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交 通 部 運 輸 研 究 所

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<tr><td colspan="4">摘要：
當沈箱式直立防波堤受到水平力作用，由於水平力的作用位置可能在物體質心以上或質心以下，可能產生不同的反應行為。其反應的型態包括靜止(rest)、滑動(slide)、擺動(rock)、滑動與擺動(slide and rock)。本研究係直立式防波堤反應行為評估準則研究的初期研究，在本研究中成功地建立自由剛性塊體受水平力作用時各種可能反應型態的準則, 有助於提供類似結構物設計時的運動行為或穩定分析。</td></tr>
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ABSTRACT:

When a vertical breakwater subjected to a horizontal force action, due to the acting position might above or below the center of mass of the structure, the response mode of the structure might be one of the following behaviors, including rest mode, slide mode, rock mode, or slide and rock mode. This research, as a preliminary research of dynamical behaviors of vertical structure, successfully establishes the criteria of the behaviors of a freestanding rigid body subjected to a horizontal force which is beneficial to engineering design in predicting the behaviors of such structure.

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本報告係國立台灣海洋大學河海工程系為執行交通部運輸研究所港灣技術研究中心的委託研究計畫”花蓮港港池共振機制研究(1/4)”(計畫編號：MOTC-IOT-94-H3DB003)需要，邀請美國北卡羅萊納州立大學，土木、營建及環境工程系榮譽退休教授董啟超博士進行有關防波堤受波浪作用力之反應行為的相關研究。研究進行中, 董教授曾就研究內容於民國 94 年 5 月 9 日至 5 月 10 日期間在台中梧棲港灣技術研究中心舉辦演講。

本報告包含中文版及英文版(請參考附錄)兩部分。報告原文係以英文撰寫，為便利國內專家學者及學子參考，在蔡仲景先生的協助下，以及董教授的最後校稿下，將其翻譯成中文。其中，英文部分基於計畫需要，先於海洋大學河海工程系水動力實驗室印製報告(報告編號：HDL-003)。

目　錄

圖目錄

符號表

a	純量，在式(3.18)及(4.42)中定義
B	物體的半個寬度
b	純量，在式(3.19)及(4.43)中定義
C	物體的質量中心
c	純量，在式(3.20)及(4.44)中定義
d	純量，在式(3.21)及(4.45)中定義
F	水平外力
f_x	水平反力
f_y	垂直反力
g	重力加速度
H	物體的半個高度
h	外力F作用線與質心的距離
I	物體對質心C的慣性力矩
k	用來表示外力F 相對於物體重量的比值的無因次且非負數值
k'	用來表示外力F作用高度h相對於物體高度H的比值的無因次且非負數值
k_A	在$k-\mu$平面上對應於點A的k值

$k_{B'}$	點 B' 的橫座標，是圖 4.4(a)、4.5(a) 、4.6(a)及 4.7(a)中曲線 $\mu^*(k)$ 和直線 AB 的交點
m	物體的總質量
RE	代表靜止狀態區間的符號
RO	代表物體依點 O 晃動狀態區間的符號
SL	代表滑動狀態區間的符號
SRO_-	代表當 $\ddot{x}_O \leq 0$ 時，物體依點 O 滑動-晃動狀態區間的符號
SRO_+	代表當 $\ddot{x}_O \geq 0$ 時，物體依點 O 滑動-晃動狀態區間的符號
x	物體質心 C 的水平位移，以向右為正
x_O	物體底部 O 點的水平位移，以向右為正
$x_{O'}$	物體底部 O' 點的水平位移，以向右為正
y	物體質心 C 的垂直位移，以向上為正
Δ	在點 A 處，$\mu^*(k)$ 的斜率和直線 AB 的斜率 $\lvert k' \rvert$ 的差值，如圖 4.4(a)、4.4(b)、4.6(a)及 4.6(b)所示
$\gamma = H / B$,	物體的細長比或形狀比
θ	物體的旋轉角，以逆時針方向為正
μ	物體與底床間的摩擦係數
$\mu^*(k)$	由式(3.24)及(4.40)定義的曲線
$\mu^{**}(k)$	由式(4.96) 及(4.97)定義的曲線

$\mu_{B'}$ 點 B' 的縱座標，是圖 4.4(a)、4.5(a)、4.6(a)及 4.7(a)中曲線 $\mu^*(k)$ 及直線 AB 的交點

ξ 反力 f_y 作用線與物體質心 C 的距離

$|\ |$ 絕對值

第一章 前　言

在海岸水域裡，不同型式的防波堤常被用在港灣設施上來抵擋波浪的侵襲；然而多年來，由於波浪力的作用，防波堤的破壞案例不斷發生。有許多種破壞類型已經被觀測到而且進行分析以瞭解波浪作用下的行為。至今這些研究已經獲得相當多的重要資訊並用來指導相關的設計工作。本報告不準備回顧這些龐大數量的相關文獻。

在本研究中，我們將嘗試著辯識當一個直立式沈箱受到仿如碎波作用的作用歷時甚短的水平衝擊力作用下的各種反應模態以及這些反應模態的啟動條件。該沈箱是以一個自由放置在剛性水平底床的剛性物體來模擬，在其接觸面之間的摩擦阻力屬於 Coulomb 型式。

依據 Shenton(1996)有關一個剛性物體受到類似地震的底床振動的研究指出放置於水平運動中的水平剛性底床上的自由剛性物體可能引發靜止(rest)、滑動(slide)、晃動(rock)、或滑動-晃動(slide-rock)等四種初始模態。

雖然本研究有關一個受到單一力量作用的剛性物體的各種反應模態啟動準則的推導主要是參考 Shenton(1996)的研究方法。但是，本研究的結果不同於 Shenton 的推導結果，而且顯得更複雜。

本文之各種反應模態啟動準則係導自於剛性物體放置在空氣中的情況下；當物體是浸入在水中時，將會額外承受浮力及上揚力的作用，目前本研究上不考慮這兩個外力 ，它們的影響將會在後續的研究工作中進行討論。

由於推導的工作相當繁瑣，各種反應模態啟動準則將分成力量作用於質心以上及質心以下兩部分來進行討論。

第二章 模　型

如圖 2.1 所示，考慮放置在水平底床上具有單位厚度、寬度為$2B$、高度為$2H$ 的矩形立方剛性物體。物體的質量假設為均勻分佈且總質量為m。在物體與底床面之間存在著 Coulomb 摩擦阻力且其摩擦係數為μ，並假設靜摩擦係數與動摩擦係數為相同。該物體在靜止模態下開始受到一個水平力F 的作用(以物體重量mg 為參考的強度值k 來表示水平力$F = mgk$，其中g 為重力加速度而k 則為一大於 0 的無因次比例值)，F 的作用點到質心C 的垂直距離為h (從質心C 以向上為正量測)，同樣地以與H 的比值表示為$h = k'H$。當水平力F 作用在物體的質心C 的上方時，$0 \le k' \le 1$；反之，當水平力F 作用在物體的質心C 的下方時，$-1 \le k' \le 0$。質心C 在水平及垂直方向的位移分別為x 及y，並取向右及向上為正。當物體處於靜止模態時，原點位於質心之上。令物體的旋轉角為θ，並取逆時針方向為正。物體所承受的反力以f_x 及f_y 表示。垂直反力f_y 的作用點至質心C 的水平距離以ξ 表示，並以自C 點起向右為正。

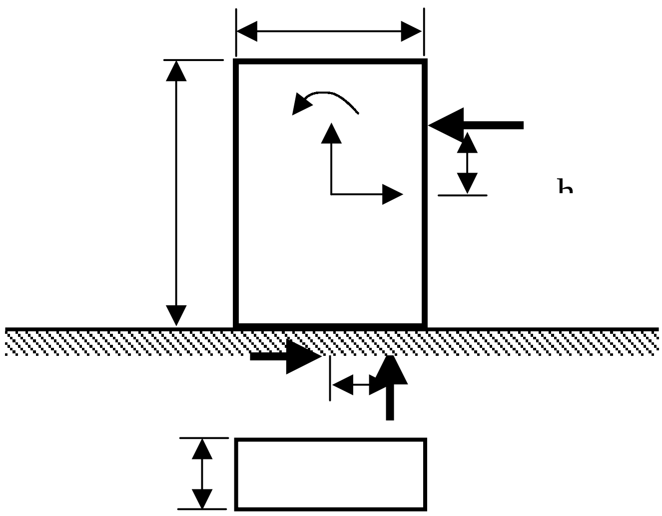

圖 2.1 模型

第三章 自由物體承受質心上方水平力作用下的反應模態啟動準則

3.1 靜止模態

當一個物體受到水平力F作用時(如圖 2.1 所示)仍保持靜止模態時的運動平衡方程式為

$$f_x = F \quad \text{.....(3.1)}$$

$$f_y = mg \quad \text{.....(3.2)}$$

由於物體不旋轉，如果對質心C取力矩的話，則其合力矩應該為 0

$$f_x H + f_y \xi + Fh = 0 \quad \text{.....(3.3)}$$

討論：

由於物體持續與水平底面接觸，所以其垂直反力 $f_y \geq 0$。由式(3.2)可知，此條件已經滿足。

由於物體處於靜止模態，水平反力 將不會大於 Coulomb 摩擦力的極限值，也就是 $f_x \leq \mu f_y$。

由於垂直反力 f_y 必須始終作用在物體底部的OO'之間，因此$|\xi| \leq B$。絕對值符號是因為ξ可能為正、負或 0 值。在後面的推導將會證明物體靜止模態下，$\xi \leq 0$ (也就是 f_y 作用在質心C的左側)。

由於$F = mgk$，由式(3.1)、(3.2)及條件 $f_x \leq \mu f_y$ 可得

$$\mu \geq k \quad \text{.....(3.4)}$$

將式(3.1)及(3.2)代入式(3.3)可得$\xi = -k(H+h) = -k(1+k')H$，由於$k$及$k'$都為正值，可知$\xi \leq 0$。因此，用$|\xi| \leq B$的條件要求可導得

$$k \le \frac{1}{\gamma(1+k')} \qquad (3.5)$$

其中，$\gamma = H/B$為物體的細長比。

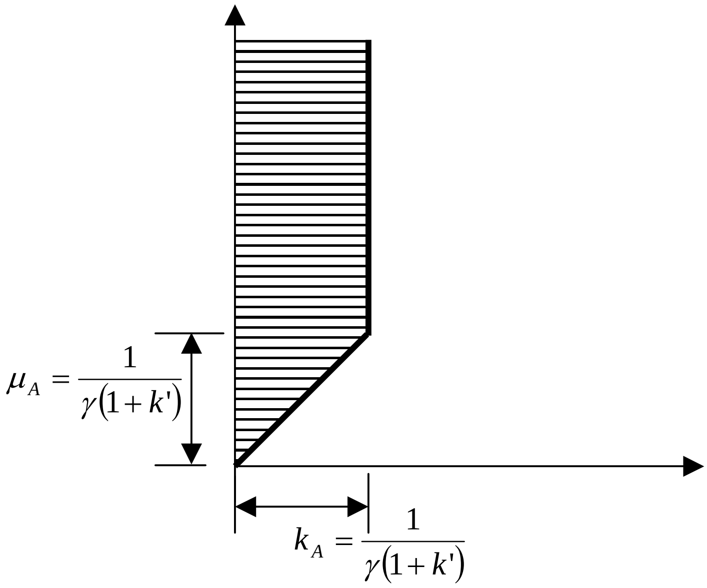

圖 3.1 靜止模態區間

由以上的推導可知，當物體受到水平力作用而仍處於靜止模態下時，式(3.4)及(3.5)必須被滿足。如圖 3.1 中所示，取水平軸為k (外力F的強度比例)及垂直軸為μ(摩擦係數)由線段OA ($\mu = k$)、線段AB ($k = \frac{1}{\gamma(1+k')}$)及垂直軸所圍成的陰影區域代表靜止區。也就是說，一個細長比(slenderness)或形狀比(aspect ratio)為$\gamma = \frac{H}{B}$的物體靜置於一底床摩擦係數為μ的水平底床上，如果受到一個強度為k的水平力作用在已知位置k'，我們可以很快地判斷物體是否仍處於靜止模態。由圖 3.1 可以看到，當細長比愈大時，線段AB越接近μ軸，而陰影區域就越狹窄，物體就越難處於靜止模態。而另一方面，如果能夠降低物體的細長比或是k'值，則屬於靜止模態的區間就越大。控制物體處於靜止模態的條件是防波堤設計者所必須擁有的重要資訊。

3.2 滑動模態

當物體受到水平力作用而開始要發生滑動時，其運動方程式為

$$m\ddot{x} = f_x - F \quad \text{.......(3.6)}$$

$$f_y = mg \quad \text{.......(3.7)}$$

由於物體不旋轉，如果對質心C取力矩的話，則其合力矩應該為 0

$$f_x H + f_y \xi + Fh = 0 \quad \text{.......(3.8)}$$

本報告中，變數符號上端的點表示是對時間的微分。上式中，式(3.7)及(3.8)與式(3.2)及(3.3)相同，但式(3.6)則與式(3.1)不同，因為物體開始要滑動，因此雖然位移$x = 0$，但加速度$\ddot{x} \neq 0$。

在本情況下，下列條件必須被滿足：

在物體瀕臨滑動的模態下，$f_x = \mu f_y$。

$f_y \geq 0$的條件會自動滿足。

$|\xi| \leq B$的條件則需要$\mu \leq \dfrac{1}{\gamma} - kk'$。

將 $f_x = \mu f_y$ 、 $h = k'H$ 及 $F = mgk$ 代入式(3.8)中，我們可以得到$\xi = -(\mu + kk')H$，而由於μ、k及k'始終大於或等於 0，因此ξ始終小於或等於 0，也就是$\xi = -(\mu + kk')H \leq 0$。這是當水平力$F$作用在質心$C$的上方所必然造成的結果。

圖 3.2 的$k - \mu$關係圖中，$\mu = \dfrac{1}{\gamma} - kk'$由線段$CD$來代表，滿足$\mu \leq \dfrac{1}{\gamma} - kk'$條件的點位於$OCD$區域間。由於$OCA$($\mu \geq k$)區域與靜止模態有關，當要啟動滑動模態時，$k$及$\mu$的值必須對應到$OAD$區域 (圖 3.2 中的陰影區域)內的一個點。從式(3.6)，由於$\mu \leq k$，我們可以得到$\ddot{x} = g(\mu - k) \leq 0$。由於目前考慮的的滑動模態，因此從靜止模態開始滑動的物體中所有點的速度及加速度都必須要相同，而且方向向左。值得注意的是當$k' = 0$

時(也就是F作用在物體的質心上)，條件可以簡化成$\mu \le \frac{1}{\gamma}$，是一個與k無關的常數值，因此CD線段是水平的。這種情況下的滑動模態區間遠大於F作用在質心上方的情況。

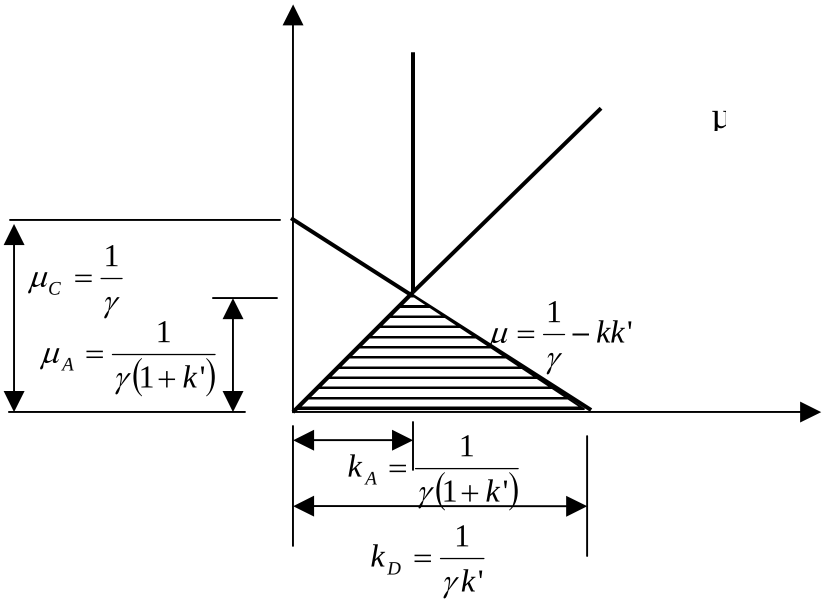

圖 3.2 滑動模態區間

3.3 晃動模態

當物體受到水平力作用而開始要發生晃動時，其運動方程式為

$$m\ddot{x} = f_x - F \quad \text{(3.9)}$$

$$m\ddot{y} = f_y - mg \quad \text{(3.10)}$$

由於物體開始要發生晃動，反力f_y將會作用在O點上(如圖 2.1 所示)。對質心C取力矩，其運動方程式為

$$I\ddot{\theta} = f_x H - f_y B + Fh \quad \text{(3.11)}$$

其中，$I = \frac{1}{3}m(B^2 + H^2)$為物體對質心$C$的質量慣性矩(mass moment of inertia)。當即將要發生晃動時，位移x、y及θ仍然保持為 0，但是它們的加速度就不是如此。點C的加速度$\ddot{x}$及$\ddot{y}$都與物體的角加速度$\ddot{\theta}$

有關，如下式所示

$$\ddot{x} = -H\ddot{\theta} \quad \text{......(3.12)}$$

及

$$\ddot{y} = B\ddot{\theta} \quad \text{......(3.13)}$$

將式(3.9)、(3.10)、(3.12)及(3.13)，以及$F = mgk$與$h = k'H$代入式(3.11)中，可得

$$\ddot{\theta} = \frac{3g}{4B(1+\gamma^2)}[k\gamma(1+k')-1] \quad \text{......(3.14)}$$

將上式反代回式(3.9)、(3.10)、(3.12)及(3.13)中，可得

$$f_x = \frac{mg}{4(1+\gamma^2)}[k(4+\gamma^2-3\gamma^2k')+3\gamma] \quad \text{......(3.15)}$$

及

$$f_y = \frac{mg}{4(1+\gamma^2)}[3\gamma(1+k')k+1+4\gamma^2] \geq 0 \quad \text{......(3.16)}$$

由於$f_y \geq 0$，物體仍和水平底床接觸。由於要滿足$\ddot{\theta} \geq 0$的條件，必須

$$k \geq \frac{1}{\gamma(1+k')} \quad \text{......(3.17)}$$

為了方便起見，令

$$a = 4+\gamma^2-3\gamma^2k' \quad \text{......(3.18)}$$

$$b = 3\gamma \quad \text{......(3.19)}$$

$$c = 3\gamma(1+k') \quad \text{......(3.20)}$$

及

$$d = 1+4\gamma^2 \quad \text{......(3.21)}$$

在四個參數中，由於γ及k'始終大於或等於 0，b、c及d也都始終大於或等於 0，但是a中因為第三項為負值，使得a不一定大於或等於0。因此，式(3.17)至(3.21)代入式(3.15)及(3.16)中可得

$$f_x = \frac{mg}{4(1+\gamma^2)}(ak+b) \qquad (3.22)$$

及

$$f_y = \frac{mg}{4(1+\gamma^2)}(ck+d) \geq 0 \qquad (3.23)$$

我們可以看到，雖然f_y始終會大於或等於 0，但由於式(3.18)中的a並不一定大於 0，f_x可能會有小於 0 的情形。

當物體要開始發生晃動時，水平反力f_x不可以超過摩擦力的上限(否則就發生滑動)，也就是$|f_x| \leq \mu f_y$。從式(3.15)及(3.16)或式(3.22)及(3.23)，本條件可改寫成

$$\mu \geq \frac{\left|(4+\gamma^2-3\gamma^2 k')k+3\gamma\right|}{3\gamma(1+k')k+1+4\gamma^2} = \frac{|ak+b|}{ck+d} = \mu^*(k) \qquad (3.24)$$

由於a及f_x的正負值性質，函數$\mu^*(k)$會有下列三種不同的變化情形：

Case 1: $a \geq 0$ (或者相當於$k' \leq \frac{4+\gamma^2}{3\gamma^2}$)，連帶地$f_x \geq 0$。如此，對所有的$k$值而言，$\mu^*(k) = \frac{ak+b}{ck+d}$。

Case2: $a \leq 0$ (或者相當於$k' \geq \frac{4+\gamma^2}{3\gamma^2}$)但是$f_x \geq 0$。如此，$k \leq \frac{b}{|a|}$且

$$\mu^*(k) = \frac{ak+b}{ck+d} = \frac{-|a|k+b}{ck+d}。$$

Case 3: $a \leq 0$ (或者相當於$k' \geq \frac{4+\gamma^2}{3\gamma^2}$)但是$f_x \leq 0$。如此，$k \geq \frac{b}{|a|}$且

$$\mu^*(k) = \frac{\left|(-|a|k+b)\right|}{ck+d}。$$

由於$0\le k'\le 1$，我們必須確認$\frac{4+\gamma^2}{3\gamma^2}\le 1$，也就是當$\gamma\le\sqrt{2}$時不會發生晃動。

以下將對不同情況下$\mu^*(k)$的性質進行討論：

Case 1:

在此情況下，我們可以證明$k=k_A=\frac{1}{\gamma(1+k')}=\frac{3}{c}=\mu*(k_A)$顯示$\mu^*(k)$會通過$A$點(如圖 3.1 中所示)。而當$k$趨近於無窮大時，由於$a$及 c都大於或等於 0，$\mu*(k)=\frac{4+\gamma^2-3\gamma^2k'}{3\gamma(1+k')}=\frac{a}{c}\ge 0$。$\mu^*(k)$曲線的斜率可以表為$\frac{d\mu*}{dk}=\frac{ad-bc}{(ck+d)^2}$，式中， $ad-bc=4(1+\gamma^2)(1+\gamma^2-3\gamma^2k')$是一個與變數$k$無關的常數，而$ck+d=3\gamma(1+k')k+1+4\gamma^2\ge 0$。$\mu^*(k)$的斜率會隨著$k$值的增加而減小。當$k\to\infty$時趨近於 0，而且$\mu^*(k)$在點$A$處的斜率為$\left.\frac{d\mu*}{dk}\right|_{k_A}=\frac{1+\gamma^2-3\gamma^2k'}{4(1+\gamma^2)}$。當$k'\le\frac{1+\gamma^2}{3\gamma^2}$時，$\left.\frac{d\mu*}{dk}\right|_{k_A}\ge 0$；不過當$k'\ge\frac{1+\gamma^2}{3\gamma^2}$時，$\left.\frac{d\mu*}{dk}\right|_{k_A}\le 0$。

圖 3.3a 及 3.3b 中的兩條$\mu=\mu^*(k)$曲線是分別屬於$0\le k'\le\frac{1+\gamma^2}{3\gamma^2}$及$\frac{1+\gamma^2}{3\gamma^2}\le k'\le\frac{4+\gamma^2}{3\gamma^2}\le 1$兩種情況。

對於給定的k' and γ，假如外力 強度k及摩擦力係數μ在$k-\mu$平面中所對應的點落在圖 3.3a 及 3.3b 中的陰影區域的話，物體將會發生晃動反應。從這些圖中，可以看到假如外力 F遠離物體質心C的話，k'會變大(參考圖 3.3b)，而且晃動模態的陰影區域也會大於外力 較接近物體質心C的情形(參考圖 2.1)，因為此時的k'較小(參考圖 3.3a)。

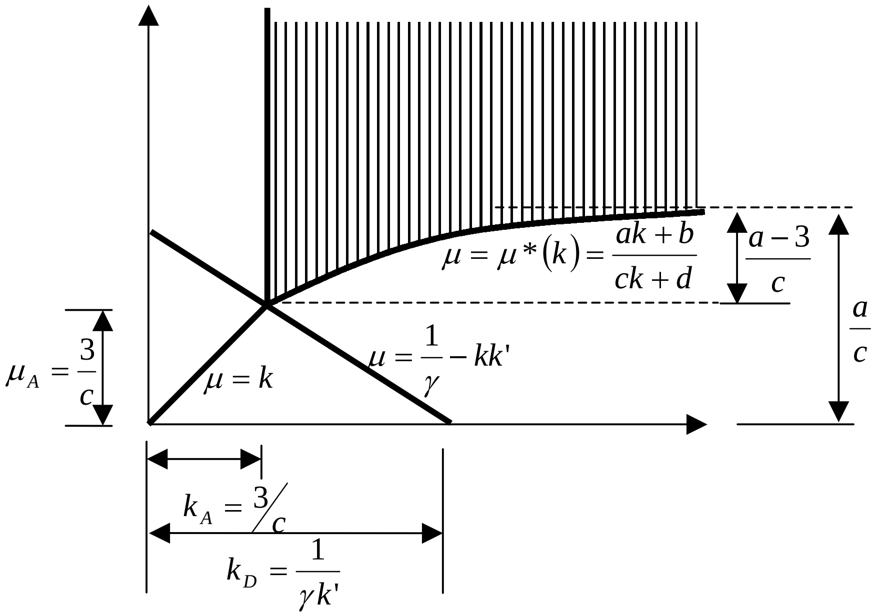

圖 3.3a 晃動模態區間，case 1， $0 \le k' \le \frac{1+\gamma^2}{3\gamma^2}$ **，** $f_x \ge 0$ **，** $a \ge 0$ **時**

