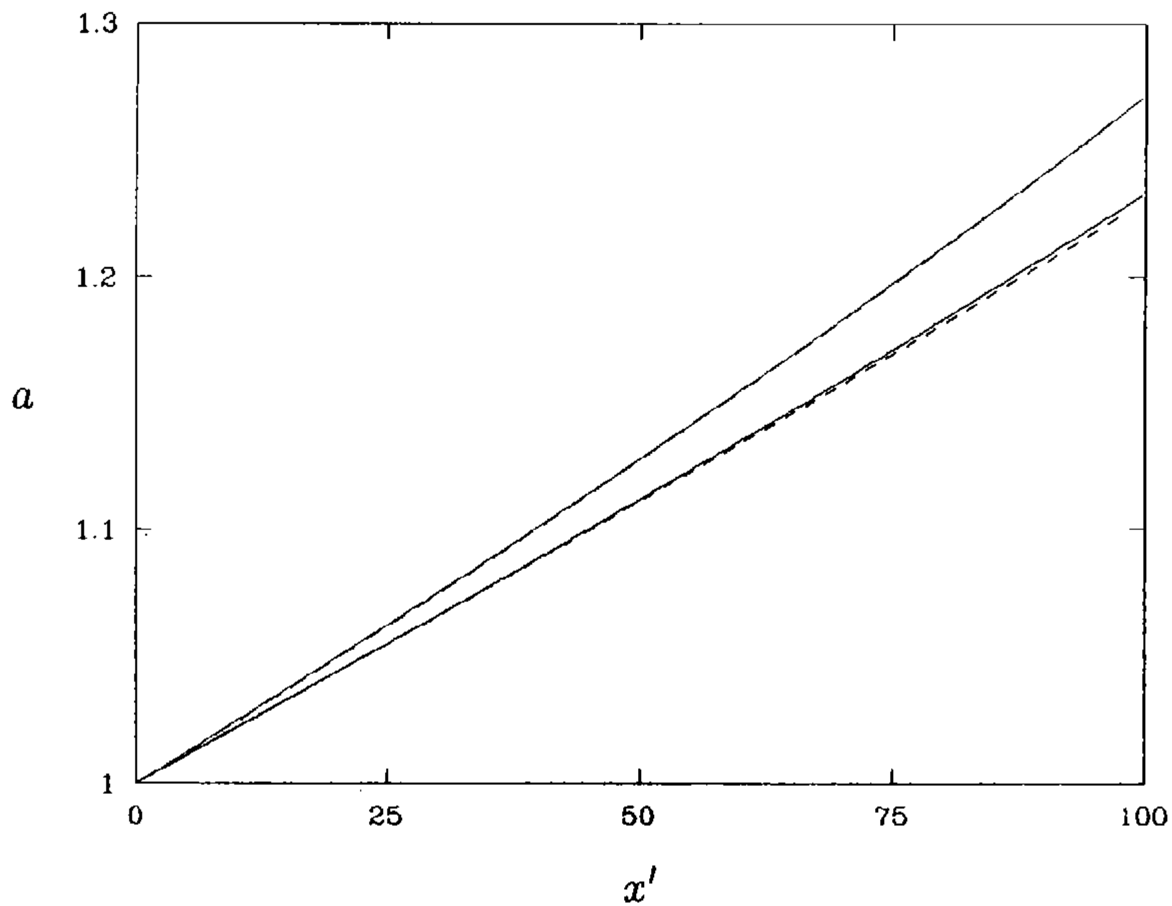

Fig. 4. The numerical solutions of equations (3.26) (narrow solid line), (4.6) (wide solid line) and (4.3) (short-dashed line) at the points marked with circles in figure 5. The long-dashed line represents the numerical solutions of (3.26) using a rectangular mesh.