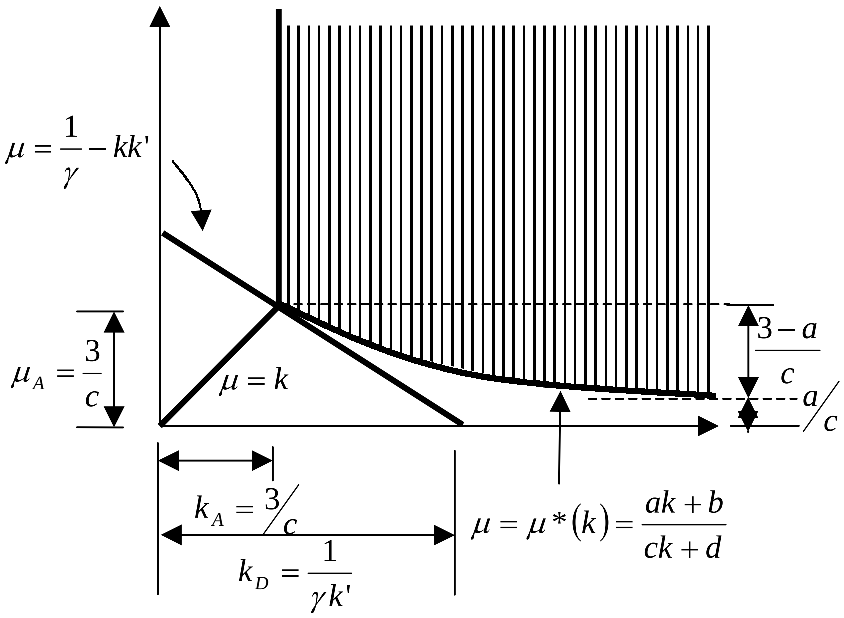

圖 3.3b 晃動模態區間，case 1， $\frac{1+\gamma^2}{3\gamma^2} \le k' \le \frac{4+\gamma^2}{3\gamma^2} \le 1$ **，** $f_x \ge 0$ **，** $a \ge 0$ 時

Case 2:

在此情況下，$\mu^*(k)=\frac{-|a|k+b}{ck+d}$。由於 $\mu^*(k)$ 一定不可為負值，曲線方程式只能存在於 $k\le\frac{b}{|a|}=\frac{3\gamma}{3\gamma^2k'-(4+\gamma^2)}$。如 case 1 中的討論，$\mu^*(k)$ 會通過點 A (如圖 3.1 所示)而且它的斜率方程式是 $\frac{d\mu*}{dk}=\frac{4(1+\gamma^2)(1+\gamma\ \ -3\gamma^2k')}{[3\gamma(1+k')k+1+4\gamma^2]^2}$。在點 A 處的斜率為 $\left.\frac{d\mu*}{dk}\right|_{k_A}=\frac{1+\gamma^2-3\gamma^2k'}{4(1+\gamma^2)}$。在本狀況中是考慮 $k'\ge\frac{4+\gamma^2}{3\gamma^2}$，而且 $\frac{1+\gamma^2}{3\gamma^2}\le\frac{4+\gamma^2}{3\gamma^2}$，因此曲線 $\mu=\mu^*(k)$ 在點 A 處的斜率將始終小於或等於 0。圖 3.4 顯示 $\mu=\mu^*(k)$ 的曲線，如果 k 及 μ 所對應的點落在陰影區域的話，物體將會發生晃動反應。

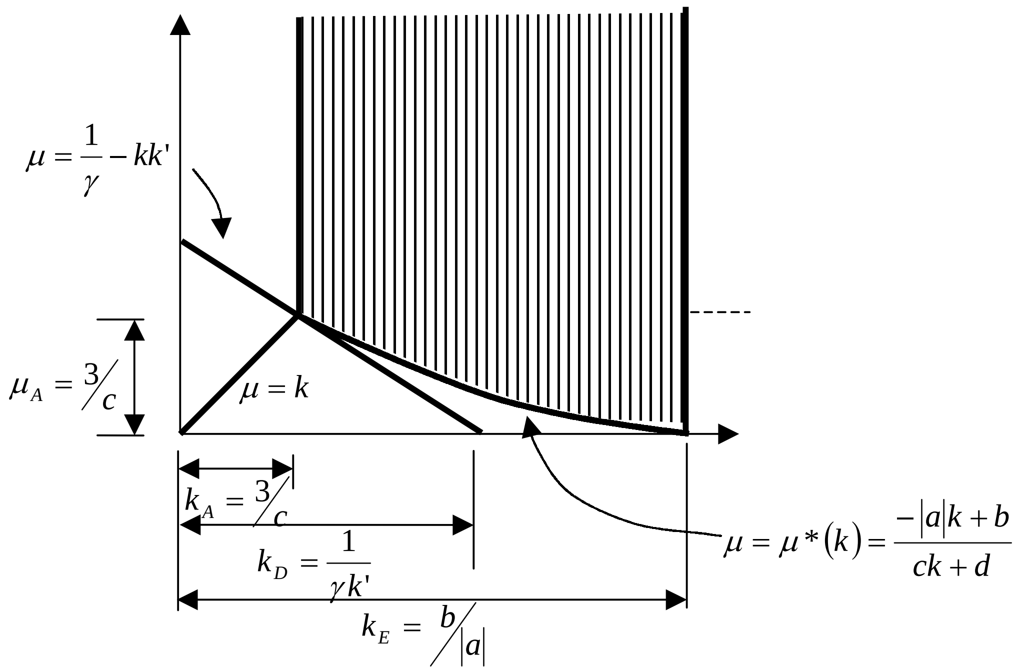

圖 3.4 晃動模態區間，case 2，$\frac{4+\gamma^2}{3\gamma^2}\le k'\le 1$**，** $f_x\ge 0$**，** $a\le 0$**時**

Case 3:

在此情況下，$\mu^*(k)=\dfrac{\left|(-|a|k+b)\right|}{ck+d}=\dfrac{-(ak+b)}{ck+d}=-\dfrac{[(4+\gamma^2-3\gamma^2k')k+3\gamma]}{3\gamma(1+k')k+1+4\gamma^2}$存在於 $f_x\le 0$ 或 $-|a|k+b\le 0$ 的條件下，也就是曲線 $\mu^*(k)$ 只能用在 $k\ge\dfrac{b}{|a|}$ 的情形下。因此，曲線 $\mu^*(k)$ 將不會通過點 A。當 $k=\dfrac{b}{|a|}$ 時，$\mu^*(k)=0$；而當 $k\to\infty$ 時，由於目前考慮的是 $k'\ge\dfrac{4+\gamma^2}{3\gamma^2}$ 的情形，$\mu^*(k)=\dfrac{|a|}{c}=\dfrac{3\gamma^2k'-(4+\gamma^2)}{3\gamma(1+k')}\ge 0$。當 $\gamma\le\sqrt{\dfrac{7}{2}}$ 時，$\mu^*(k)\le\mu_A$；反之，當 $\gamma\ge\sqrt{\dfrac{7}{2}}$ 時，$\mu^*(k)\ge\mu_A$。曲線 $\mu=\mu^*(k)$ 的斜率方程式是 $\dfrac{d\mu^*}{dk}=\dfrac{|a|d+bc}{(ck+d)^2}\ge 0$，斜率隨著 k 的增加而減小並在 $k\to\infty$ 時趨近於 0。圖 3.5 顯示 $\mu=\mu^*(k)$ 的曲線。

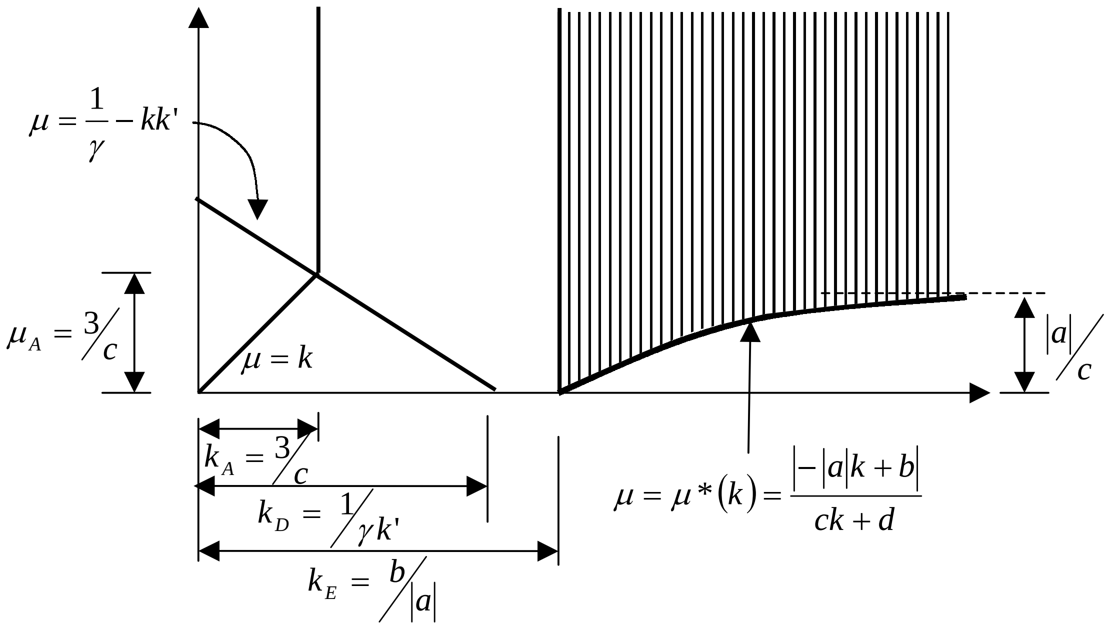

圖 3.5 晃動模態區間，case 3， $\dfrac{4+\gamma^2}{3\gamma^2}\le k'\le 1$**，** $f_x\le 0$**，** $a\le 0$**時**

假如我們將圖 3.4 及 3.5 與圖 3.3 比較的話, 將會發現外力 F 作用在較遠離物體質心C的位置(也就是k'比較大)，如此發生晃動的機會就相對變高。

3.4 滑動-晃動模態

當物體對點O處於即將發生滑動及晃動的瞬間，反力 f_y 會作用在點O，而運動平衡方程式為

$$m\ddot{x} = f_x - F \quad \text{.......(3.25)}$$

$$m\ddot{y} = f_y - mg \quad \text{.......(3.26)}$$

對質心C取力矩，其運動方程式為

$$I\ddot{\theta} = f_x H - f_y B + Fh \quad \text{.......(3.27)}$$

上述的方程式與晃動模態的初始運動方程式，即式(3.9)、(3.10)及(3.11)相同，不同的地方是：在晃動模態中，$\ddot{x} = -H\ddot{\theta}$ (如式(3.12)所示)且$|f_x| \le \mu f_y$；而如果快速地檢視滑動及晃動模態, 目前的$\ddot{x}$並不是只與$\ddot{\theta}$相關的函數，而且

$$|f_x| = \mu f_y \quad \text{.......(3.28)}$$

但是

$$\ddot{y} = B\ddot{\theta} \quad \text{.......(3.29)}$$

就如同式(3.13)所述。

利用式(3.26)、(3.27)、(3.28)與(3.29)，以及$F = mgk$、$h = k'H$與$I = \frac{1}{3}m(H^2 + B^2)$，我們可以得到

$$\ddot{\theta} = \frac{3g}{B}\frac{\mu\gamma - 1 + kk'\gamma}{4 + \gamma^2 - 3\mu\gamma} \quad \text{.......(3.30)}$$

因為，$\ddot{\theta} \ge 0$，必須有下列的條件：

$$\mu\gamma - 1 + kk'\gamma \ge 0 \text{ 及 } 4 + \gamma^2 - 3\mu\gamma \ge 0 \quad \text{..(3.31)}$$

或

$$\mu\gamma - 1 + kk'\gamma \le 0 \text{ 及 } 4 + \gamma^2 - 3\mu\gamma \le 0 \quad \text{..(3.32)}$$

方程式(3.31)及(3.32)相當於

$$\mu \ge \frac{1}{\gamma} - kk' \text{ 及 } \mu \le \frac{4+\gamma^2}{3\gamma} \quad \text{..(3.33)}$$

或

$$\mu \le \frac{1}{\gamma} - kk' \text{ 及 } \mu \ge \frac{4+\gamma^2}{3\gamma} \quad \text{..(3.34)}$$

式(3.33)可改寫為

$$\frac{1}{\gamma} - kk' \le \mu \le \frac{4+\gamma^2}{3\gamma} \quad \text{..(3.35)}$$

且式(3.34)可改寫為

$$\frac{4+\gamma^2}{3\gamma} \le \mu \le \frac{1}{\gamma} - kk' \quad \text{..(3.36)}$$

我們可以證明當 $\gamma = 2$ 時，$\frac{4+\gamma^2}{3\gamma}$ 的最小值為 1.33。由於 $\frac{1}{\gamma} \le \frac{4+\gamma^2}{3\gamma}$，式(3.36)將無法被滿足。因此，只有當式(3.35)成立時，$\ddot{\theta} \ge 0$ 才會被滿足；而根據式(3.26) 及(3.29)，只有 $\ddot{\theta} \ge 0$ 時，$f_y \ge 0$。

在此，我們要檢查在 F 作用下物體滑動的方向。該方向是要由當物體發生轉動時，點 O 的水平速度 $\dot{x}_O$ 來決定。由於物體原先是處於靜止模態，因此可以檢驗當物體開始要發生滑動-晃動時，點 O 的加速度 $\ddot{x}_O$。

由於 $\ddot{x}_O = \ddot{x} + H\ddot{\theta}$，又從式(3.25)可以得到 $\ddot{x}$，透過代數的過程，可以證明

$$\ddot{x}_O = \frac{g}{4+\gamma^2-3\mu\gamma}\{\mu[3\gamma(1+k')k+1+4\gamma^2]-[(4+\gamma^2-3\gamma^2k')k+3\gamma]\}$$
$$= \frac{g}{4+\gamma^2-3\mu\gamma}[\mu(ck+d)-(ak+b)]$$
(3.37)

由於 $\ddot{x}_O \le 0$ (物體向左邊滑動且 $f_x \ge 0$)，必須有下列的條件

$4+\gamma^2-3\mu\gamma \ge 0$ 及 $\mu(ck+d)-(ak+b) \le 0$ ....(3.38)

或

$4+\gamma^2-3\mu\gamma \le 0$ 及 $\mu(ck+d)-(ak+b) \ge 0$ ....(3.39)

方程式(3.38)可改寫為

$\mu \le \dfrac{4+\gamma^2}{3\gamma}$ 及 $\mu \le \dfrac{ak+b}{ck+d} = \mu^*(k)$ ....(3.40)

而式(3.39)可改寫為

$\mu \ge \dfrac{4+\gamma^2}{3\gamma}$ 及 $\mu \ge \dfrac{ak+b}{ck+d} = \mu^*(k)$ ....(3.41)

由於 $\ddot{\theta} \ge 0$，從式(3.35)，我們必須滿足 $\mu \le \dfrac{4+\gamma^2}{3\gamma}$，因此式(3.41)是不成立的。式(3.40)及(3.33)及 $\dot{x}_O \le 0$ ($f_x \ge 0$)提供了滑動-晃動模態的啟動條件如下式所示

$\dfrac{1}{\gamma} - kk' \le \mu \le \mu^*(k) = \dfrac{ak+b}{ck+d} = \dfrac{(4+\gamma^2-3\gamma^2k')k+3\gamma}{3\gamma(1+k')k+1+4\gamma^2}$ ....(3.42)

就如前節所述，我們必須區分出當 $a \ge 0$，$k' \le \dfrac{4+\gamma^2}{3\gamma^2}$ (case 1)的模態及當 $a \le 0$，$k' \ge \dfrac{4+\gamma}{3\gamma^2}$ (case 2)的模態。第一種模態有兩種可能：$k' \le \dfrac{1+\gamma^2}{3\gamma^2}$ ($\mu^*(k)$ 在點 A 處的斜率 ≥ 0)及 $k' \ge \dfrac{1+\gamma^2}{3\gamma^2}$ ($\mu^*(k)$ 在點 A 處的斜率 ≤ 0)。

圖 3.6a、3.6b 及 3.7 顯示當$\ddot{x}_O \le 0$及$f_x \ge 0$時，物體啟動滑動-晃動模態的相對區域。

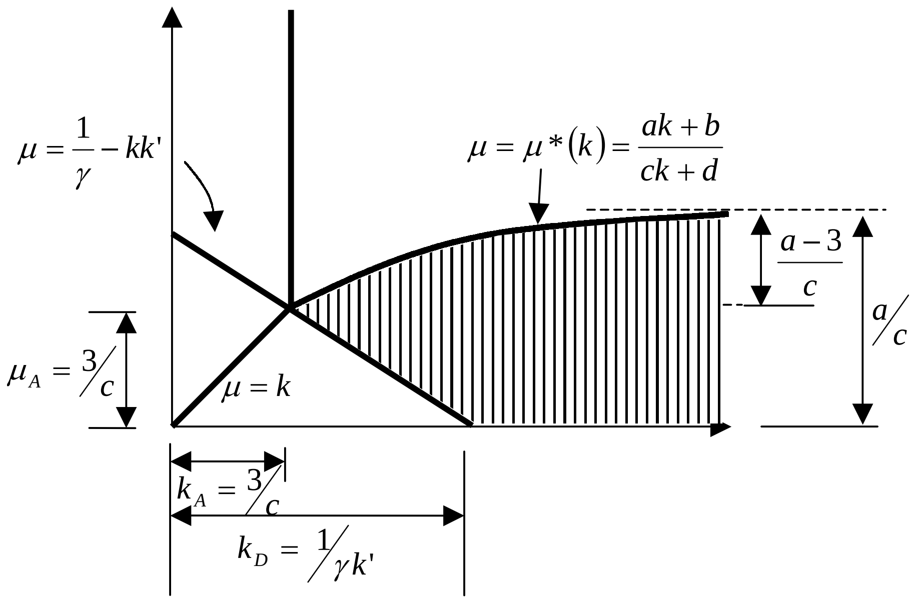

圖 3.6a 滑動-晃動模態區間，case 1， $0 \le k' \le \frac{1+\gamma^2}{3\gamma^2}$**，** $f_x \ge 0$**，** $a \ge 0$**時**

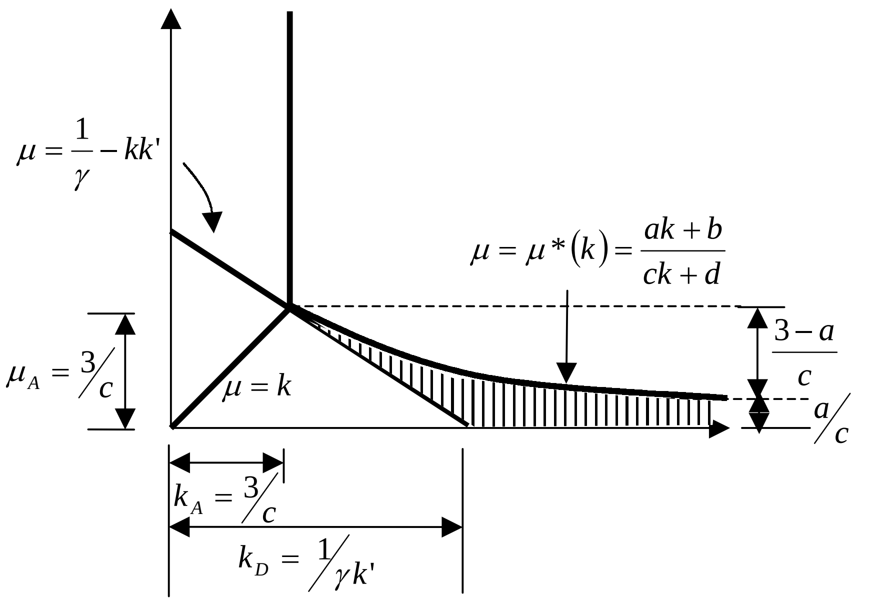

圖 3.6b 滑動-晃動模態區間，case 1， $\frac{1+\gamma^2}{3\gamma^2} \le k' \le \frac{4+\gamma^2}{3\gamma^2} \le 1$**，** $f_x \ge 0$**，** $a \ge 0$**時**

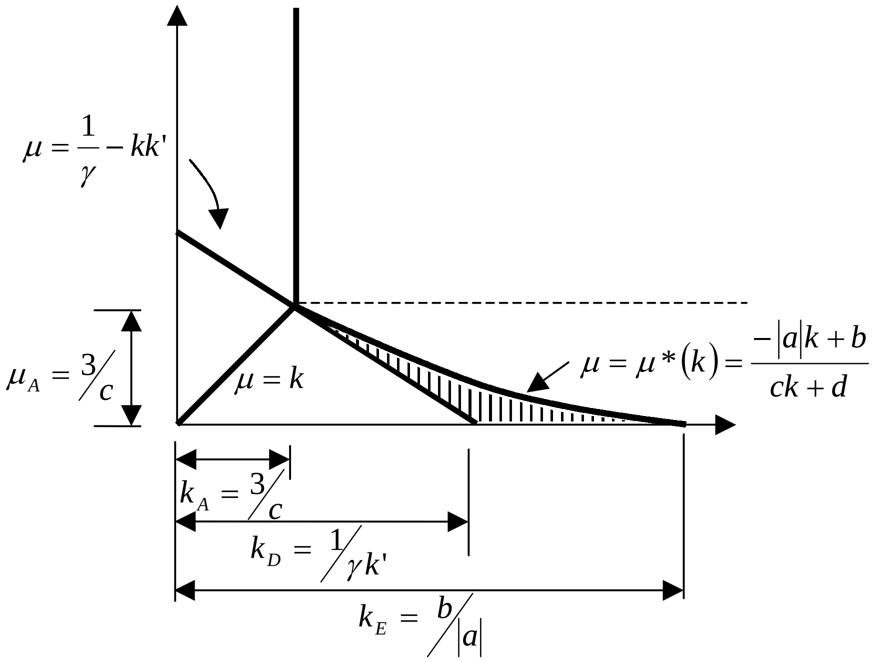

圖 3.7 滑動-晃動模態區間，case 2， $\frac{4+\gamma^2}{3\gamma^2} \le k' \le 1$， $f_x \ge 0$， $a \le 0$**時**

在 $\ddot{x} \ge 0$(物體向右邊移動)以及 $f_x \le 0$ (f_x 向左方作用)的情形中，我們重複推導式(3.30)中 $\ddot{\theta}$ 的過程，可以得到本情形中的方程式

$$\ddot{\theta} = \frac{3g}{B(4+\gamma^2+3\mu\gamma)}(kk'\gamma - \mu\gamma - 1) \quad \text{...(3.43)}$$

由於 $\ddot{\theta} \ge 0$，必須滿足

$$\mu \le -\frac{1}{\gamma} + kk' \quad \text{...(3.44)}$$

藉由式(3.43)及 $\ddot{x}_O = \ddot{x} + H\ddot{\theta}$ 的關係，可以得到

$$\begin{aligned} \ddot{x}_O &= -\frac{g}{4+\gamma^2+3\mu\gamma}\{\mu[3\gamma(1+k')k+1+4\gamma^2]+[4+\gamma^2-3\gamma^2k')k+3\gamma]\} \\ &= -\frac{g}{4+\gamma^2+3\mu\gamma}[\mu(ck+d)+(ak+b)] \end{aligned} \quad (3.45)$$

由於 $\ddot{x}_O \geq 0$，必須 $\mu \leq \frac{\left|-|a|k+b\right|}{ck+d} = \mu^*(k)$ 而且 $k \geq \frac{b}{|a|}$。

圖 3.8 顯示 $\mu = \mu^*(k)$ 的曲線。其中可以清楚地看到曲線 $\mu = \mu^*(k)$ 上方的點線，以及表示在 $\ddot{x}_O \geq 0$ 情形時會啟動滑動-晃動模態的陰影區間。

綜合而言，圖 3.9a、3.9b、3.10 及 3.11 中所顯示的靜止、晃動及滑動-晃動模態依序分別代表 $0 \leq k' \leq \frac{1+\gamma^2}{3\gamma^2}$、$f_x \geq 0$、$a \geq 0$；$\frac{1+\gamma^2}{3\gamma^2} \leq k' \leq \frac{4+\gamma^2}{3\gamma^2} \leq 1$、$f_x \geq 0$、$a \geq 0$；$\frac{4+\gamma^2}{3\gamma^2} \leq k' \leq 1$、$f_x \geq 0$、$a \leq 0$ 及 $\frac{4+\gamma^2}{3\gamma^2} \leq k' \leq 1$、$f_x \leq 0$、$a \leq 0$ 等情形。為了簡潔，圖中的各個運動模態分別以符號 RE (靜止模態)、SL (滑動模態)、RO (對點 O 晃動模態)、SRO_- (對點 O 滑動-晃動模態，但 $\dot{x}_o \leq 0$)及 SRO_+ (對點 O 滑動-晃動模態，但 $\dot{x}_o \geq 0$)。

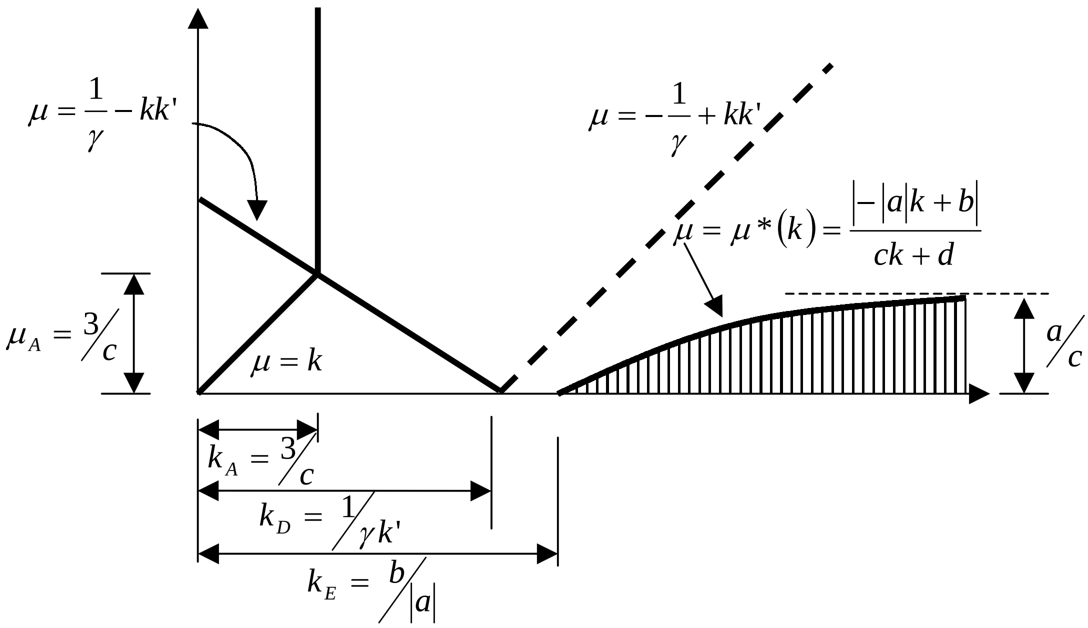

圖 3.8 滑動-晃動模態區間，case 3， $\frac{4+\gamma^2}{3\gamma^2} \leq k' \leq 1$，$f_x \leq 0$，$a \leq 0$ **時**

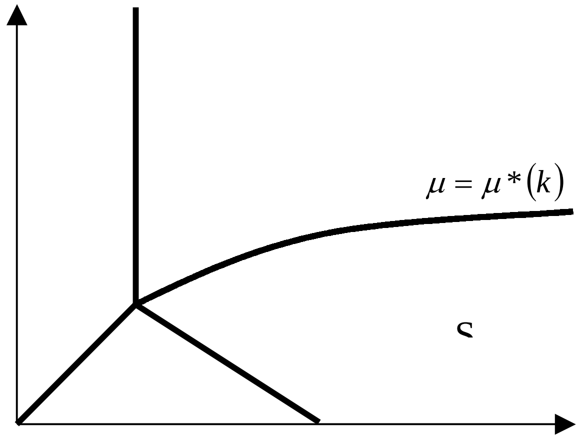

圖 3.9a 所有的模態，case 1， $0 \le k' \le \frac{1+\gamma^2}{3\gamma^2}$ **，** $f_x \ge 0$ **，** $a \ge 0$ **時**

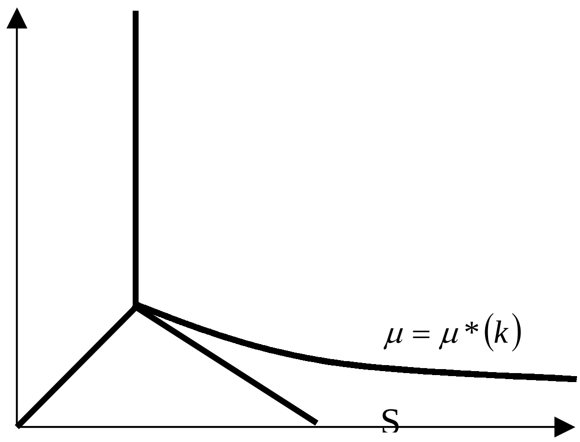

圖 3.9b 所有的模態，case 1， $\frac{1+\gamma^2}{3\gamma^2} \le k' \le \frac{4+\gamma^2}{3\gamma^2} \le 1$ **，** $f_x \ge 0$ **，** $a \ge 0$ **時**

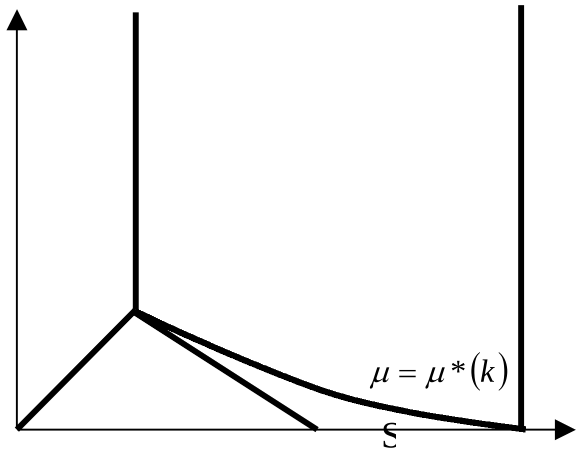

圖 3.10 所有的模態，case 2， $\frac{4+\gamma^2}{3\gamma^2} \le k' \le 1$ **，** $f_x \ge 0$ **，** $a \le 0$ **時**

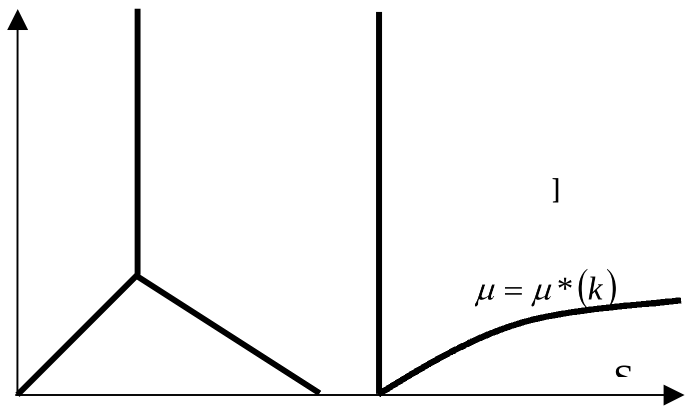

圖 3.11 所有的模態，case 3， $\frac{4+\gamma^2}{3\gamma^2} \le k' \le 1$ **，** $f_x \le 0$ **，** $a \le 0$ **時**

第四章 自由物體承受質心下方水平力作用下的反應模態啟動準則

4.1 靜止模態

當一個物體受到水平力F作用時(如圖 4.1 所示)仍保持靜止模態時的平衡方程式為

$$f_x = F \qquad \text{.........(4.1)}$$

$$f_y = mg \qquad \text{.........(4.2)}$$

由於物體不旋轉，如果對質心C取力矩的話，則其合力矩應該為 0

$$f_x H + f_y \xi - F|k'|H = 0 \qquad \text{.........(4.3)}$$

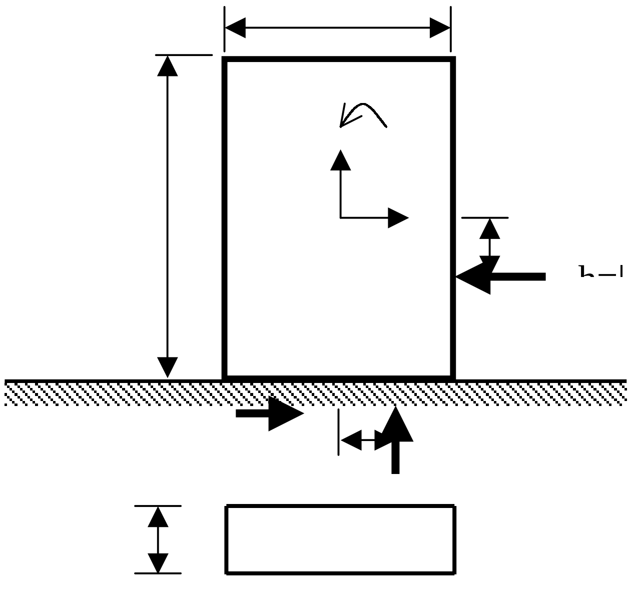

圖 4.1 模型

在本情況下，下列條件必須被滿足：

由於物體持續與水平底面接觸，我們必須有

$$f_y \geq 0 \quad \text{...}(4.4)$$

由於物體處於靜止模態，水平反力f_x將不會大於 Coulomb 摩擦力的極限值μf_y，也就是

$$f_x \leq \mu f_y \quad \text{...}(4.5)$$

由於垂直反力f_y必須始終作用在物體底部的OO'之間，因此

$$|\xi| \leq B \quad \text{...}(4.6)$$

依式(4.2)，$f_y \geq 0$的條件永遠滿足。而從式(4.1)及(4.2)可知條件(4.5)的成立需要

$$\mu \geq k \quad \text{...}(4.7)$$

將式(4.1)及(4.2)代入式(4.3)中，可以得到

$$\xi = -kH(1-|k'|) \leq 0 \quad \text{...}(4.8)$$

因為

$$0 \leq |k'| \leq 1 \quad \text{...}(4.9)$$

方程式(4.8)表示當物體受到F力的作用而仍保持靜止模態，反

力 f_y 向質心C的左方作用。

條件(4.6)需要

$$k \le \frac{1}{\gamma(1-|k'|)} \qquad (4.10)$$

條件(4.7)及(4.10)確保在F力的作用而物體仍保持靜止模態，圖 4.2 中$k-\mu$平面上的陰影區域顯示其對應的點。我們可以看到靜止模態區域會隨著$|k'|$值的增加而增加。以就是說，當外力F愈接近物體的底部時，物體愈能保持在靜止模態。此種現象和外力F作用於物體質心C上方的情形不同。在後面的推導也可以看到，外力F作用於物體質心C的上方或下方會有相當不同的行為。

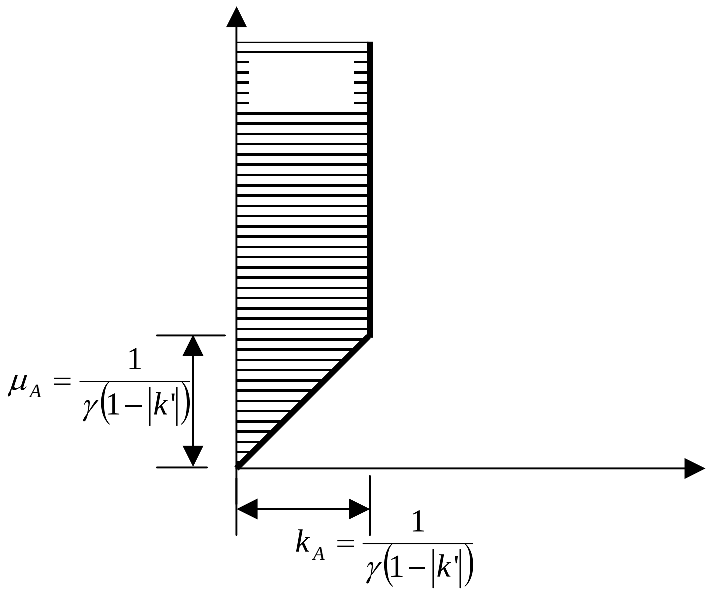

圖 4.2 靜止模態區間

4.2 滑動模態

當物體受到水平力作用而開始要發生滑動時，其運動方程式為

$$m\ddot{x} = f_x - F \quad \text{.......(4.11)}$$

$$f_y = mg \quad \text{.......(4.12)}$$

由於物體不旋轉，如果對質心C取力矩的話，則其合力矩應該為 0

$$f_x H + f_y \xi - F|k'|H = 0 \quad \text{.......(4.13)}$$

在本情況下，下列條件必須被滿足：

由於物體瀕臨滑動的模態

$$f_x = \mu f_y \quad \text{.......(4.14)}$$

由於垂直反力f_y不可以為負值。也就是

$$f_y \geq 0 \quad \text{.......(4.15)}$$

由於f_y等於物體的重量，式(4.15)的條件永遠成立。

由於f_y必須始終作用在物體底部的OO'之間，因此

$$|\xi| \leq B \quad \text{.......(4.16)}$$

由方程式(4.13)可得

$$\xi = (k|k'| - \mu)H \quad \text{.......(4.17)}$$

式中，ξ的正負號由$k|k'| - \mu$的值來決定。如果外力F對點C的力

矩超過 f_x 及 f_y 所產生的力矩，則 $k|k'| \geq \mu$ 。在此情形下，$\xi \geq 0$，即反力 f_y 作用在點 C 的右方。反過來說，假如 $k|k'| \leq \mu$，則 $\xi \leq 0$，即 f_y 作用在點 C 的左方。要滿足式(4.16)的條件要求會有下列兩種情形：

當 $k|k'| \geq \mu$ 時，

$$\mu \geq -\frac{1}{\gamma} + k|k'| \quad \text{.......(4.18)}$$

當 $k|k'| \leq \mu$ 時，

$$\mu \leq \frac{1}{\gamma} + k|k'| \quad \text{.......(4.19)}$$

圖 4.3 中繪製了平行線 DC

$$\mu = -\frac{1}{\gamma} + k|k'| \quad \text{.......(4.20)}$$

直線 AB

$$\mu = \frac{1}{\gamma} + k|k'| \quad \text{.......(4.21)}$$

與點線

$$\mu = k|k'| \quad \text{.......(4.22)}$$

圖中的陰影區域與物體滑動模態反應的啟動有關，它由水平軸、直線 OA、直線 DC (式(4.20))及直線 AB (式(4.21))所圍成。在直線 $\mu = k|k'|$ (式(4.22))以上的陰影區域屬於 $\xi \leq 0$ (f_y 作用在點 C 的左方)

的情形；而直線$\mu = k|k'|$(式(4.22))以下的陰影區域屬於$\xi \ge 0$ (f_y作用在點C的右方)的情形。直線AB及直線DC，以及直線$\mu = k|k'|$都延伸到無窮遠處。

當滑動模態被啟動時，由於$\mu \le k$，由式(4.11)可得

$$m\ddot{x} = mg(\mu - k) \le 0 \quad \text{...(4.23)}$$

上式表示當物體承受到向左方作用的力F，其加速度$\ddot{x} \le 0$。由於目前所討論的是滑動模態，因此物體上的所有的點都必須有相同的加速度大小與向左的方向；同時由於物體初期是處於靜止模態，物體上的所有的點的速度也都必須有相同而且朝向左方。

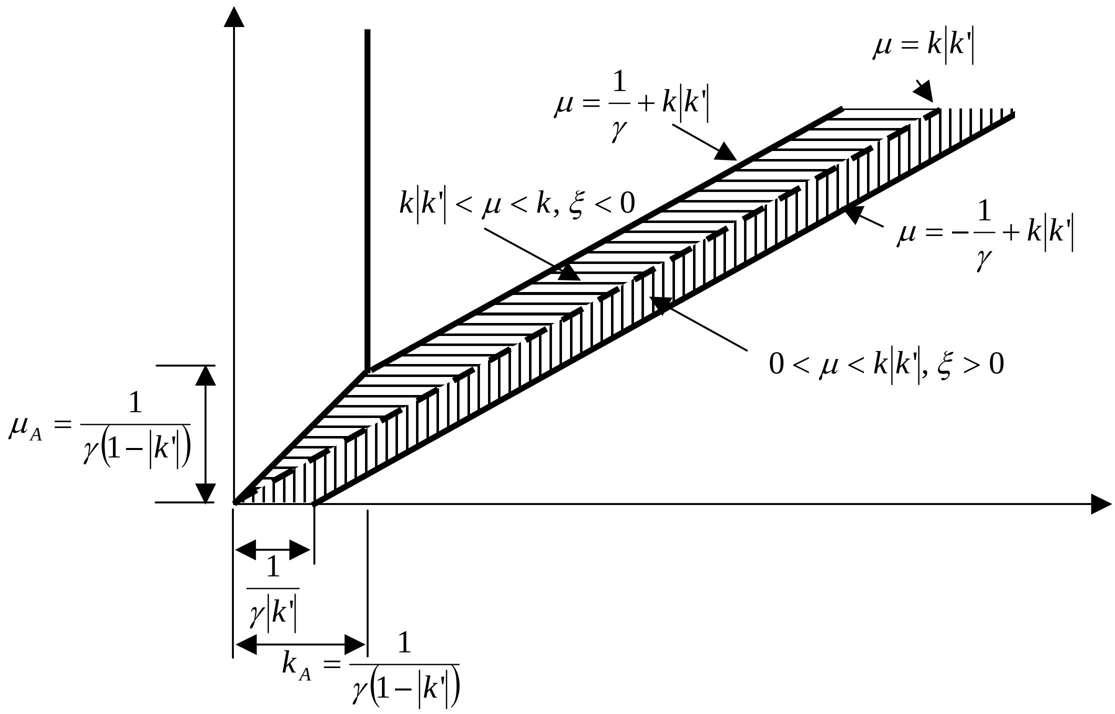

圖 4.3 滑動模態區間

從圖 4.3，我們注意到，直線DC與k軸的交點D可能在點A的左邊或右邊。我們可以證明，當$|k'| \le \frac{1}{2}$時，點D會在點A的右邊；否則，它會在點A的左邊。同樣地，$|k'|$值越大，AB、DC及$\mu = k|k'|$等直線就越陡。再次地證明，外力F作用於物體質心C的上方或下方的滑動模態啟動準則有明顯的不同。

4.3 依點 O 晃動的模態

當物體受到水平力作用而開始要發生晃動(沒有滑動)時，其運動方程式為

$$m\ddot{x} = f_x - F \qquad (4.24)$$

$$m\ddot{y} = f_y - mg \qquad (4.25)$$

由於物體開始要發生晃動，反力f_y將會作用在O點上(如圖 4.1 所示)。對質心C取力矩，其運動方程式為

$$I\ddot{\theta} = f_x H - f_y B - F|k'|H \qquad (4.26)$$

其中，$I = \frac{1}{3}m(H^2 + B^2)$為物體對質心$C$的質量慣性矩。

從式(4.24)可得

$$f_x = m\ddot{x} + F \qquad (4.27)$$

以及從式(4.25)可得

$$f_y = m\ddot{y} + mg \qquad (4.28)$$

當物體即將繞著點O旋轉時，$\ddot{x}$及$\ddot{y}$與$\ddot{\theta}$的關係為

$$\ddot{x} = -H\ddot{\theta} \quad \text{.......(4.29)}$$

及

$$\ddot{y} = B\ddot{\theta} \quad \text{.......(4.30)}$$

如此，從式(4.27)及(4.28)可得

$$f_x = m(gk - H\ddot{\theta}) \quad \text{.......(4.31)}$$

及

$$f_y = m(g + B\ddot{\theta}) \quad \text{.......(4.32)}$$

繞著點O開始旋轉的條件為

$$\ddot{\theta} \geq 0 \quad \text{.......(4.33)}$$

$$f_y \geq 0 \quad \text{.......(4.34)}$$

及

$$|f_x| \leq \mu |f_y| \quad \text{.......(4.35)}$$

從式(4.26)、(4.31)及(4.32)可得

$$\ddot{\theta} = \frac{3g}{4B(1+\gamma^2)}[k\gamma(1-|k'|)-1] \quad \text{.......(4.36)}$$

條件(4.33)的成立需要

$$k \geq \frac{1}{\gamma(1-|k'|)} \quad \text{(4.37)}$$

也就是說，物體繞著點O晃動的啟動條件是只有當k及μ值在$k-\mu$平面上所對應的點落在直線AE的右側才會發生(如圖 4.2 及圖 4.3)。

在決定了$\ddot{\theta}$以後，式(4.31)及(4.32)可改寫為

$$f_x = \frac{mg}{4(1+\gamma^2)}\{k[4+\gamma^2+3\gamma^2|k'|]+3\gamma\} \geq 0 \quad \text{(4.38)}$$

及

$$f_y = \frac{mg}{4(1+\gamma^2)}[3\gamma(1-|k'|)k+1+4\gamma^2] \geq 0 \quad \text{(4.39)}$$

上二式顯示f_x向右方作用，而且$f_y \geq 0$(式(4.34))的條件可以被滿足。

為了讓$|f_x| \leq \mu|f_y|$，我們必須有下列的關係

$$\mu \geq \frac{[4+\gamma^2+3\gamma^2|k'|]k+3\gamma}{3\gamma(1-|k'|)k+1+4\gamma^2} = \mu^*(k) \quad \text{(4.40)}$$

也就是說，對於物體繞著點O晃動而不滑動的情況，摩擦係數必須非常得大。為了簡單起見，我們令$\mu^*(k)$表示成

$$\mu^*(k) = \frac{ak+b}{ck+d} \quad \text{(4.41)}$$

其中，

$$a = 4 + \gamma^2 + 3\gamma^2 |k'| \quad \text{...(4.42)}$$

$$b = 3\gamma \quad \text{...(4.43)}$$

$$c = 3\gamma(1 - |k'|) \quad \text{...(4.44)}$$

及

$$d = 1 + 4\gamma^2 \quad \text{...(4.45)}$$

曲線$\mu^*(k)$具有下列的性質：

從式(4.41)，當$k = \frac{1}{\gamma(1-|k'|)} = k_A = \frac{3}{c}$時，$\mu = \mu^*(k) = \mu_A = \frac{3}{c}$。顯示曲線$\mu^*(k)$會通過點$A$。當$k \to \infty$時，

$$\mu^*(k) = \frac{a}{c} = \frac{4 + \gamma^2 + 3\gamma^2 |k'|}{3\gamma(1 - |k'|)} \geq 0 \quad \text{...(4.46)}$$

曲線$\mu^*(k)$的斜率為

$$\frac{d\mu^*}{dk} = \frac{ad - bc}{(ck + d)^2} = \frac{4(1+\gamma^2)(1+\gamma^2 + 3\gamma^2 |k'|)}{[3k\gamma(1 - |k'|) + 1 + 4\gamma^2]^2} \geq 0 \quad \text{.............................(4.47)}$$

是一個單純的隨k增加而減小的函數，而且其分子部分與k無關。當$k \to \infty$時，$\frac{d\mu^*}{dk} = 0$。

假如在點A處比較$\mu^*(k)$的斜率及直線AB ($\mu=\frac{1}{\gamma}+k|k'|$)的斜率$|k'|$，我們會發現其差值為$\Delta=\frac{(1+\gamma^2)-(4+\gamma^2)|k'|}{4(1+\gamma^2)}$。由此，產生兩種可能：$\Delta\geq 0$($|k'|\leq\frac{1+\gamma^2}{4+\gamma^2}$)及$\Delta\leq 0$($|k'|\geq\frac{1+\gamma^2}{4+\gamma^2}$)。圖 4.4a 及 4.4b 中分別繪製了兩種情形的曲線$\mu^*(k)$、直線AB ($\mu=\frac{1}{\gamma}+k|k'|$)以及直線$DC$ ($\mu=-\frac{1}{\gamma}+k|k'|$) (在此圖及隨後的圖形中，直線$CD$在$k'\geq\frac{1}{2}$的時候，點$D$是畫在點$A$的左側，其說明可參考 4.2 滑動模態的末段文字)。對於前面的情形($\Delta\geq 0$)，曲線$\mu^*(k)$及直線AB ($\mu=\frac{1}{\gamma}+k|k'|$)在點$A$相交，而且點$B'$的座標可以表示成如下所示

$$k=\frac{1+\gamma^2}{3\gamma|k'|}=k_{B'} \qquad \text{.........(4.48)}$$

及

$$\mu=\frac{4+\gamma^2}{3\gamma}=\mu_{B'} \qquad \text{.........(4.49)}$$

對於後面的情形($\Delta\leq 0$)，可以證明$k_{B'}\leq k_A$。在圖 4.4a 及 4.4b 中的陰影區域與物體的晃動模態有關(繞著點O)。我們可以發現晃動模態的區間與滑動模態的區間有重疊的現象。由於這些運動模態必須相互區隔，這種區間重疊的情形必須要解決。此一問題將會在 4.7 節的討論中說明。在外力F作用在質心C的上方的情形中，根據式(3.18a)所描述的事實：a值可以大於、等於或小於 0，然而在本情況中，式(4.42)中的a值則永遠大於或等於 0，所以這兩種情形起動物體繞著點O晃動的準則並不相同。