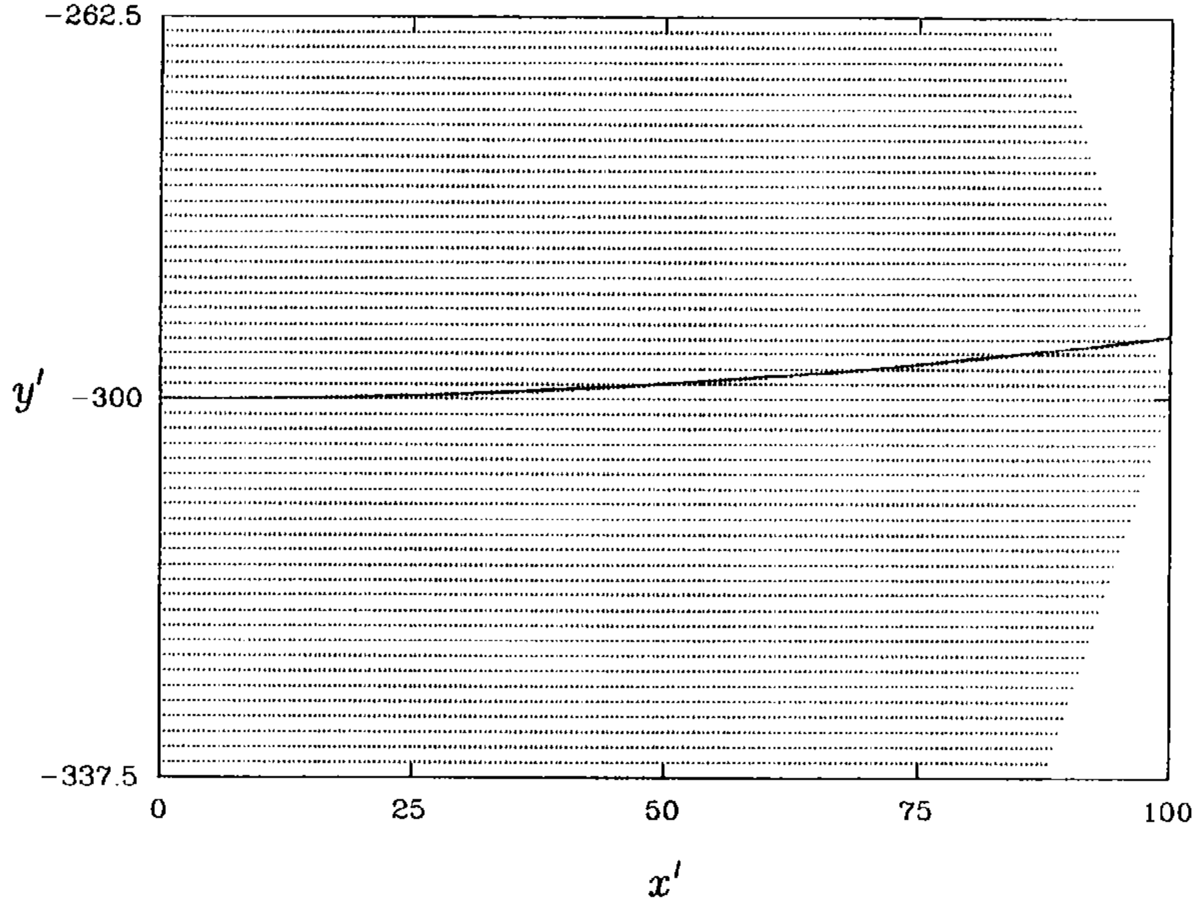

Fig. 5. The mesh points where the solution values are calculated. The line connecting the points marked with circles represents a curve whose tangent is everywhere parallel to $\mathbf{k}$.

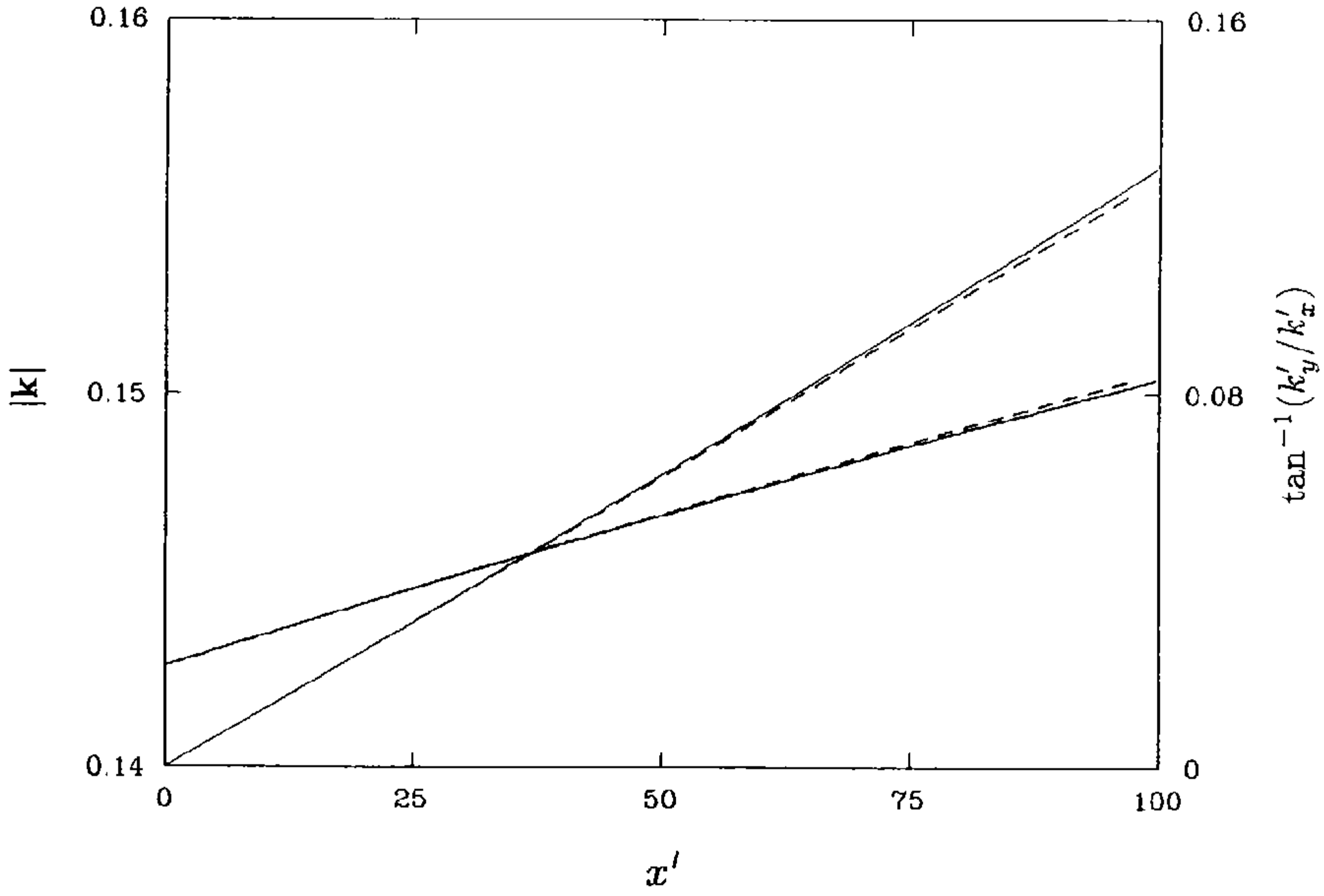

Fig. 6. The solution values of the magnitude (wide lines) and the direction (narrow lines) of $\mathbf{k}$ at the points marked with circles in figure 5. The dashed lines represent those using a rectangular mesh.