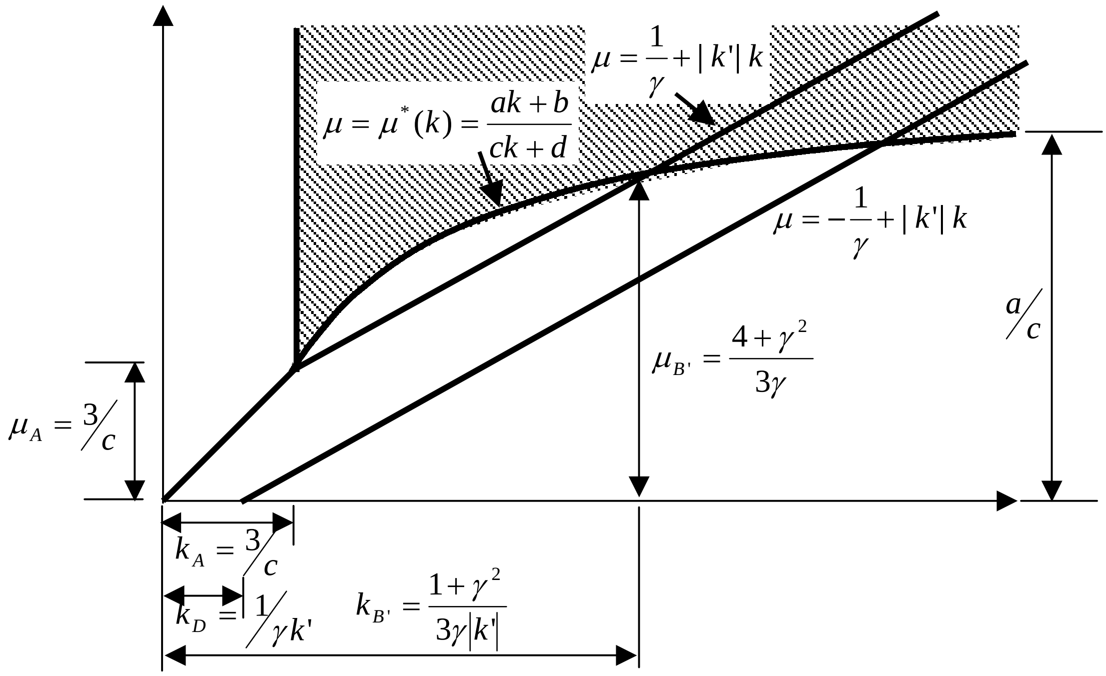

圖 4.4a 晃動模態區間， $|k'| \le \frac{1+\gamma^2}{4+\gamma^2}$ **時**

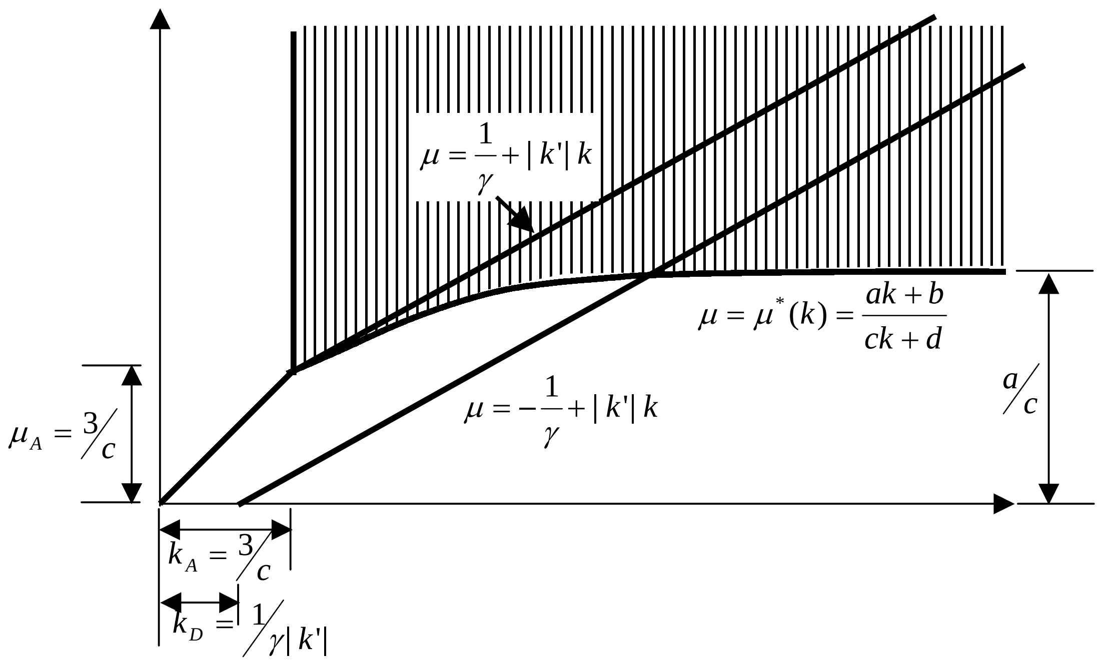

圖 4.4b 晃動模態區間， $|k'| \ge \frac{1+\gamma^2}{4+\gamma^2}$ **時**

4.4 依點 O’晃動的模態

雖然當向左方的外力作用下，物體是明顯地不會繞著O'點晃動，(如圖 4.1 所示)，不過我們仍然進行討論與證明。

當物體受到水平力作用而開始要發生晃動(沒有滑動)時，其運動平衡方程式為

$$m\ddot{x} = f_x - F \quad \text{.......}(4.50)$$

$$m\ddot{y} = f_y - mg \quad \text{.......}(4.51)$$

由於物體開始要發生晃動，反力f_y將會作用在O'點上。對質心C取力矩，其合力矩為

$$I\ddot{\theta} = f_x H + f_y B - F|k'|H \quad \text{.......}(4.52)$$

當物體即將要繞著O'開始晃動時，由於θ是以逆時針為正

$$\ddot{x} = -H\ddot{\theta} \quad \text{.......}(4.53)$$

及

$$\ddot{y} = -B\ddot{\theta} \quad \text{.......}(4.54)$$

因此，從式(4.50)及(4.51)，可得

$$f_x = m(gk - H\ddot{\theta}) \quad \text{.......}(4.55)$$

$$f_y = m(g - B\ddot{\theta}) \quad \text{.......}(4.56)$$

考慮 $I=\dfrac{1}{3}m(H^2+B^2)$ 及 $F=mgk$，以及式(4.55)及(4.56)，從式(4.52)可以得到

$$\ddot{\theta}=\frac{3g}{4B(1+\gamma^2)}[1+k\gamma(1-|k'|)]>0 \quad \text{.......}(4.57)$$

此一關係說明物體依 O’點晃動的模式是不可能產生的。

4.5 依點 O 滑動-晃動模態

當物體對點 O 處於即將發生向左滑動及晃動的瞬間，反力 f_y 會作用在點 O，而運動平衡方程式為

$$m\ddot{x}=f_x-F \quad \text{.......}(4.58)$$

$$m\ddot{y}=f_y-mg \quad \text{.......}(4.59)$$

及

$$I\ddot{\theta}=f_xH-f_yB-F|k'|H \quad \text{.......}(4.60)$$

在此滑動-晃動模態 下

$$f_x=\mu f_y \quad \text{.......}(4.61)$$

及

$$\ddot{y}=B\ddot{\theta} \quad \text{.......}(4.62)$$

因此，從式(4.59)及(4.62)可得

$$f_y = m\ddot{y} + mg = m(g + B\ddot{\theta}) \quad \text{(4.63)}$$

將式(4.61)及(4.63)中的 f_x 及 f_y 代入式(4.60)，並代入 $F = mgk$，可以得到

$$\ddot{\theta} = \frac{3g}{B}\frac{(\mu\gamma - 1 - k|k'|\gamma)}{(4 + \gamma^2 - 3\mu\gamma)} \quad \text{(4.64)}$$

要啟動物體對點 O 的滑動-晃動模態，必須要

$$\ddot{\theta} \geq 0 \quad \text{(4.65)}$$

及

$$f_y \geq 0 \quad \text{(4.66)}$$

從式(4.63)可知，只要條件 $\ddot{\theta} \geq 0$ 被滿足，式(4.66)的條件就會被滿足。如果要式(4.65)成立的話，式(4.64)的分子與分母，必須同時都大於或等於 0，或者同時都小於或等於 0。也就是

$$\mu \geq \frac{1}{\gamma} + k|k'| \quad \text{(4.67)}$$

及

$$\mu \leq \frac{4 + \gamma^2}{3\gamma} = \mu_{B'} \quad \text{(4.68)}$$

或

$$\mu \leq \frac{1}{\gamma} + k|k'| \quad \text{(4.69)}$$

及

$$\mu \geq \frac{4+\gamma^2}{3\gamma} = \mu_{B'} \qquad (4.70)$$

為了滿足式(4.67)及(4.68)，必須

$$\frac{4+\gamma^2}{3\gamma} \geq \frac{1}{\gamma} + k|k'| \qquad (4.71)$$

或

$$k \leq \frac{1+\gamma^2}{3\gamma|k'|} = k_{B'} \qquad (4.72)$$

從式(4.48)可以看到上式的確等於$k_{B'}$。

同樣地，為了滿足式(4.69)及(4.70)，必須

$$k \geq \frac{1+\gamma^2}{3\gamma|k'|} = k_{B'} \qquad (4.73)$$

為了清晰起見，我們再次強調為了滿足$\ddot{\theta} \geq 0$條件，我們必須滿足式(4.67) ($\mu \geq \frac{1}{\gamma} + k|k'|$)與式(4.72) ($k \leq k_{B'}$)或式(4.69)($\mu \leq \frac{1}{\gamma} + k|k'|$)與式(4.73) ($k \geq k_{B'}$)中間的一組條件。

當物體開始要向左滑動時，物體在點O的速度必須小於或等於0($\dot{x}_O \leq 0$)。因為物體是從靜止開始啟動的，條件$\dot{x}_O \leq 0$相當於條件$\ddot{x}_O \leq 0$。我們可以證明當物體處於這種滑動-晃動模態的運動時，

$$\ddot{x}_O = \ddot{x} + H\ddot{\theta} \qquad (4.74)$$

從式 (4.58)、(4.61)及(4.63)，因為

$$\ddot{x} = \mu(g + B\ddot{\theta}) - gk \qquad (4.75)$$

我們可以從式(4.74)及(4.64)得到

$$\ddot{x}_O = \frac{g}{4+\gamma^2-3\mu\gamma}\{\mu[3\gamma(1-|k'|)k+1+4\gamma^2]-[(4+\gamma^2+3\gamma^2|k'|)k+3\gamma]\} \qquad (4.76)$$

假如括號裡的分子部分以及分母部分的符號相反的話，$\ddot{x}_O \le 0$。也就是必須有

$$\mu \le \frac{(4+\gamma^2+3\gamma^2|k'|)k+3\gamma}{3\gamma(1-|k'|)k+1+4\gamma^2} = \mu^*(k) \qquad (4.77)$$

及

$$\mu \le \frac{4+\gamma^2}{3\gamma} = \mu_{B'} \qquad (4.78)$$

或是，對等地，

$$k \le \frac{1+\gamma^2}{3\gamma|k'|} = k_{B'} \qquad (4.79)$$

或

$$\mu \ge \mu^*(k) \qquad (4.80)$$

及

$$\mu \geq \frac{4+\gamma^2}{3\gamma} = \mu_{B'} \quad \text{.......(4.81)}$$

或是，對等地，

$$k \geq \frac{1+\gamma^2}{3\gamma|k'|} = k_{B'} \quad \text{.......(4.82)}$$

結合式(4.67)與(4.72) ($\ddot{\theta} \geq 0$的情形)及式(4.77)與(4.79) ($\ddot{x}_O \leq 0$的情形)可得到圖 4.5a 中的陰影區域為當$|k'| \leq \frac{1+\gamma^2}{4+\gamma^2}$ ($\Delta \geq 0$，Δ為點A處$\mu^*(k)$的斜率與 4.3 節依點O產生晃動模態中定義的直線AB的斜率$|k'|$的差值)時啟動依點O發生滑動-晃動模態的區間。

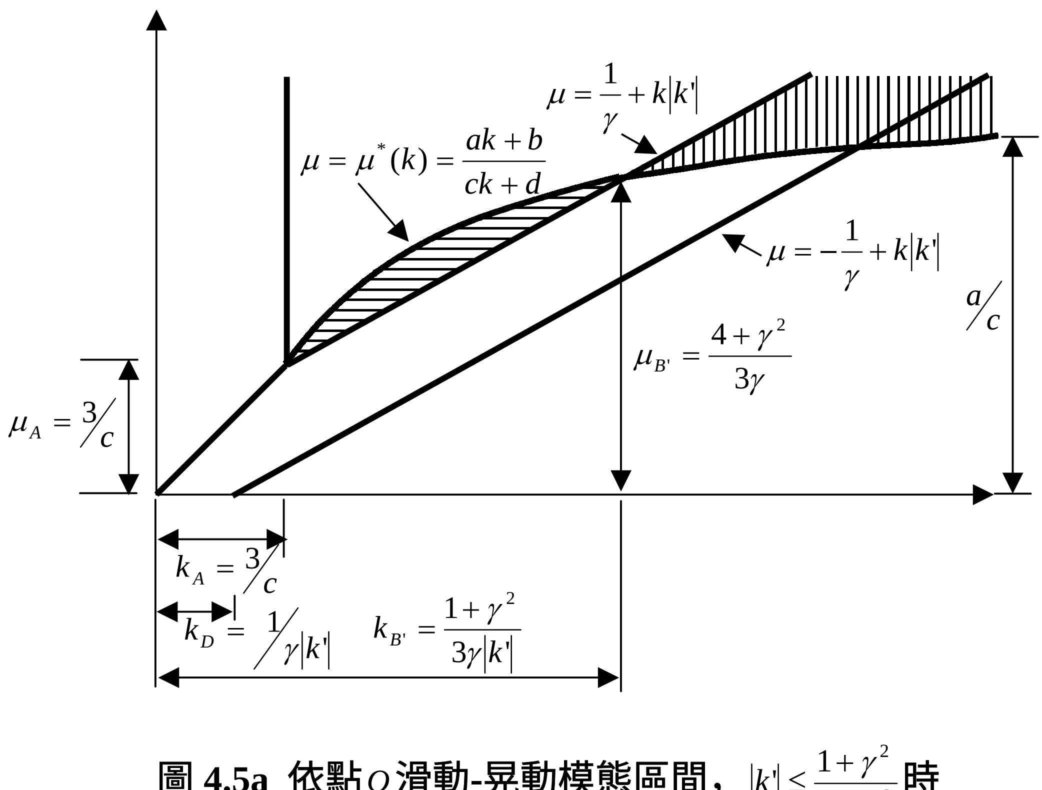

圖 4.5a 依點O滑動-晃動模態區間，$|k'| \leq \frac{1+\gamma^2}{4+\gamma^2}$時

結合式(4.69)與(4.73) ($\ddot{\theta} \geq 0$的情形)及式(4.80)與(4.82) ($\ddot{x}_O \leq 0$的情形)可得到圖 4.5b 的陰影區域為當$|k'| \geq \frac{1+\gamma^2}{4+\gamma^2}$ ($\Delta \leq 0$)時啟動依點O發生滑動-晃動模態的區間。再一次地，我們看到依點O滑動-晃動模態的區間與依點O晃動模態的區間有重疊的情形，同樣也會在 4.7 節中討論。

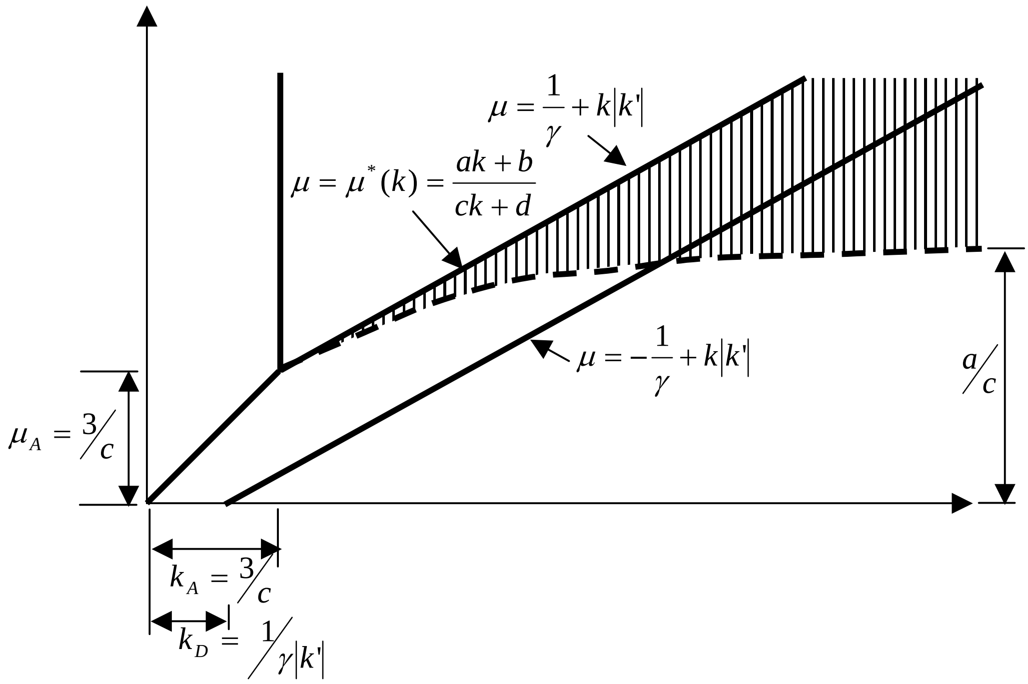

圖 4.5b 依點O滑動-晃動模態區間，$|k'| \geq \frac{1+\gamma^2}{4+\gamma^2}$時

4.6 依點 O'滑動-晃動模態

本節我們檢視依點O'瞬間發生滑動-晃動模態運動的啟動條件(如圖 Fig.4.1 所示)。

運動方程式可寫為

$$m\ddot{x} = f_x - F \qquad (4.83)$$

$$m\ddot{y} = f_y - mg \quad \text{..........(4.84)}$$

由於物體開始要發生滑動-晃動，反力 f_y 將會作用在 O' 點上。對質心 C 取力矩，其運動方程式為

$$I\ddot{\theta} = f_x H + f_y B - F|k'|H \quad \text{..........(4.85)}$$

此時

$$f_x = \mu f_y \quad \text{..........(4.86)}$$

以及，由於旋轉中心為 O'

$$\ddot{y} = -B\ddot{\theta} \quad \text{..........(4.87)}$$

從式(4.84)

$$f_y = m(g - B\ddot{\theta}) \quad \text{..........(4.88)}$$

將式(4.86)及(4.88)代入式(4.85)中，並引入 $F = mgk$，可得

$$\ddot{\theta} = \frac{3g}{B(4+\gamma^2+3\mu\gamma)}(\mu\gamma + 1 - k|k'|\gamma) \quad \text{..........(4.89)}$$

對於這種反應行為的發生，必須符合下列條件：

$$\ddot{\theta} \le 0 \quad \text{..........(4.90)}$$

及

$$f_y \ge 0 \quad \text{..........(4.91)}$$

由於$\ddot{\theta} \le 0$，從式(4.89)，我們必須有

$$\mu \le -\frac{1}{\gamma} + k|k'| \quad \text{.......(4.92)}$$

根據式(4.88)，假如式(4.90)能被滿足的話，式(4.91)也將一樣被滿足。

對於物體向左邊滑動的情形，必須要$\dot{x}_{O'} \le 0$或者是$\ddot{x}_{O'} \le 0$。因為

$$\ddot{x}_{O'} = \ddot{x} + H\ddot{\theta} \quad \text{.......(4.93)}$$

及從式(4.83)、(4.86)與(4.88)獲得

$$\ddot{x} = g(\mu - k) - \mu B\ddot{\theta} \quad \text{.......(4.94)}$$

利用式(4.89)的關係，從式(4.93)可得

$$\ddot{x}_{O'} = \frac{g}{4+\gamma^2+3\mu\gamma}\{\mu[(1+4\gamma^2)-3\gamma(1-|k'|)k]-[(4+\gamma^2+3\gamma^2|k'|)k-3\gamma]\} \quad \text{.......(4.95)}$$

當$\ddot{x}_{O'} \le 0$時，必須要

$$\mu \le \frac{(4+\gamma^2+3\gamma|k'|)k-3\gamma}{1+4\gamma^2-3\gamma(1-|k'|)k} \equiv \mu^{**}(k) \quad \text{.......(4.96)}$$

上式是變數k的函數。現在來研究$\mu^{**}(k)$的性質。首先，可以看到

$$\mu^{**}(k) = \frac{ak-b}{d-ck} \quad \text{.......(4.97)}$$

其中，a、b、c及d是由式(4.42)、(4.43)、(4.44)及(4.45)所定義的常數值。

在$k=0$處，$\mu^{**}=-\frac{b}{d}\le 0$。在$k=\frac{d}{c}$處，$\mu^{**}$趨近於無窮大。而在$k=\frac{b}{a}$處，$\mu^{**}=0$。曲線$\mu^{**}(k)$的斜率是$\frac{d\mu^{**}}{dk}=\frac{ad-bc}{(d-ck)^2}=\frac{4(1+\gamma^2)(1+\gamma^2+3\gamma^2|k'|)}{(d-ck)^2}\ge 0$，當$k\to\frac{d}{c}$時，斜率趨於無窮大。

從由$\ddot{\theta}\le 0$條件所得的式(4.92)及由$\ddot{x}_{O'}\le 0$條件所得的式(4.96),，圖 4.6a 及 4.6b 分別顯示當$\Delta\ge 0$及$\Delta\le 0$ (Δ的定義參考 4.3 節所述)時依點O'發生滑動-晃動模態反應的對應區間。我們可以證明曲線$\mu^{**}(k)$與k軸的交點F在曲線$\mu=-\frac{1}{\gamma}+k|k'|$與$k$軸的交點$D$的左邊。

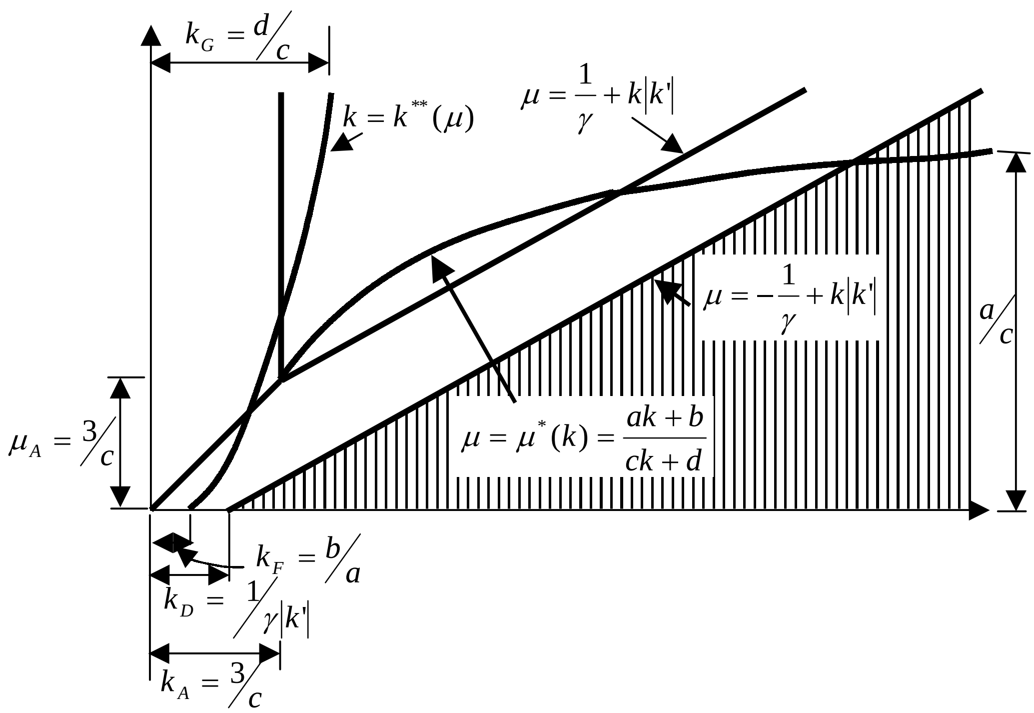

圖 4.6a 依點O'滑動-晃動模態區間，$|k'|\le\frac{1+\gamma^2}{4+\gamma^2}$時

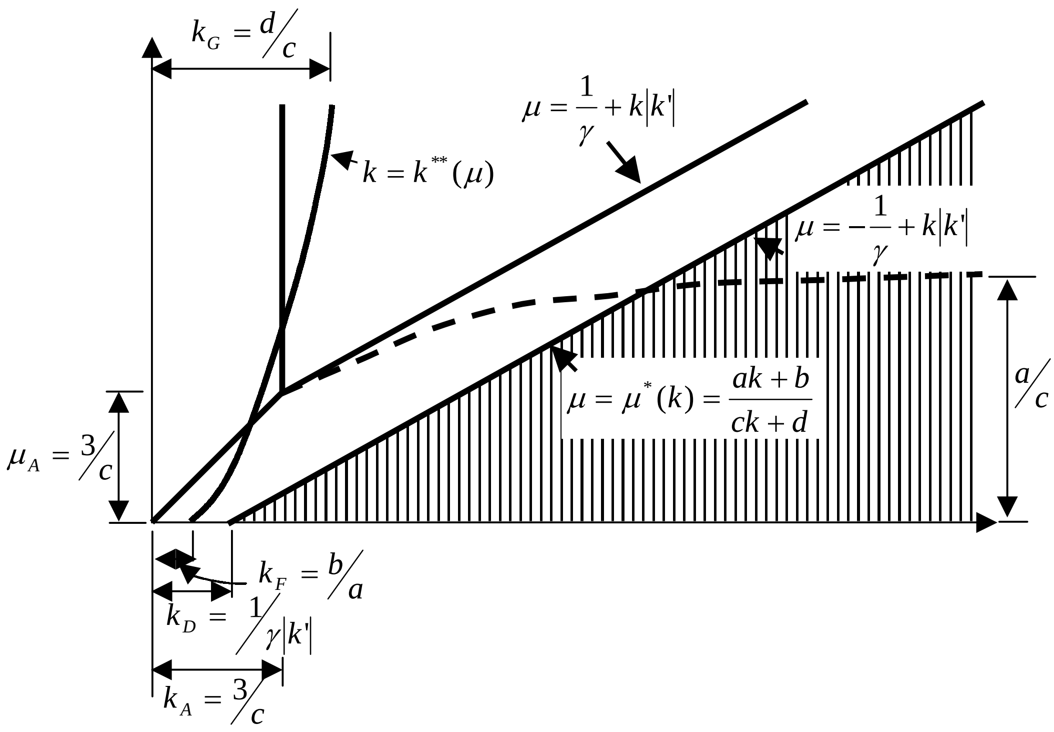

圖 4.6b 依點 O' 滑動-晃動模態區間，$|k'| \geq \frac{1+\gamma^2}{4+\gamma^2}$ 時

4.7 討　論

以上各章節分析了當一個放置於水平剛性底床上的剛性物體受到一個向左作用於物體質心下方的水平力時可能產生的各種不同反應模態的啟動條件。與水平力作用於物體質心上方的情形(第三章)不同的是，在 $k-\mu$ 平面上不同的反應模態的對應區間有重疊的現象，導致必須要進一步分析以取定這些重疊的區間應如何處理。

在圖 4.3 中，當一組 k 及 μ 的值對應到陰影區域裡一點，物體便會發生滑動。反力 f_y 作用在底部 OO' 之間(如圖 4.1 所示)。當 f_y 作用於點 O 時，其 k 及 μ 的值落在邊界 AB；當 f_y 作用於點 O' 時，其 k 及 μ 的值落在邊界 CD。因此，在此區間內，除了當 k 及 μ 的值對應到邊界 AB 及 DC 上的點以外，物體不會發生依點 O 或點 O' 晃動的情形。根據這點理由，在圖 4.4a 依點 O 晃動模態的 $BB'C'C$ 區間及圖 4.4b 的

$BAC'C$ 區間應該被去除掉。根據同樣的理由，在圖 4.4a 及圖 4.4b 中的 $CC'C''$ 區段也不會發生依點 O 晃動的情形。因為線段 DC ($C'C$)表示當反力 f_y 作用在 O' ($\xi = B$)的情形。

同樣的理由, 在圖 4.5a 中物體依點 O 滑動-晃動的情形中, $BB'C'C$ 區間及 $CC'C''$ 區間應該被去除掉。而圖 Fig.4.5b 中所代表的情形中，物體依點 O 發生滑動-晃動的情形根本不會發生，且區間 $BAC'C''$ 不應該被考慮。對於物體依點 O' 滑動-晃動的情形，在圖 4.6a 及 4.6b 中的整個陰影區域是依點 O' 滑動-晃動模態的有效區域。

圖 4.7a 及 4.7b 分別顯示當 $|k'| \le \frac{1+\gamma^2}{4+\gamma^2}$ 時及當 $|k'| \ge \frac{1+\gamma^2}{4+\gamma^2}$ 時的最後結果。為了簡單起見，各個反應模態分別以不同的符號表示，例如 RE (靜止模態)、SL (滑動模態)、RO (依點 O 晃動模態)、SRO_- (依點 O 滑動-晃動模態，當 $\dot{x} \le 0$ 時)及 SRO'_- (依點 O' 滑動-晃動模態，當 $\dot{x} \le 0$ 時)。

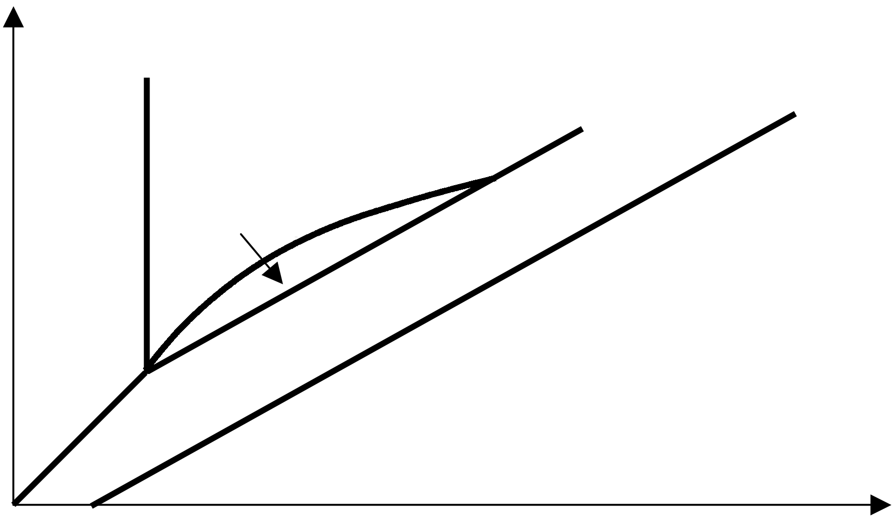

圖 4.7a 所有模態， $|k'| \le \frac{1+\gamma^2}{4+\gamma^2}$ **時**

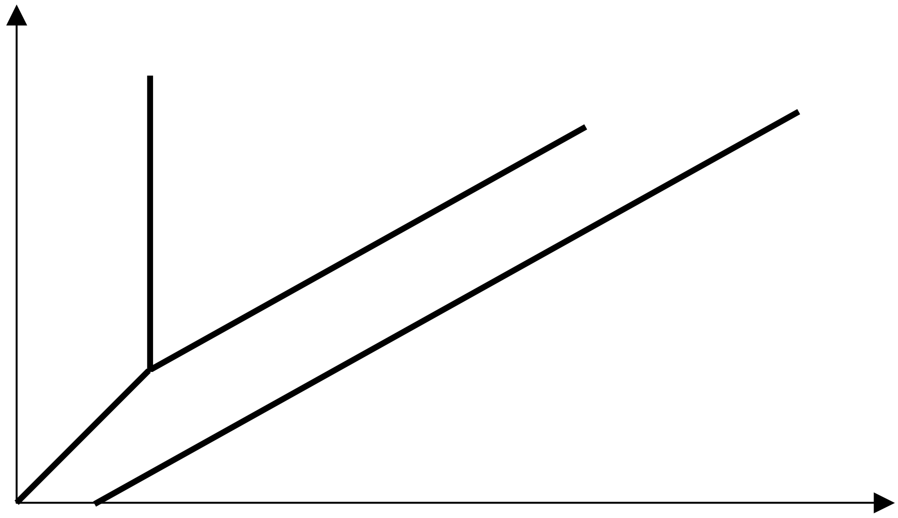

圖 4.7b 所有模態， $|k'| \geq \dfrac{1+\gamma^2}{4+\gamma^2}$ **時**

第五章 結　論

本研究的動機主要是想瞭解當一個沈箱受到碎波波力作用時會發生什麼樣的反應。本研究是一個初階的研究，我們以一個放置於粗糙底床的剛性物體來模擬一個沈箱，及以一水平力來模擬碎波波力的作用。我們成功地辨識及推導沈箱可能發生的各種不同反應模態的啟動條件。

推導過程中所使用的方程式是平面剛性物體的三個運動方程式。雖然模型很簡單，但分析的結果顯示了物體的複雜行為。雖然這些條件的推導相當繁雜，但其結果可以用圖形的型式，在$k-\mu$平面上(k與外力F的大小有關而μ為物體與底床間的摩擦係數)，提供了一個很簡單的方法來表現物體各種反應模態的啟動條件。

第六章 致　謝

本研究是在交通部運輸研究所港灣技術研究中心的委託研究計畫”花蓮港港池共振機制研究(1/4)”(計畫編號：MOTC-IOT-94-H3DB003)的經費支援下進行的。計畫參與人員在此感謝港灣研究中心的支持，與邱永芳主任的大力協助。同時也感謝海洋大學蔡仲景先生的中文翻譯與校稿。

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Shenton, HW. Criteria for initiation of slide, rock and slide-rock rigid body modes. Journal of Engineering Mechanics(ASCE) 1996;122(7):690-693

附　錄　A

Behavior of a Freestanding Rigid Body Subjected to a Horizontal Force

by

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I. Introduction

In coastal waters, breakwaters of various types are used. Over the years, damage of breakwaters has occurred, primarily due to wave actions. Many kinds of damage have been observed and efforts have been made to understand behavior of breakwaters to wave actions. These studies have yielded valuable information to guide their design.

In this report, no attempt has been made to review the extensive literature of the subject.

In this study, we seek to identify the various modes of response of a vertical caisson and the conditions under which these modes of response are initiated when the caisson is subjected to a single concentrated horizontal force of short duration mimicking the action of a breaking wave.

The caisson is modeled as a freestanding rigid body resting on a rigid horizontal base which exerts a Coulomb type frictional resistance to the body.

The approach to the determination of criteria for initiation of modes of response of a rigid body due to a force follows that of Shenton (1996) in which a rigid body is subjected to earthquake-like base excitations. The results in this study, however, are quite different and much more complex than those of Shenton's.

Criteria for initiation of modes of response are obtained for a rigid body placed in air. When the body is immersed in water, buoyancy force and uplift force are involved. These forces are not included in this study; their effects will be considered and reported later.

Because the derivations are rather involved, criteria for initiation of the various modes of response are presented separately for the cases of the force acting above and below center of mass of the body.

II. Model

Consider a plane, rectangular rigid body with a rectangular footprint (see Fig.2.1). Width of the body is $2B$, its height, $2H$, and its depth equals unity. Mass of the body is assumed to be uniformly distributed and total mass is m. The body rests on a horizontal base. Between the body and the base exists a Coulomb friction force with coefficient of friction equal to μ. No distinction is made between static and kinetic coefficients of friction. The body is initially at rest and is subjected to a horizontal force F which is expressed as a multiple of weight of the body, mgk, where g is gravitational acceleration and k is a non-dimensional positive quantity. The distance between line of action of force F and center of mass C is denoted by h, measured positive upward from C, expressed as a fraction of H. That is, $h = k'H$. When F is applied above center of mass of the body, $0 \le k' \le 1$ and when F is below C, $-1 \le k' \le 0$. Displacement of C in horizontal and vertical directions are respectively x and y, considered positive to the right and upward, with origin at center of mass of the body when it is at rest; rotation of the body is θ considered positive counter-clockwise. Reaction forces are f_x and f_y. Distance between line of action of f_y and C is ξ positive to the right of C.

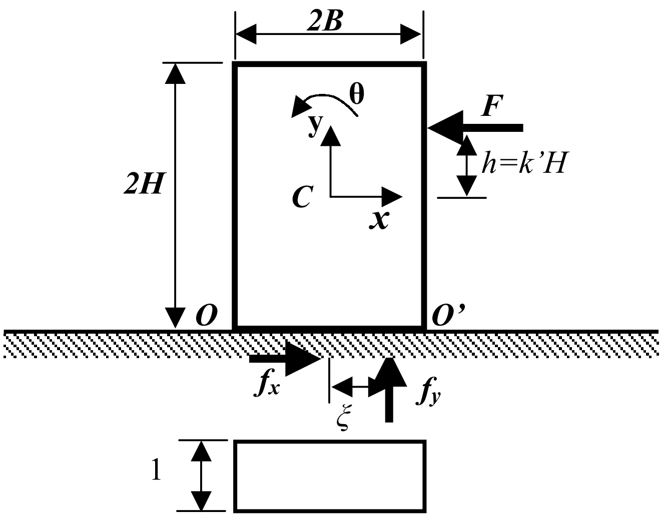

Fig.2.1 Model

III. Criteria for initiation of modes of response of a freestanding rigid body under the action of a horizontal force applied above center of mass

III.1. Rest

When the body is at rest under the action of F (see Fig.2.1), equations of equilibrium are:

$$f_x = F \tag{3.1}$$

$$f_y = mg \tag{3.2}$$

and, by taking moment of the forces about point C,

$$f_x H + f_y \xi + Fh = 0 \tag{3.3}$$

For the body to be in contact with the base, $f_y \geq 0$. For the body to be at rest, horizontal resistance must not exceed limiting Coulomb friction force: $|f_x| \leq \mu |f_y|$. Since f_x may be positive or negative or equal to zero, the absolute value sign is used. The absolute value sign for f_y may be eliminated since, from (3.2), $f_y \geq 0$ always. Finally, f_y always lies within the base (OO') of the body. This requirement is satisfied if $|\xi| \leq B$. Again, the absolute value sign is used because ξ may be positive or negative or equal to zero. As will be shown shortly, in rest mode, $|\xi| \leq 0$ (f_y acts to the left of point C.)

Since $F = mgk$, from (3.1) and (3.2), the condition $|f_x| \leq \mu |f_y|$ gives

$$\mu \geq k \tag{3.4}$$

From (3.3), $\xi = -k(H + h) = -k(1 + k')H \leq 0$. The requirement that $|\xi| \leq B$ therefore gives

$$k \le \frac{1}{\gamma(1+k')} \tag{3.5}$$

where $\gamma = H/B$ is aspect or slenderness ratio of the body and is a measure of its slenderness. Thus, for the body to be at rest, conditions (3.4) and (3.5) must be satisfied. It is convenient to view these conditions graphically. In Fig.3.1, horizontal axis is k (measure of the strength of F) and vertical axis is μ (coefficient of friction). The two conditions (3.4) and (3.5) are both satisfied if values of k and μ correspond to a point in the shaded region enclosed by lines OA ($\mu = k$), AB ($k = \frac{1}{\gamma(1+k')}$) and the vertical axis. Thus, for a body of given aspect ratio $\gamma = \frac{H}{B}$ resting on a horizontal base, subjected to a horizontal force of known strength k and known line of action (as given by k'), one can determine, at a glance, whether the body remains at rest or not once value of coefficient of friction μ can be ascertained. From Fig.3.1, it is seen that the larger the aspect ratio γ, the closer is line AB to the μ axis, the narrower is the shaded region, and the less likely will the body be at rest. Similarly, for larger value of k', line of action of F is higher, line AB is closer to the μ axis, the shaded region is narrower, and it is less likely that the body will be at rest. On the other hand, by lowering aspect ratio γ of the body and/or value of k', the region corresponding to rest mode may be made larger.

Conditions governing rest mode of the body is a piece of information a designer of breakwater must have.

III.2. Slide

Equations of motion are:

$$m\ddot{x} = f_x - F \tag{3.6}$$

$$f_y = mg \tag{3.7}$$

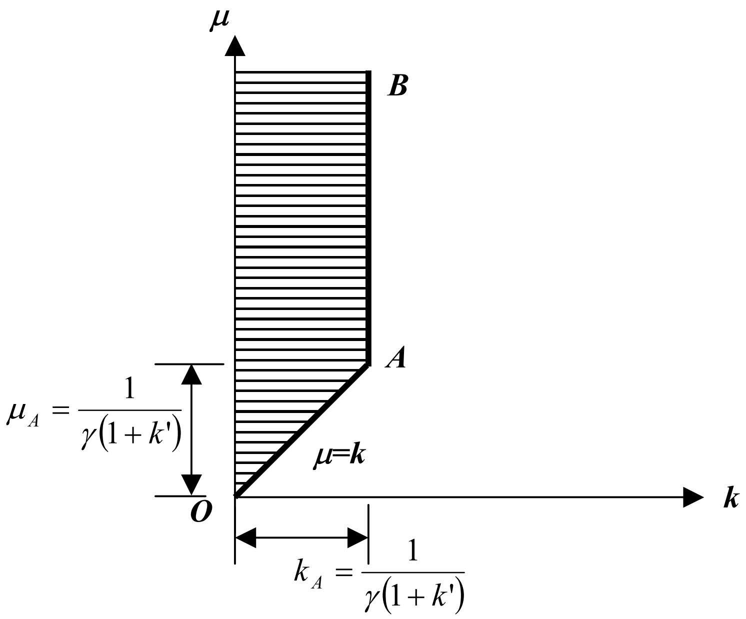

Fig.3.1 Rest region

and, by taking moment of the forces about point C,

$$f_x H + f_y \xi + Fh = 0 \tag{3.8}$$

Here and hereafter, over-dot denotes differentiation with respect to time. While (3.7) and (3.8) are the same as (3.2) and (3.3), (3.6) differs from (3.1) since the body is about to slide so that $\ddot{x} \neq 0$ although x is equal to zero. On the verge of sliding, $f_x = \mu f_y$. By substituting $f_x = \mu f_y$, $h = k'H$ and $F = mgk$ into (3.8), we get $\xi = -(\mu + kk')H \leq 0$ always, a result that stems from the fact that F acts above the center of mass of the body. Condition $f_y \geq 0$ is automatically satisfied. Condition $|\xi| \leq B$ requires $\mu \leq \frac{1}{\gamma} - kk'$. In Fig.3.2, with k the horizontal axis and μ the vertical axis, $\mu = \frac{1}{\gamma} - kk'$ is represented by the line CD. Condition $\mu \leq \frac{1}{\gamma} - kk'$ is represented by region OCD. Since region OCA ($\mu \geq k$) corresponds to rest mode, for slide mode to be initiated, values of k and μ must correspond to a point in shaded region OAD. From (3.6), we have $\ddot{x} = g(\mu - k) \leq 0$, because $\mu \leq k$. Since sliding is the mode of response being considered, accelerations and velocities of all points in the body are the same and point to the left, the body being initially at rest. It may be noted that for $k'=0$, (F acts through center of mass of the body), the condition becomes $\mu \le \frac{1}{\gamma}$, a constant, independent of k, so that line CD is horizontal; the region corresponding to slide mode in this case is larger than if F acts above center of mass of the body.

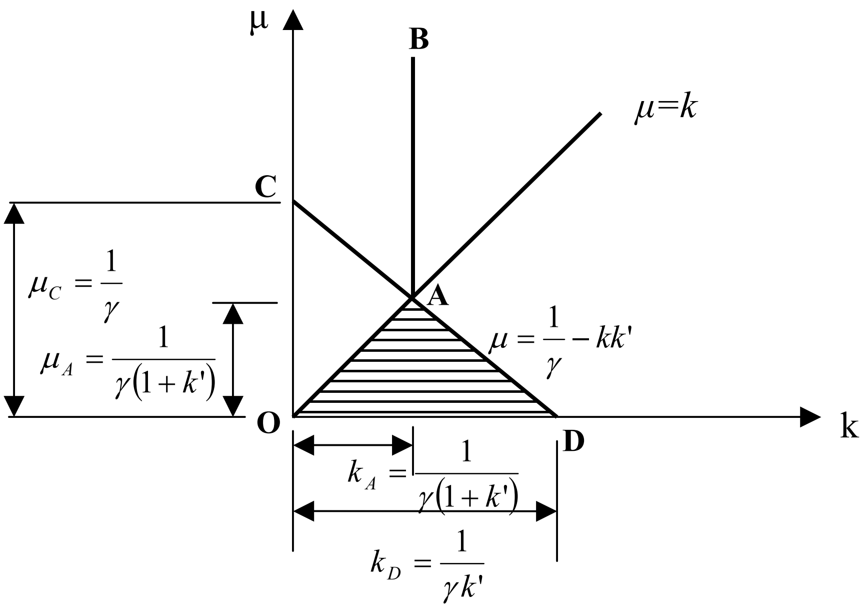

Fig.3.2 Slide region

III.3. Rock

Equations of motion are:

$$m\ddot{x} = f_x - F \tag{3.9}$$

$$m\ddot{y} = f_y - mg \tag{3.10}$$

Noting that f_y acts at point O (see Fig.2.1) about which the body rotates, by taking moment of forces about the point C,

$$I\ddot{\theta} = f_x H - f_y B + Fh \tag{3.11}$$

where $I = \frac{1}{3}m(B^2 + H^2)$ is mass moment of inertia of the body about its center of mass C. When rocking is impending, while coordinates x, y and θ remain equal to zero, their accelerations do not. Accelerations $\ddot{x}$ and $\ddot{y}$ of point C are related to angular acceleration $\ddot{\theta}$ of the body as

$$\ddot{x} = -H\ddot{\theta} \tag{3.12}$$

and

$$\ddot{y} = B\ddot{\theta} \tag{3.13}$$

By substituting (3.9), (3.10), (3.12) and (3.13) into (3.11), noting that $F = mgk$ and $h = k'H$, we get

$$\ddot{\theta} = \frac{3g}{4B(1+\gamma^2)}[k\gamma(1+k') - 1] \tag{3.14}$$

By substituting (3.14) into (3.12), (3.13) and (3.9), (3.10), we have

$$f_x = \frac{mg}{4(1+\gamma^2)}[k(4+\gamma^2 - 3\gamma^2 k') + 3\gamma] \tag{3.15}$$

and

$$f_y = \frac{mg}{4(1+\gamma^2)}[3\gamma(1+k')k + 1 + 4\gamma^2] \geq 0 \tag{3.16}$$

The body remains in contact with the base since $f_y \geq 0$. For $\ddot{\theta} \geq 0$, we must have

$$k \geq \frac{1}{\gamma(1+k')} \tag{3.17}$$

For convenience, let

$$a = 4 + \gamma^2 - 3\gamma^2 k' \tag{3.18}$$

$$b = 3\gamma \tag{3.19}$$

$$c = 3\gamma(1+k') \tag{3.20}$$

and

$$d = 1 + 4\gamma^2 \tag{3.21}$$

so that

$$f_x = \frac{mg}{4(1+\gamma^2)}(ak + b) \tag{3.22}$$

and

$$f_y = \frac{mg}{4(1+\gamma^2)}(ck + d) \geq 0 \tag{3.23}$$

We see that while $f_y \geq 0$ always, f_x may not greater than zero, because the quantity a in (3.18) is not always greater than zero.

For rocking to be initiated, horizontal resistant force f_x must not exceed limiting friction force. This condition, $|f_x| \leq \mu f_y$, gives, from (3.15) and (3.16) or (3.22) and (3.23),

$$\mu \geq \frac{|(4+\gamma^2 - 3\gamma^2 k')k + 3\gamma|}{3\gamma(1+k')k + 1 + 4\gamma^2} = \frac{|ak+b|}{ck+d} = \mu^*(k) \tag{3.24}$$

The function $\mu^*(k)$ behaves in a variety of ways depending on the sign of the quantity a and that of f_x. There are in fact three possible cases that may arise. They are:

Case 1: $a \geq 0$ (or equivalently, $k' \leq \frac{4+\gamma^2}{3\gamma^2}$), so $f_x \geq 0$. Thus, $\mu^*(k) = \frac{ak+b}{ck+d}$ for all values of k.

Case2: $a \leq 0$ (or equivalently, $k' \geq \frac{4+\gamma^2}{3\gamma^2}$), but $f_x \geq 0$. Thus,

$$\mu^*(k) = \frac{ak+b}{ck+d} = \frac{-|a|k+b}{ck+d} \text{ for } k \leq \frac{b}{|a|}.$$

Case 3: $a \le 0$ (or equivalently, $k' \ge \frac{4+\gamma^2}{3\gamma^2}$), and $f_x \le 0$. Thus,

$$\mu^*(k) = \frac{\left|(-|a|k + b)\right|}{ck+d} \text{ for } k \ge \frac{b}{|a|}.$$

Since $0 \le k' \le 1$, we must make sure that $\frac{4+\gamma^2}{3\gamma^2} \le 1$. This means that for $\gamma \le \sqrt{2}$, no rocking motion can be initiated.

Properties of $\mu^*(k)$ are examined for each of the above three cases.

Case 1:

It may be verified that $k = k_A = \frac{1}{\gamma(1+k')} = \frac{3}{c} = \mu^*(k_A)$. This means that curve $\mu^*(k)$ passes point A (see Fig.3.1). As k approaches infinity, $\mu^*(k) = \frac{4+\gamma^2 - 3\gamma^2 k'}{3\gamma(1+k')} = \frac{a}{c} \ge 0$ always since both the quantities a and c are greater or equal to zero. Slope of $\mu^*(k)$ is $\frac{d\mu^*}{dk} = \frac{ad-bc}{(ck+d)^2}$ where $ad-bc = 4(1+\gamma^2)(1+\gamma^2-3\gamma^2 k')$ is a constant, independent of k, and $ck+d = 3\gamma(1+k')k + 1 + 4\gamma^2 \ge 0$. Thus, slope of $\mu^*(k)$ is a decreasing function of k and approaches zero as k approaches infinity. Also, slope of $\mu^*(k)$ at point A is $\left.\frac{d\mu^*}{dk}\right|_{k_A} = \frac{1+\gamma^2-3\gamma^2 k'}{4(1+\gamma^2)}$. When $k' \le \frac{1+\gamma^2}{3\gamma^2}$, $\left.\frac{d\mu^*}{dk}\right|_{k_A} \ge 0$ whereas when $k' \ge \frac{1+\gamma^2}{3\gamma^2}$, $\left.\frac{d\mu^*}{dk}\right|_{k_A} \le 0$.

In Figs 3.3a and 3.3b curves $\mu = \mu^*(k)$ are sketched respectively for the cases of $0 \le k' \le \frac{1+\gamma^2}{3\gamma^2}$ and $\frac{1+\gamma^2}{3\gamma^2} \le k' \le \frac{4+\gamma^2}{3\gamma^2} \le 1$.

For given values of k' and γ, if values of k (force) and μ coefficient of friction correspond to a point in the $k-\mu$ plane that falls in one of the shaded regions in Figs.3.3a and 3.3b, rock mode of response would ensue. From these figures, it can be seen that if F is far above center of mass of the body, k' is large (Fig.3.3b) and the shaded region of rock mode is larger than if the force is closer to point C (see Fig.2.1) in which case, value of k' is smaller (Fig.3.3a).

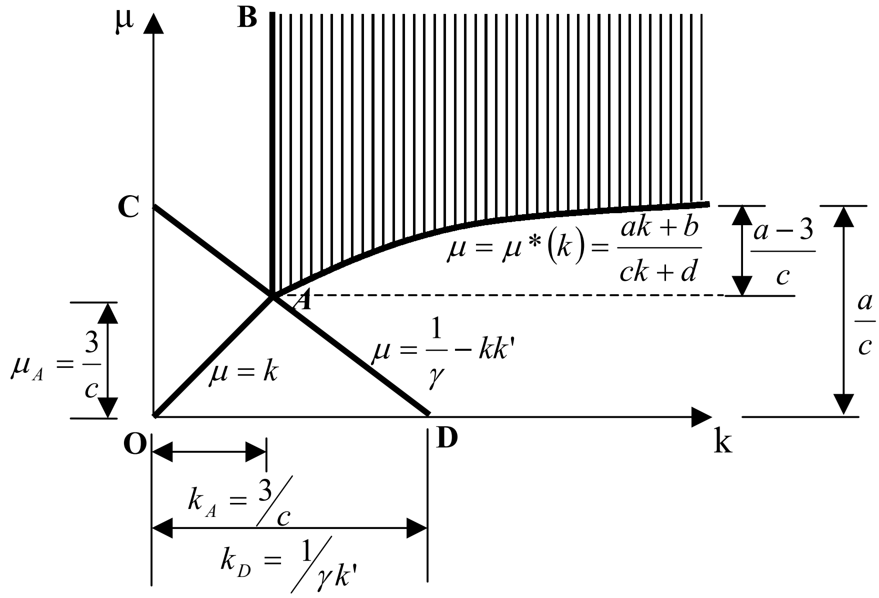

Fig.3.3a Rock region, case 1, $0 \le k' \le \frac{1+\gamma^2}{3\gamma^2}$, $f_x \ge 0$, $a \ge 0$

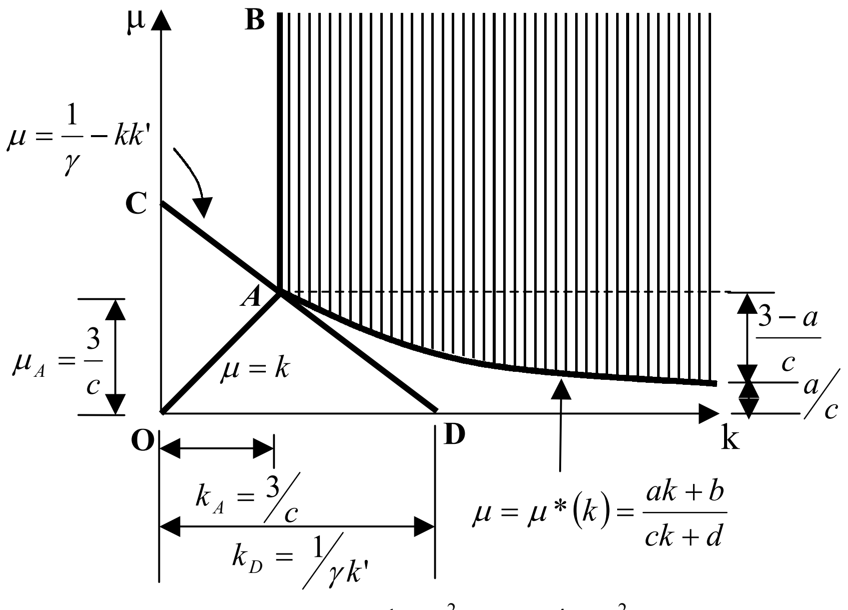

Fig.3.3b Rock region, case 1, $\frac{1+\gamma^2}{3\gamma^2} \le k' \le \frac{4+\gamma^2}{3\gamma^2} \le 1$, $f_x \ge 0$, $a \ge 0$

Case 2:

In this case, $\mu^*(k) = \dfrac{-|a|k + b}{ck + d}$. Since $\mu^*(k)$ must be nonnegative, the function is valid only if $k \le \dfrac{b}{|a|} = \dfrac{3\gamma}{3\gamma^2 k' - (4 + \gamma^2)}$. As in case 1, $\mu^*(k)$ passes point A (see Fig.3.1) and its slope is given by $\dfrac{d\mu^*}{dk} = \dfrac{4(1+\gamma^2)(1+\gamma \ \ -3\gamma^2 k')}{[3\gamma(1+k')k + 1 + 4\gamma^2]^2}$. At point A, $\left.\dfrac{d\mu^*}{dk}\right|_{k_A} = \dfrac{1+\gamma^2 - 3\gamma^2 k'}{4(1+\gamma^2)}$. Since in the case under consideration, $k' \ge \dfrac{4+\gamma^2}{3\gamma^2}$, and since $\dfrac{1+\gamma^2}{3\gamma^2} \le \dfrac{4+\gamma^2}{3\gamma^2}$, slope of $\mu = \mu^*(k)$ at point A is always less than or equal to zero. Curve $\mu = \mu^*(k)$ is sketched in Fig.3.4. Rock mode is initiated if values of k and μ correspond to a point that lies in the shaded region.

Case 3:

In this case, $\mu^*(k) = \dfrac{|(-|a|k + b)|}{ck + d} = \dfrac{-(ak + b)}{ck + d} = -\dfrac{[(4+\gamma^2 - 3\gamma^2 k')k + 3\gamma]}{3\gamma(1+k')k + 1 + 4\gamma^2}$, valid for $f_x \le 0$ or $-|a|k + b \le 0$, thus the curve $\mu^*(k)$ applies for $k \ge \dfrac{b}{|a|}$ only. In this case, $\mu^*(k)$ does not pas the point A. At $k = \dfrac{b}{|a|}$, $\mu^*(k) = 0$ and as k approaches infinity, $\mu^*(k) = \dfrac{|a|}{c} = \dfrac{3\gamma^2 k' - (4+\gamma^2)}{3\gamma(1+k')} \ge 0$ since for the case under consideration $k' \ge \dfrac{4+\gamma^2}{3\gamma^2}$. Slope of $\mu = \mu^*(k)$ is $\dfrac{d\mu^*}{dk} = \dfrac{|a|d + bc}{(ck+d)^2} \ge 0$ which decreases as k increases and approaches zero as k approaches infinity. The curve $\mu = \mu^*(k)$ is sketched in Fig.3.5.

If we compare Figs.3.4 and 3.5 with Fig.3.3, we see that as F acts far above center of mass C (k' large), chance of rocking being initiated is high.

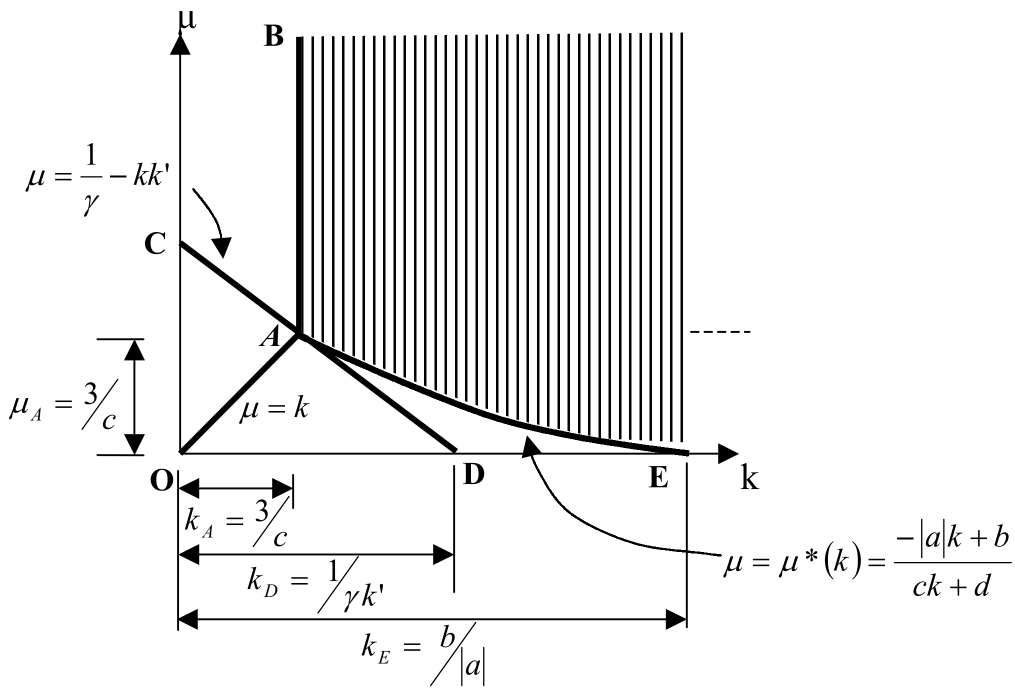

Fig.3.4 Rock region, case 2, $\frac{4+\gamma^2}{3\gamma^2} \le k' \le 1$, $f_x \ge 0$, $a \le 0$

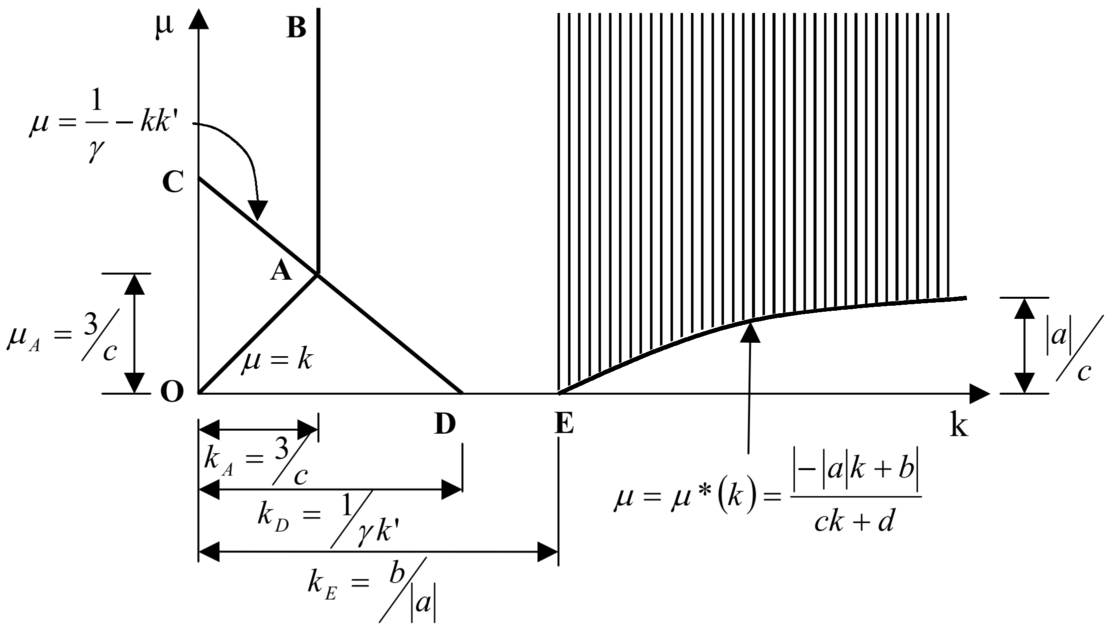

Fig.3.5 Rock region, case 3, $\frac{4+\gamma^2}{3\gamma^2} \le k' \le 1$, $f_x \le 0$, $a \le 0$

III.4. Slide-rock

When the body is on the verge of sliding and rocking about point O simultaneously, f_y acts at O and equations of motion are:

$$m\ddot{x} = f_x - F \tag{3.25}$$

$$m\ddot{y} = f_y - mg \tag{3.26}$$

and

$$I\ddot{\theta} = f_x H - f_y B + Fh \tag{3.27}$$

These equations are the same as (3.9), (3.10) and (3.11), equations of motion for initiation of rock mode. The difference is that in the case of rock mode, $\ddot{x} = -H\ddot{\theta}$ (see (3.12)) and $|f_x| \le \mu f_y$. In the present case, on the brink of slide-rock mode, $\ddot{x}$ is not related to $\ddot{\theta}$ as given in (3.12), and

$$|f_x| = \mu f_y \tag{3.28}$$

but

$$\ddot{y} = B\ddot{\theta} \tag{3.29}$$

as in (3.13).

By using (3.26), (3.27), (3.28) and (3.29), noting that $F = mgk$, $h = k'H$, and $I = \frac{1}{3}m(H^2 + B^2)$, we get

$$\ddot{\theta} = \frac{3g}{B}\frac{\mu\gamma - 1 + kk'\gamma}{4 + \gamma^2 - 3\mu\gamma} \tag{3.30}$$

For $\ddot{\theta} \ge 0$, we must have either

$$\mu\gamma - 1 + kk'\gamma \ge 0 \text{ and } 4 + \gamma^2 - 3\mu\gamma \ge 0 \tag{3.31}$$

or

$$\mu\gamma - 1 + kk'\gamma \le 0 \text{ and } 4 + \gamma^2 - 3\mu\gamma \le 0 \tag{3.32}$$

Equations (3.31) and (3.32) are equivalent to

$$\mu \ge \frac{1}{\gamma} - kk' \text{ and } \mu \le \frac{4+\gamma^2}{3\gamma} \tag{3.33}$$

or

$$\mu \le \frac{1}{\gamma} - kk' \text{ and } \mu \ge \frac{4+\gamma^2}{3\gamma} \tag{3.34}$$

(3.33) may be rewritten as

$$\frac{1}{\gamma} - kk' \le \mu \le \frac{4+\gamma^2}{3\gamma} \tag{3.35}$$

and (3.34) as

$$\frac{4+\gamma^2}{3\gamma} \le \mu \le \frac{1}{\gamma} - kk' \tag{3.36}$$

It may be shown that the minimum value of $\frac{4+\gamma^2}{3\gamma}$ is 1.33 when $\gamma = 2$. Since $\frac{1}{\gamma} \le \frac{4+\gamma^2}{3\gamma}$, (3.36) can not be satisfied. Thus, $\ddot{\theta} \ge 0$ is satisfied if (3.35) holds true, and, if $\ddot{\theta} \ge 0$, then $f_y \ge 0$, on account of (3.26) and (3.29). It remains to examine the body's direction of sliding under the action of F. This is determined by velocity $\dot{x}_O$ at point O of the body about which rotation takes place. Since the body is originally at rest, it suffices to examine acceleration $\ddot{x}_O$ of point O when slide-rock motion is impending.

Noting that $\ddot{x}_O = \ddot{x} + H\ddot{\theta}$ and $\ddot{x}$ is given by (3.25), it may be verified, after some algebra, that

$$\ddot{x}_O = \frac{g}{4+\gamma^2-3\mu\gamma}\{\mu[3\gamma(1+k')k+1+4\gamma^2]-[(4+\gamma^2-3\gamma^2 k')k+3\gamma]\}$$
$$= \frac{g}{4+\gamma^2-3\mu\gamma}[\mu(ck+d)-(ak+b)] \tag{3.37}$$

For $\ddot{x}_O \le 0$ (body sliding to the left and $f_x \ge 0$), we must have

$$4+\gamma^2-3\mu\gamma \ge 0 \text{ and } \mu(ck+d)-(ak+b) \le 0 \tag{3.38}$$

or

$$4+\gamma^2-3\mu\gamma \le 0 \text{ and } \mu(ck+d)-(ak+b) \ge 0 \tag{3.39}$$

Equation (3.38) may be rewritten as

$$\mu \le \frac{4+\gamma^2}{3\gamma} \text{ and } \mu \le \frac{ak+b}{ck+d} = \mu^*(k) \tag{3.40}$$

and (3.39) as

$$\mu \ge \frac{4+\gamma^2}{3\gamma} \text{ and } \mu \ge \frac{ak+b}{ck+d} = \mu^*(k) \tag{3.41}$$

Since, for $\ddot{\theta} \ge 0$, we must have, from (3.35), $\mu \le \frac{4+\gamma^2}{3\gamma}$, so (3.41) is not valid. (3.40) and (3.33) give condition for slide-rock to be initiated with $\dot{x}_O \le 0$ ($f_x \ge 0$) as

$$\frac{1}{\gamma} - kk' \le \mu \le \mu^*(k) = \frac{ak+b}{ck+d} = \frac{(4+\gamma^2-3\gamma^2 k')k+3\gamma}{3\gamma(1+k')k+1+4\gamma^2} \tag{3.42}$$

As explained in the last section (Rock), we must distinguish between the cases of $k' \le \frac{4+\gamma^2}{3\gamma^2}$ (case 1) for $a \ge 0$, and $k' \ge \frac{4+\gamma}{3\gamma^2}$ (case 2) for $a \le 0$. The first case has two sub-cases: $k' \le \frac{1+\gamma^2}{3\gamma^2}$ (slope of $\mu^*(k)$ at point A is ≥ 0) and $k' \ge \frac{1+\gamma^2}{3\gamma^2}$ (slope of $\mu^*(k)$ at point A is ≤ 0).

Figs.3.6a, 3.6b and 3.7 give the regions corresponding to initiation of slide-rock mode when $\ddot{x}_O \le 0, f_x \ge 0$.

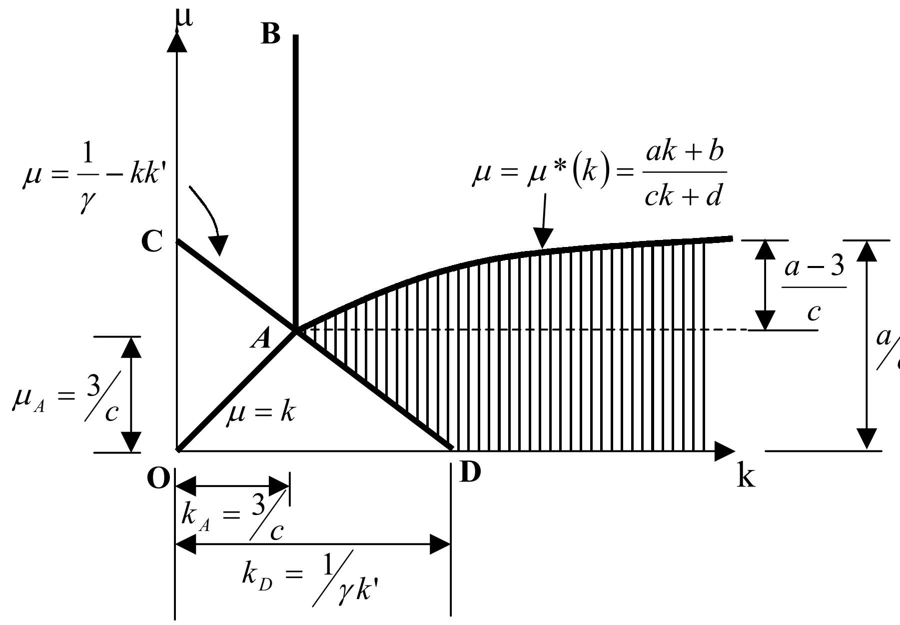

Fig.3.6a Slide-rock region, case 1, $0 \le k' \le \frac{1+\gamma^2}{3\gamma^2}$, $f_x \ge 0$, $a \ge 0$

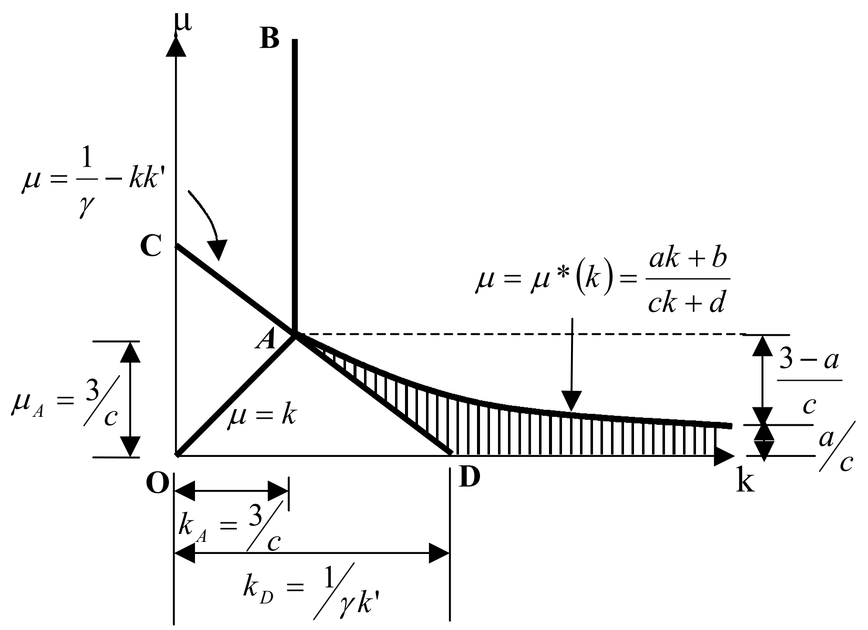

Fig.3.6b Slide-rock region, case 1, $\frac{1+\gamma^2}{3\gamma^2} \le k' \le \frac{4+\gamma^2}{3\gamma^2} \le 1$, $f_x \ge 0$, $a \ge 0$

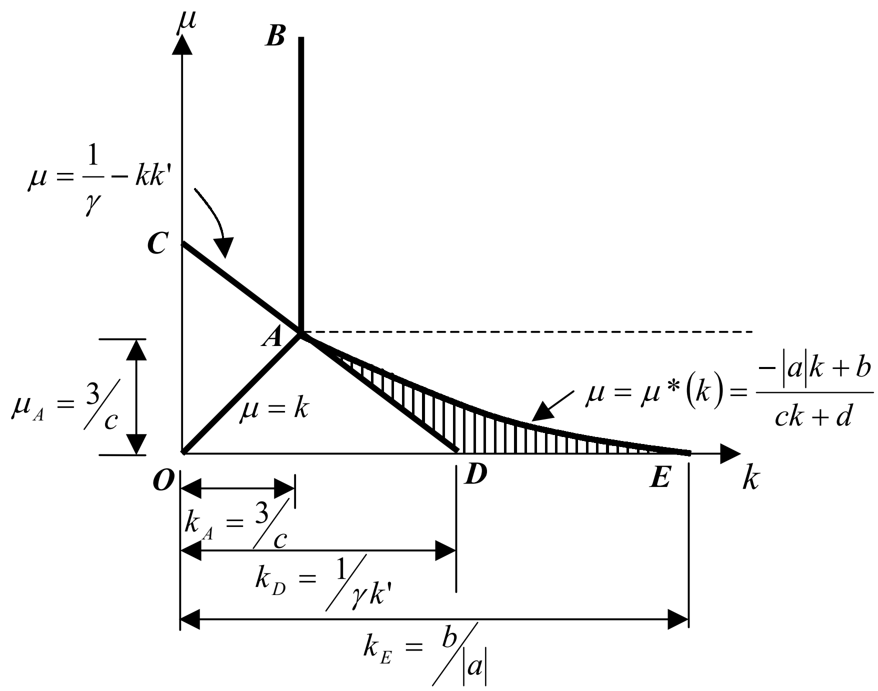

Fig.3.7 Slide-rock region, case 2, $\frac{4+\gamma^2}{3\gamma^2} \le k' \le 1$, $f_x \ge 0$, $a \le 0$

For the case of $\ddot{x} \ge 0$ (body moving to the right) and $f_x \le 0$ (f_x pointing to the left), we repeat the steps leading to the expression for $\ddot{\theta}$ in (3.30), we get, for the present case,

$$\ddot{\theta} = \frac{3g}{B(4+\gamma^2+3\mu\gamma)}(kk'\gamma - \mu\gamma - 1) \tag{3.43}$$

For $\ddot{\theta} \ge 0$, it is necessary that

$$\mu \le -\frac{1}{\gamma} + kk' \tag{3.44}$$

By using (3.43) and the relation $\ddot{x}_O = \ddot{x} + H\ddot{\theta}$ we get

$$\begin{aligned} \ddot{x}_O &= -\frac{g}{4+\gamma^2+3\mu\gamma}\{\mu[3\gamma(1+k')k+1+4\gamma^2]+[4+\gamma^2-3\gamma^2k')k+3\gamma]\} \\ &= -\frac{g}{4+\gamma^2+3\mu\gamma}[\mu(ck+d)+(ak+b)] \end{aligned} \tag{3.45}$$

For $\ddot{x}_O \geq 0$, we must have $\mu \leq \dfrac{\left|-|a|k+b\right|}{ck+d} = \mu^*(k)$ for $k \geq \dfrac{b}{|a|}$.

The curve $\mu = \mu^*(k)$ is presented in Fig.3.8. It is easy to see that the dotted line lies above curve $\mu = \mu^*(k)$ and the shaded region represents the case for initiation of slide-rock mode with $\ddot{x}_O \geq 0$.

To summarize, regions of rest, slide, rock and slide-rock are shown in Figs.3.9a, 3.9b, 3.10 and 3.11 respectively for the cases of $0 \leq k' \leq \dfrac{1+\gamma^2}{3\gamma^2}$, $f_x \geq 0$, $a \geq 0$; $\dfrac{1+\gamma^2}{3\gamma^2} \leq k' \leq \dfrac{4+\gamma^2}{3\gamma^2} \leq 1$, $f_x \geq 0$, $a \geq 0$; $\dfrac{4+\gamma^2}{3\gamma^2} \leq k' \leq 1$, $f_x \geq 0$, $a \leq 0$ and $\dfrac{4+\gamma^2}{3\gamma^2} \leq k' \leq 1$, $f_x \leq 0$, $a \leq 0$.

For brevity, the modes are represented by the symbols *RE* (rest), *SL* (slide), *RO* (rock about O), SRO_- (slide-rock about O with $\dot{x}_o \leq 0$) and SRO_+ (slide-rock about O with $\dot{x}_o \geq 0$).

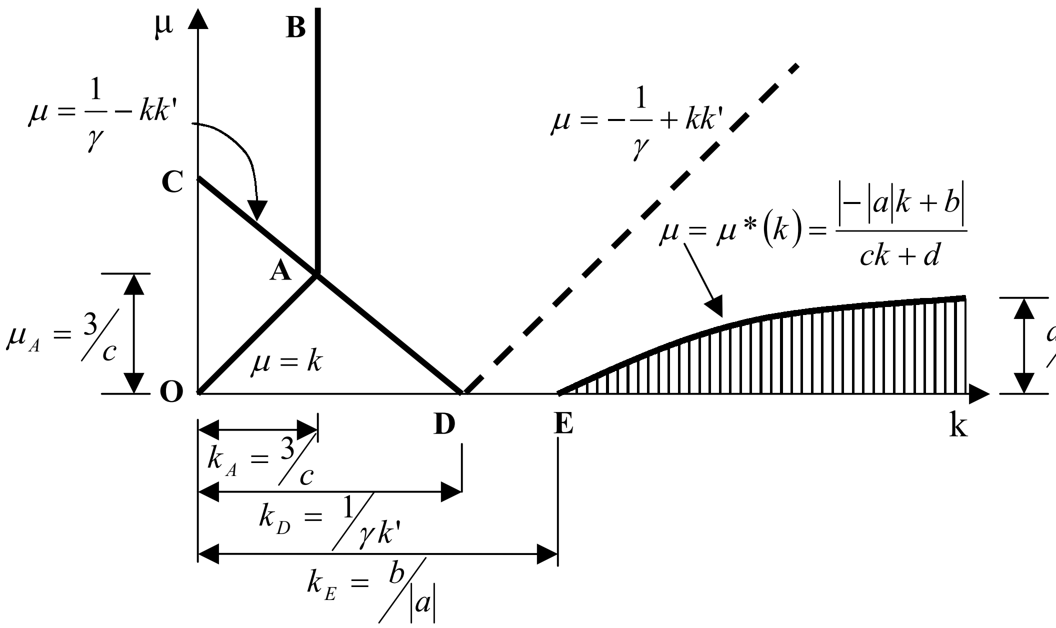

Fig.3.8 Slide-rock region, case 3, $\dfrac{4+\gamma^2}{3\gamma^2} \leq k' \leq 1$, $f_x \leq 0$, $a \leq 0$

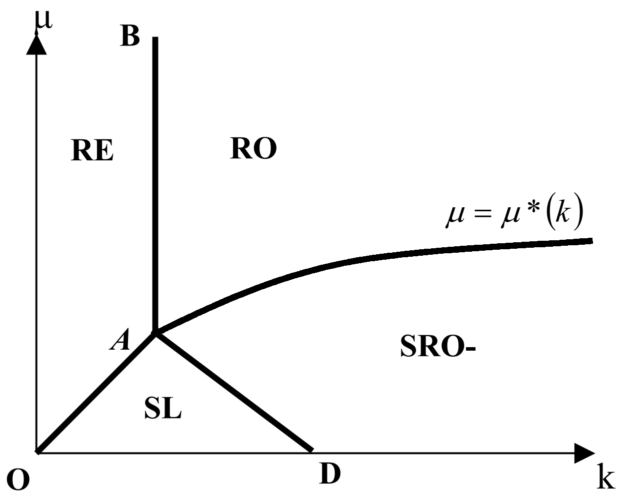

Fig.3.9a All modes, case 1, $0 \le k' \le \dfrac{1+\gamma^2}{3\gamma^2}$, $f_x \ge 0$, $a \ge 0$

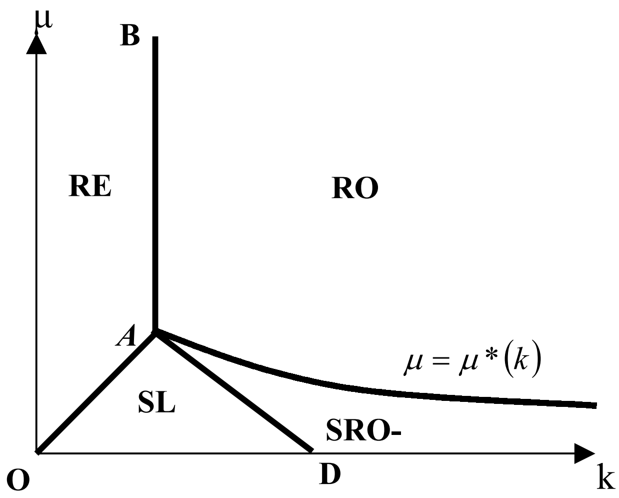

Fig.3.9b All modes, case 1, $\dfrac{1+\gamma^2}{3\gamma^2} \le k' \le \dfrac{4+\gamma^2}{3\gamma^2} \le 1$, $f_x \ge 0$, $a \ge 0$

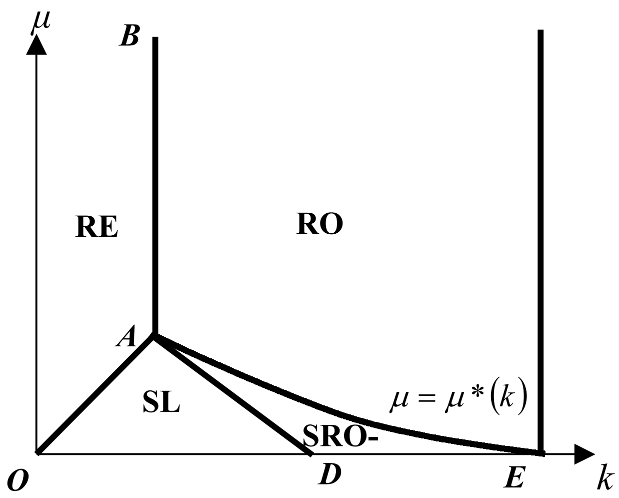

Fig.3.10 All modes, case 2, $\frac{4+\gamma^2}{3\gamma^2} \le k' \le 1$, $f_x \ge 0$, $a \le 0$

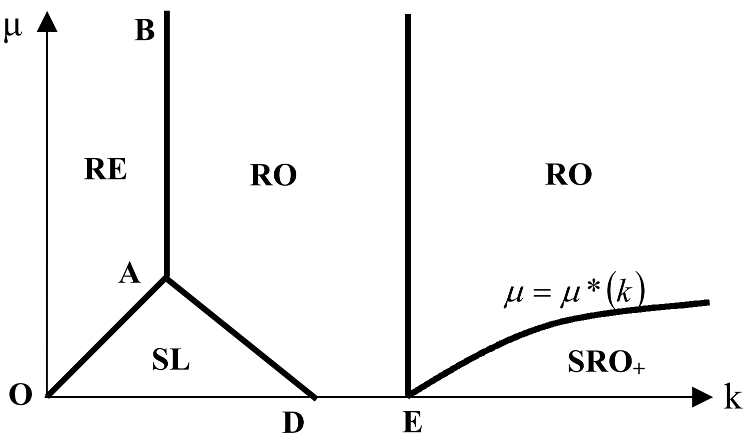

Fig.3.11 All modes, case 3, $\frac{4+\gamma^2}{3\gamma^2} \le k' \le 1$, $f_x \le 0$, $a \le 0$

IV. Criteria for initiation of modes of response of a freestanding rigid body under the action of a horizontal force applied below center of mass

IV.1. Rest

When the body is at rest under the action of F (see Fig.4.1), equations of equilibrium are:

$$f_x = F \tag{4.1}$$

$$f_y = mg \tag{4.2}$$

and, by summing moment of the forces about point C,

$$f_x H + f_y \xi - F|k'|H = 0 \tag{4.3}$$

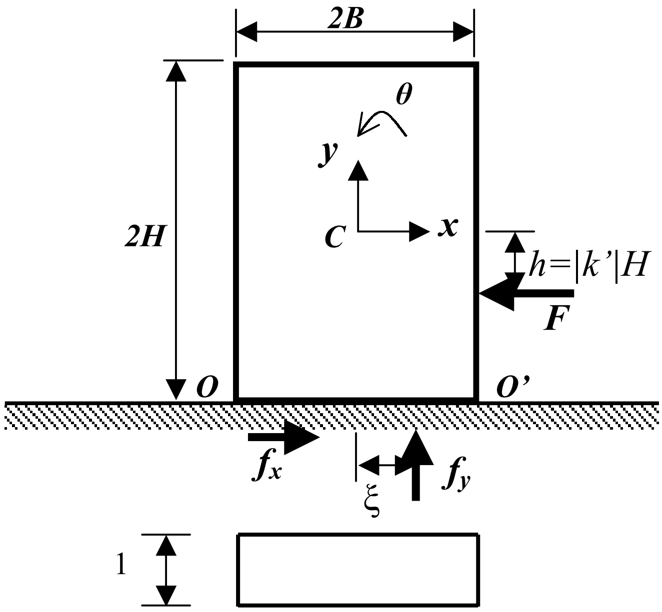

Fig.4.1 Model

For the body not to leave the base, we must have

$$f_y \geq 0 \tag{4.4}$$

For the body to be at rest, horizontal resisting force f_x must not exceed limiting frictional force μf_y. That is,

$$|f_x| \leq \mu |f_y| \tag{4.5}$$

Furthermore, f_y must necessarily act within width ($O - O'$) of base of the body. That is,

$$|\xi| \leq B \tag{4.6}$$

Condition that $f_y \geq 0$ is always satisfied on account of (4.2). Condition (4.5) requires, from (4.1) and (4.2)

$$\mu \geq k \tag{4.7}$$

By substituting (4.1) and (4.2) into (4.3), we have

$$\xi = -kH(1 - |k'|) \leq 0 \tag{4.8}$$

since

$$0 \leq |k'| \leq 1 \tag{4.9}$$

Equation (4.8) means, under the action of F, when the body is at rest, f_y acts to the left of center of mass C.

Condition (4.6) requires

$$k \leq \frac{1}{\gamma(1 - |k'|)} \tag{4.10}$$

Conditions (4.7) and (4.10) that ensure the body to remain at rest under the action of F are represented by the shaded region in $k - \mu$ plane in Fig.4.2. We see that rest region increases with increasing value of $|k'|$. That is, the likelihood of the body remaining at rest becomes higher as F is closer to the bottom of the body. This is different from the situation when F is applied above center of mass C. As shall be seen later, behavior of the body is quite different depending on whether F is applied above or below C.

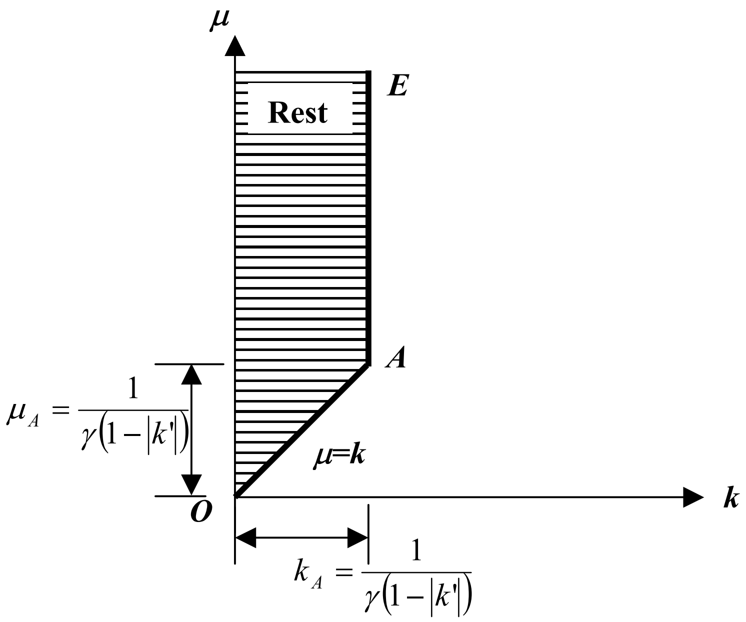

Fig.4.2 Rest region

IV.2. Slide

Equations of motion are:

$$m\ddot{x} = f_x - F \tag{4.11}$$

$$f_y = mg \tag{4.12}$$

and, by summing moment of the forces about point C,

$$f_x H + f_y \xi - F|k'|H = 0 \tag{4.13}$$

where, since sliding is impending,

$$f_x = \mu f_y \tag{4.14}$$

Again, vertical reaction force f_y must not be negative. That is,

$$f_y \geq 0 \tag{4.15}$$

Also, f_y must be within boundaries ($O - O'$) of base of the body. Thus,

$$|\xi| \leq B \tag{4.16}$$

Since f_y equals weight of the body, (4.15) is always satisfied.

Equation (4.13) gives

$$\xi = (k|k'| - \mu)H \tag{4.17}$$

Sign of ξ depends on that of $k|k'| - \mu$. If the moment of force F about C exceeds that due to f_x and f_y, $k|k'| \geq \mu$, in which case, $\xi \geq 0$ and reaction force f_y acts to the right of C. On the other hand, if $k|k'| \leq \mu$, then $\xi \leq 0$, and f_y acts to the left of C. The requirement imposed by (4.16) therefore is satisfied if, for $k|k'| \geq \mu$,

$$\mu \geq -\frac{1}{\gamma} + k|k'| \tag{4.18}$$

and, for $k|k'| \leq \mu$,

$$\mu \leq \frac{1}{\gamma} + k|k'| \tag{4.19}$$

The parallel lines DC

$$\mu = -\frac{1}{\gamma} + k|k'| \tag{4.20}$$

and AB

$$\mu = \frac{1}{\gamma} + k|k'| \tag{4.21}$$

and the dotted line

$$\mu = k|k'| \tag{4.22}$$

are drawn in Fig.4.3. The shaded region corresponds to initiation of slide mode of response. It is bounded by horizontal axis, line OA, and the lines DC ((4.20)) and AB ((4.21)). The shaded region above line $\mu = k|k'|$ ((4.22)) corresponds to the situation when $\xi \le 0$ (f_y acts to the left of C) and the region below line $\mu = k|k'|$ ((4.22)) corresponds to the situation when $\xi \ge 0$ (f_y acts to the right of C). The lines AB and DC as well as (4.22) extend to infinity without bound.

For slide mode to be initiated, from (4.11),

$$m\ddot{x} = mg(\mu - k) \le 0 \tag{4.23}$$

since $\mu \le k$. This means, under the action of the left pointing force F, $\ddot{x} \le 0$. Since sliding is the mode of response being considered, accelerations of all points in the body are the same and point to the left, and since the body is initially at rest, velocities of all the points in the body also point to the left, as expected.

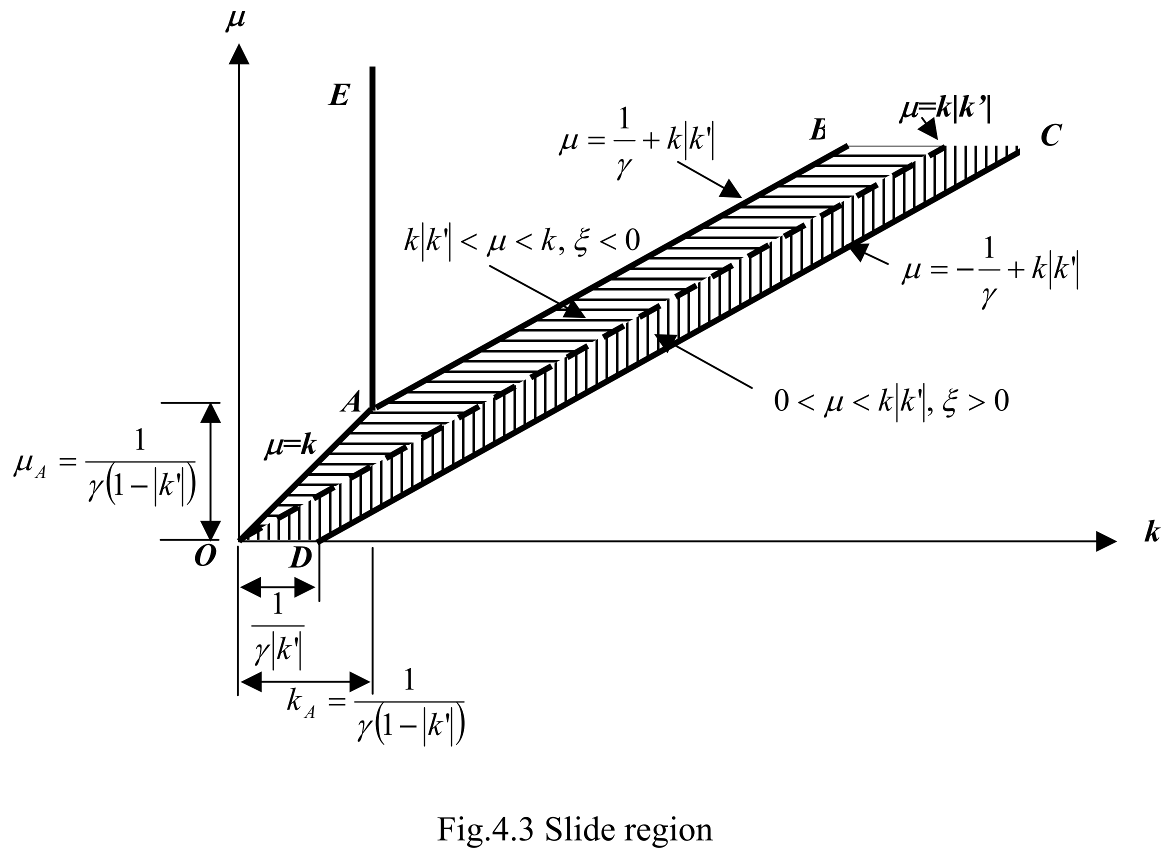

Fig.4.3 Slide region

We note that in Fig.4.3, point D where line DC intersects k-axis may be to the left or the right of point A. It may be verified that when $|k'| \le \frac{1}{2}$, point D lies to the right of A; otherwise, it lies to the left of A. Also, the larger the value of $|k'|$, the steeper the lines AB, DC and $\mu = k|k'|$. Again, criteria for initiation of the body into slide mode are different for the cases of F acting above and below C.

IV.3. Rock about the point O

Equations of motion are:

$$m\ddot{x} = f_x - F \tag{4.24}$$

$$m\ddot{y} = f_y - mg \tag{4.25}$$

and, by taking moment of the forces about point C, noting that f_y acts at point O (see Fig.4.1) about which the body rotates,

$$I\ddot{\theta} = f_x H - f_y B - F|k'|H \tag{4.26}$$

where $I = \dfrac{1}{3}m(H^2 + B^2)$ is mass moment of inertia of the body about C.

From (4.24),

$$f_x = m\ddot{x} + F \tag{4.27}$$

and, from (4.25),

$$f_y = m\ddot{y} + mg \tag{4.28}$$

When the body is about to rotate about O, $\ddot{x}$ and $\ddot{y}$ are related to $\ddot{\theta}$ as

$$\ddot{x} = -H\ddot{\theta} \tag{4.29}$$

and

$$\ddot{y} = B\ddot{\theta} \tag{4.30}$$

Thus, from (4.27) and (4.28),

$$f_x = m(gk - H\ddot{\theta}) \tag{4.31}$$

and

$$f_y = m(g + B\ddot{\theta}) \tag{4.32}$$

Conditions for impending rocking motion about point O are:

$$\ddot{\theta} \geq 0 \tag{4.33}$$

$$f_y \geq 0 \tag{4.34}$$

and

$$|f_x| \leq \mu |f_y| \tag{4.35}$$

From (4.26), (4.31) and (4.32),

$$\ddot{\theta} = \frac{3g}{4B(1+\gamma^2)}[k\gamma(1-|k'|)-1] \tag{4.36}$$

Condition (4.33) requires

$$k \geq \frac{1}{\gamma(1-|k'|)} \tag{4.37}$$

That is, rocking about O can be initiated only if values of k and μ are such that the point on the $k-\mu$ plane lies to the right of line AE in Fig.4.2 (and Fig.4.3).

Having determined $\ddot{\theta}$, (4.31) and (4.32) give

$$f_x = \frac{mg}{4(1+\gamma^2)}\{k[4+\gamma^2+3\gamma^2|k'|]+3\gamma\} \geq 0 \tag{4.38}$$

and

$$f_y = \frac{mg}{4(1+\gamma^2)}[3\gamma(1-|k'|)k+1+4\gamma^2] \geq 0 \tag{4.39}$$

This shows that f_x points to the right and the condition $f_y \geq 0$ ((4.34)) is satisfied.

In order that $|f_x| \leq \mu |f_y|$, we must have

$$\mu \geq \frac{[4+\gamma^2+3\gamma^2|k'|]k+3\gamma}{3\gamma(1-|k'|)k+1+4\gamma^2} = \mu^*(k) \tag{4.40}$$

That is, for the body to rock about point O without sliding, coefficient of friction must be sufficiently large. For simplicity, we may express $\mu^*(k)$ as

$$\mu^*(k) = \frac{ak+b}{ck+d} \tag{4.41}$$

where

$$a = 4+\gamma^2+3\gamma^2|k'| \tag{4.42}$$

$$b = 3\gamma \tag{4.43}$$

$$c = 3\gamma(1-|k'|) \tag{4.44}$$

and

$$d = 1+4\gamma^2 \tag{4.45}$$

The curve $\mu^*(k)$ has the following properties:

$\mu^*(k)$ passes point A, since, when $k = \dfrac{1}{\gamma(1-|k'|)} = k_A = \dfrac{3}{c}$, from (4.41), $\mu = \mu^*(k)$ $= \mu_A = \dfrac{3}{c}$. As $k \to \infty$,

$$\mu^* = \frac{a}{c} = \frac{4+\gamma^2+3\gamma^2|k'|}{3\gamma(1-|k'|)} \geq 0 \tag{4.46}$$

Slope of $\mu^*(k)$ is

$$\frac{d\mu^*}{dk} = \frac{ad-bc}{(ck+d)^2} = \frac{4(1+\gamma^2)(1+\gamma^2+3\gamma^2|k'|)}{[3k\gamma(1-|k'|)+1+4\gamma^2]^2} \geq 0 \tag{4.47}$$

which is a monotonically decreasing function of k, and the numerator is independent of k. As $k \to \infty$, $\dfrac{d\mu^*}{dk} = 0$.

If we compare slope of $\mu^*(k)$ at point A with $|k'|$, slope of line AB ($\mu = \frac{1}{\gamma} + k|k'|$), we see that the difference is $\Delta = \frac{(1+\gamma^2)-(4+\gamma^2)|k'|}{4(1+\gamma^2)}$. Two situations arise: $\Delta \geq 0$, ($|k'| \leq \frac{1+\gamma^2}{4+\gamma^2}$) and $\Delta \leq 0$, ($|k'| \geq \frac{1+\gamma^2}{4+\gamma^2}$). In Figs.4.4a and 4.4b, sketches of curve $\mu^*(k)$ together with lines AB ($\mu = \frac{1}{\gamma} + k|k'|$) and DC ($\mu = -\frac{1}{\gamma} + k|k'|$) are given for each of the two cases (In this and subsequent figures, line CD is drawn such that point D lies to the left of A for the case of $k' \geq \frac{1}{2}$; see comments at end of IV.2. Slide section). For the former case ($\Delta \geq 0$), $\mu^*(k)$ and AB ($\mu = \frac{1}{\gamma} + k|k'|$) intersect at point A and point B' whose coordinates may be shown to be given by

$$k = \frac{1+\gamma^2}{3\gamma|k'|} = k_{B'} \tag{4.48}$$

and

$$\mu = \frac{4+\gamma^2}{3\gamma} = \mu_{B'} \tag{4.49}$$

For the latter case ($\Delta \leq 0$), it may be verified that $k_{B'} \leq k_A$. In Figs.4.4a and 4.4b the shaded regions correspond to initiation of rock (about point O) mode. It is seen that regions of rock and slide modes overlap. Since these modes of response must preclude one another, this matter of overlapping regions must be resolved. This problem will be addressed later in IV.7. Discussion. In the case of F applied above point C, on account of the fact that the quantity a in (3.18a) may be either greater, equal to or smaller than 0, whereas in the present case, the quantity a in (4.42) is always greater or equal to 0, criteria for initiation of the body into the rock (about point O) mode for these two cases are different.

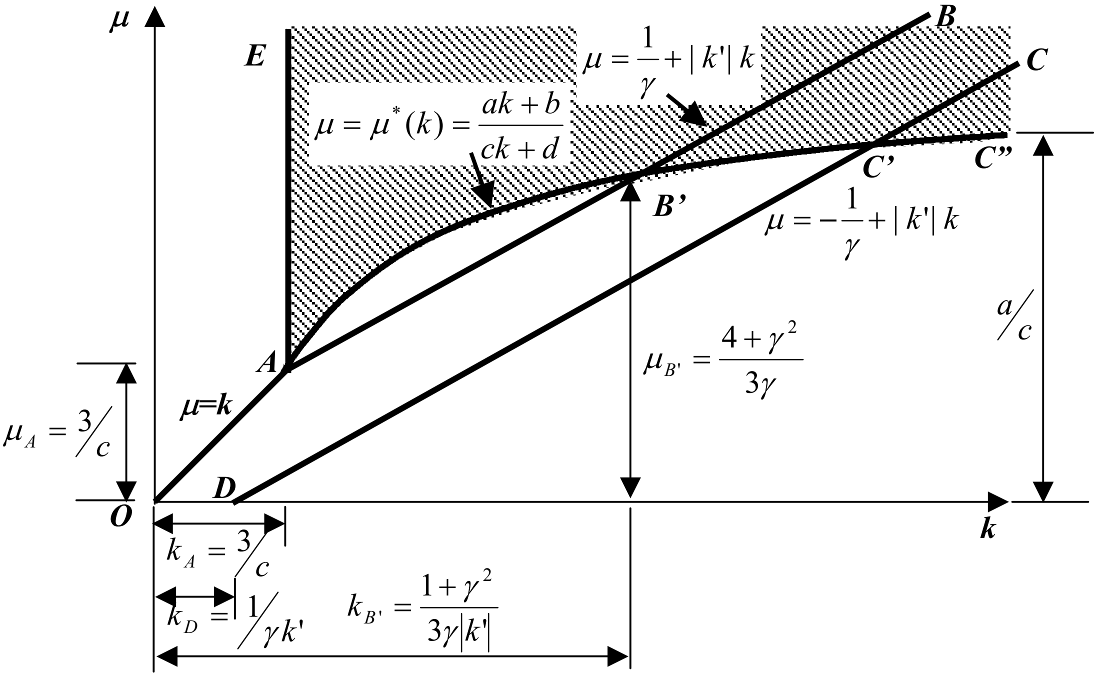

Fig.4.4a Rock region, $|k'| \le \frac{1+\gamma^2}{4+\gamma^2}$

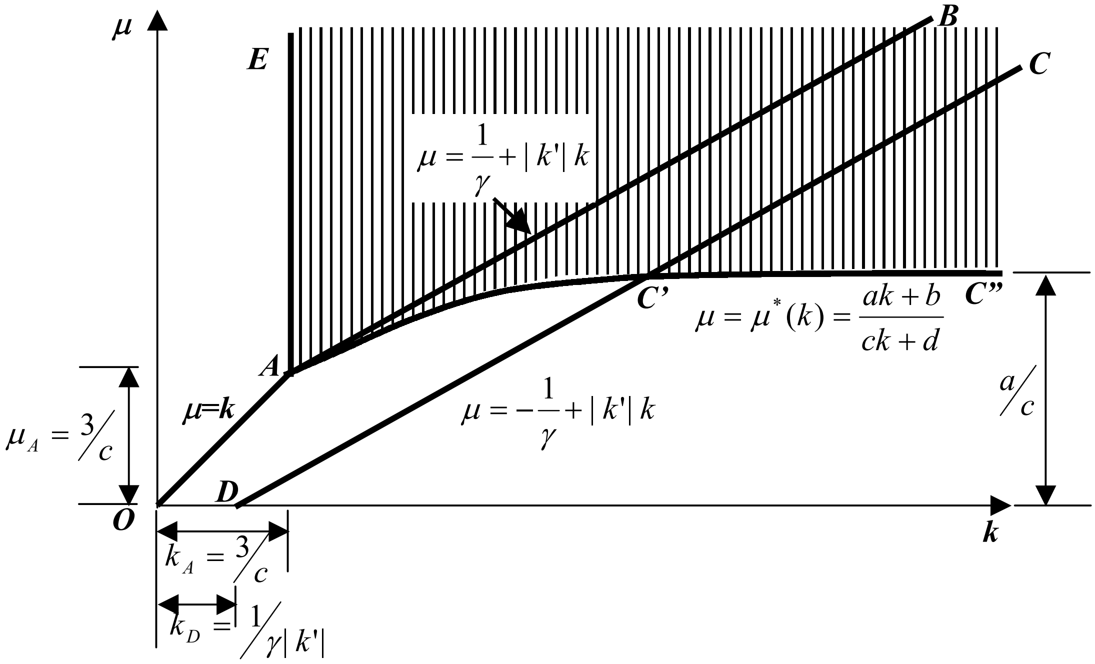

Fig.4.4b Rock region, $|k'| \ge \frac{1+\gamma^2}{4+\gamma^2}$

IV.4. Rock about the point O’

Although it is clear that the body is unlikely to be able to rock about point O' under the action of F which is directed to the left (see Fig.4.1), we will nevertheless show that this is indeed the case.

Equations of motion are:

$$m\ddot{x} = f_x - F \tag{4.50}$$

$$m\ddot{y} = f_y - mg \tag{4.51}$$

and, with f_y acting at O', by taking moment of the forces about point C.

$$I\ddot{\theta} = f_x H + f_y B - F|k'|H \tag{4.52}$$

When the body is about to rock about O', noting that θ is positive counterclockwise,

$$\ddot{x} = -H\ddot{\theta} \tag{4.53}$$

and

$$\ddot{y} = -B\ddot{\theta} \tag{4.54}$$

Thus, from (4.50) and (4.51),

$$f_x = m(gk - H\ddot{\theta}) \tag{4.55}$$

$$f_y = m(g - B\ddot{\theta}) \tag{4.56}$$

Considering $I = \frac{1}{3}m(H^2 + B^2)$, $F = mgk$, and taking into account (4.55) and (4.56), we have, from (4.52),

$$\ddot{\theta} = \frac{3g}{4B(1+\gamma^2)}[1 + k\gamma(1-|k'|)] > 0 \tag{4.57}$$

which is physically impossible to be realized.

IV.5. Slide-rock about the point O

When the body is on the verge of sliding to the left and rocking about point O simultaneously, reaction force f_y acts at point O and equations of motion are:

$$m\ddot{x} = f_x - F \tag{4.58}$$

$$m\ddot{y} = f_y - mg \tag{4.59}$$

and

$$I\ddot{\theta} = f_x H - f_y B - F|k'|H \tag{4.60}$$

In this slide-rock mode,

$$f_x = \mu f_y \tag{4.61}$$

and

$$\ddot{y} = B\ddot{\theta} \tag{4.62}$$

so that from (4.59) and (4.62)

$$f_y = m\ddot{y} + mg = m(g + B\ddot{\theta}) \tag{4.63}$$

By substituting f_x and f_y in (4.61) and (4.63) into (4.60) and noting that $F = mgk$, we have

$$\ddot{\theta} = \frac{3g}{B}\frac{(\mu\gamma - 1 - k|k'|\gamma)}{(4 + \gamma^2 - 3\mu\gamma)} \tag{4.64}$$

For slide-rock (about O) to be initiated, it is necessary that

$$\ddot{\theta} \geq 0 \tag{4.65}$$

and

$$f_y \geq 0 \tag{4.66}$$

From (4.63), it is clear that so long as condition $\ddot{\theta} \geq 0$ is satisfied, so is condition (4.66). For (4.65) to hold, from (4.64), we must have either both the numerator and the denominator larger than or equal to zero, or both of them smaller than or equal to zero. That is, either

$$\mu \geq \frac{1}{\gamma} + k|k'| \tag{4.67}$$

and

$$\mu \leq \frac{4+\gamma^2}{3\gamma} = \mu_{B'} \tag{4.68}$$

or

$$\mu \leq \frac{1}{\gamma} + k|k'| \tag{4.69}$$

and

$$\mu \geq \frac{4+\gamma^2}{3\gamma} = \mu_{B'} \tag{4.70}$$

To satisfy (4.67) and (4.68), we must have

$$\frac{4+\gamma^2}{3\gamma} \geq \frac{1}{\gamma} + k|k'| \tag{4.71}$$

or

$$k \leq \frac{1+\gamma^2}{3\gamma|k'|} = k_{B'} \tag{4.72}$$

It can be seen from (4.48) that the above is indeed equal to $k_{B'}$. Similarly, to satisfy (4.69) and (4.70), we must have

$$k \geq \frac{1+\gamma^2}{3\gamma|k'|} = k_{B'} \tag{4.73}$$

For clarity, it is repeated that to satisfy condition $\ddot{\theta} \geq 0$ we must satisfy either (4.67) ($\mu \geq \frac{1}{\gamma} + k|k'|$) and (4.72) ($k \leq k_{B'}$) or (4.69) ($\mu \leq \frac{1}{\gamma} + k|k'|$) and (4.73) ($k \geq k_{B'}$).

For the body to start to slide to the left, velocity at O must be less than or equal to zero ($\dot{x}_O \leq 0$). Since the body is initially at rest, condition $\dot{x}_O \leq 0$ is equivalent to $\ddot{x}_O \leq 0$. It may be verified that when the body is in this slide-rock about O' mode of motion,

$$\ddot{x}_O = \ddot{x} + H\ddot{\theta} \tag{4.74}$$

Since, from (4.58), (4.61), and (4.63)

$$\ddot{x} = \mu(g + B\ddot{\theta}) - gk \tag{4.75}$$

we have, from (4.74) and (4.64)

$$\ddot{x}_O = \frac{g}{4+\gamma^2 - 3\mu\gamma}\{\mu[3\gamma(1-|k'|)k + 1 + 4\gamma^2] - [(4+\gamma^2+3\gamma^2|k'|)k + 3\gamma]\} \tag{4.76}$$

$\ddot{x}_O \leq 0$ can be realized if the numerator enclosed in the curly brackets and the denominator are of opposite signs. That is, either

$$\mu \leq \frac{(4+\gamma^2+3\gamma^2|k'|)k + 3\gamma}{3\gamma(1-|k'|)k + 1 + 4\gamma^2} = \mu^*(k) \tag{4.77}$$

and

$$\mu \leq \frac{4+\gamma^2}{3\gamma} = \mu_{B'} \tag{4.78}$$

or, equivalently,

$$k \leq \frac{1+\gamma^2}{3\gamma|k'|} = k_{B'} \tag{4.79}$$

or

$$\mu \geq \mu^*(k) \tag{4.80}$$

and

$$\mu \geq \frac{4+\gamma^2}{3\gamma} = \mu_{B'} \tag{4.81}$$

or, equivalently,

$$k \geq \frac{1+\gamma^2}{3\gamma|k'|} = k_{B'} \tag{4.82}$$

Combining (4.67), (4.72) (for $\ddot{\theta} \geq 0$) and (4.77), (4.79) (for $\ddot{x}_O \leq 0$) regions for initiation of mode of slide-rock about O are given in Fig.4.5a for the case $|k'| \leq \frac{1+\gamma^2}{4+\gamma^2}$ ($\Delta \geq 0$, Δ being the difference of slope of $\mu^*(k)$ at point A and $|k'|$, slope of line AB defined in IV.3. Rock about point O). Conditions (4.69), (4.73) (for $\ddot{\theta} \geq 0$) and (4.80), (4.82) (for $\ddot{x}_O \leq 0$) combine to give the shaded region in Fig.4.5b for the case $|k'| \geq \frac{1+\gamma^2}{4+\gamma^2}$ ($\Delta \leq 0$). Again, we see that region of slide-rock about O overlaps region of slide and region of rock about O. This issue will be resolved later in IV.7. Discussion.

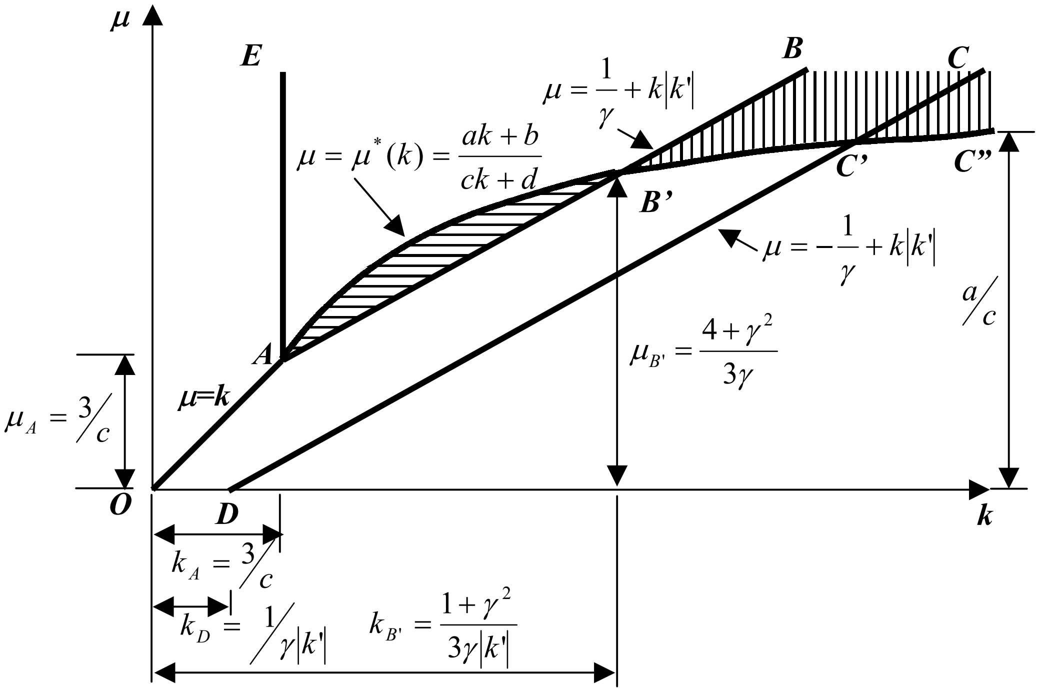

Fig.4.5a Slide-Rock (about O) region, $|k'| \le \frac{1+\gamma^2}{4+\gamma^2}$

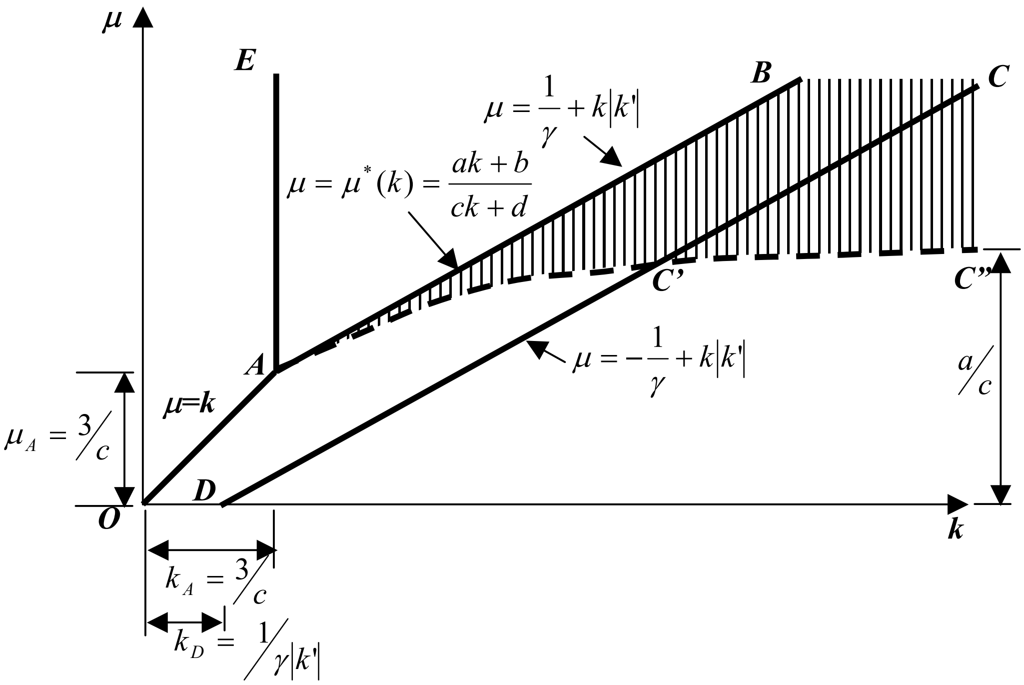

Fig.4.5b Slide-Rock (about O) region, $|k'| \ge \frac{1+\gamma^2}{4+\gamma^2}$

IV.6. Slide-rock about the point O’

It remains to examine conditions for initiation of mode of motion when sliding and rocking about point O' occur simultaneously (see Fig.4.1).

Equations of motion are:

$$m\ddot{x} = f_x - F \tag{4.83}$$

$$m\ddot{y} = f_y - mg \tag{4.84}$$

and, noting that f_y acts at O',

$$I\ddot{\theta} = f_x H + f_y B - F|k'|H \tag{4.85}$$

Here,

$$f_x = \mu f_y \tag{4.86}$$

and since center of rotation is O',

$$\ddot{y} = -B\ddot{\theta} \tag{4.87}$$

From (4.84)

$$f_y = m(g - B\ddot{\theta}) \tag{4.88}$$

By substituting (4.86) and (4.88) into (4.85), noting that $F = mgk$, we have

$$\ddot{\theta} = \frac{3g}{B(4+\gamma^2+3\mu\gamma)}(\mu\gamma + 1 - k|k'|\gamma) \tag{4.89}$$

For this mode of response to take place the following conditions must be met:

$$\ddot{\theta} \le 0 \tag{4.90}$$

and

$$f_y \ge 0 \tag{4.91}$$

From (4.89), for $\ddot{\theta} \le 0$, we must have

$$\mu \le -\frac{1}{\gamma} + k|k'| \tag{4.92}$$

If (4.90) is satisfied, so would (4.91) be, on account of (4.88).

For the body to slide to the left, we must have $\dot{x}_{O'} \le 0$ or equivalently, $\ddot{x}_{O'} \le 0$.

Since

$$\ddot{x}_{O'} = \ddot{x} + H\ddot{\theta} \tag{4.93}$$

and, from (4.83), (4.86) and (4.88),

$$\ddot{x} = g(\mu - k) - \mu B\ddot{\theta} \tag{4.94}$$

By using (4.89), from (4.93),

$$\ddot{x}_{O'} = \frac{g}{4+\gamma^2+3\mu\gamma}\{\mu[(1+4\gamma^2) - 3\gamma(1-|k'|)k] - [(4+\gamma^2+3\gamma^2|k'|)k - 3\gamma]\} \tag{4.95}$$

For $\ddot{x}_{O'} \le 0$ it is necessary that

$$\mu \le \frac{(4+\gamma^2+3\gamma|k'|)k - 3\gamma}{1+4\gamma^2 - 3\gamma(1-|k'|)k} \equiv \mu^{**}(k) \tag{4.96}$$

a function of k. We now study properties of $\mu^{**}(k)$. First, it is recognized that

$$\mu^{**}(k) = \frac{ak-b}{d-ck} \tag{4.97}$$

where the quantities a, b, c, and d are constants defined in (4.42), (4.43), (4.44) and (4.45).

At $k=0$, $\mu^{**} = -\frac{b}{d} \le 0$. At $k = \frac{d}{c}$, μ^{**} approaches infinity and at $k = \frac{b}{a}$, $\mu^{**} = 0$.

Slope of $\mu^{**}(k)$ is $\frac{d\mu^{**}}{dk} = \frac{ad-bc}{(d-ck)^2} = \frac{4(1+\gamma^2)(1+\gamma^2+3\gamma^2|k'|)}{(d-ck)^2} \ge 0$ which approaches infinity as k approaches $\frac{d}{c}$.

From (4.92) which stems from the requirement that $\ddot{\theta} \leq 0$ and (4.96), required by the condition that $\ddot{x}_{O'} \leq 0$, the regions corresponding to slide-rock about O' mode of response are given in Figs.4.6a and 4.6b for the two cases $\Delta \geq 0$ and $\Delta \leq 0$ (For definition of Δ, see IV.3. Rock about point O). It may be verified that point F of intersection of the curve $\mu^{**}(k)$ with k-axis lies to the left of point D, point of intersection of line $\mu = -\dfrac{1}{\gamma} + k|k'|$ and the k-axis.

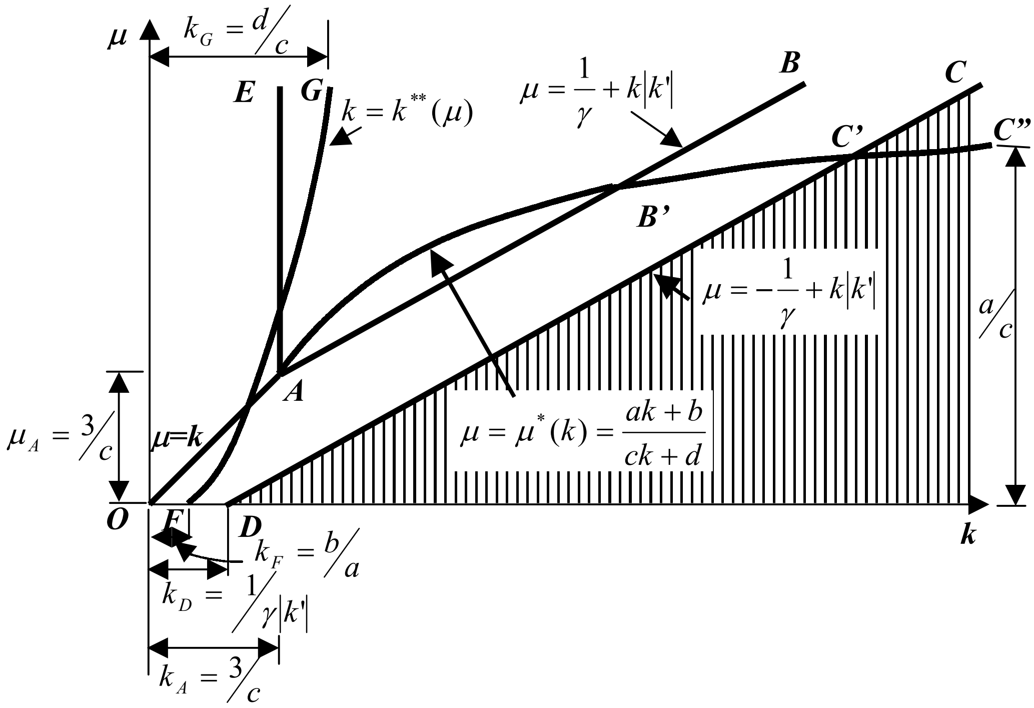

Fig.4.6a Slide-rock (about O') region, $|k'| \le \frac{1+\gamma^2}{4+\gamma^2}$

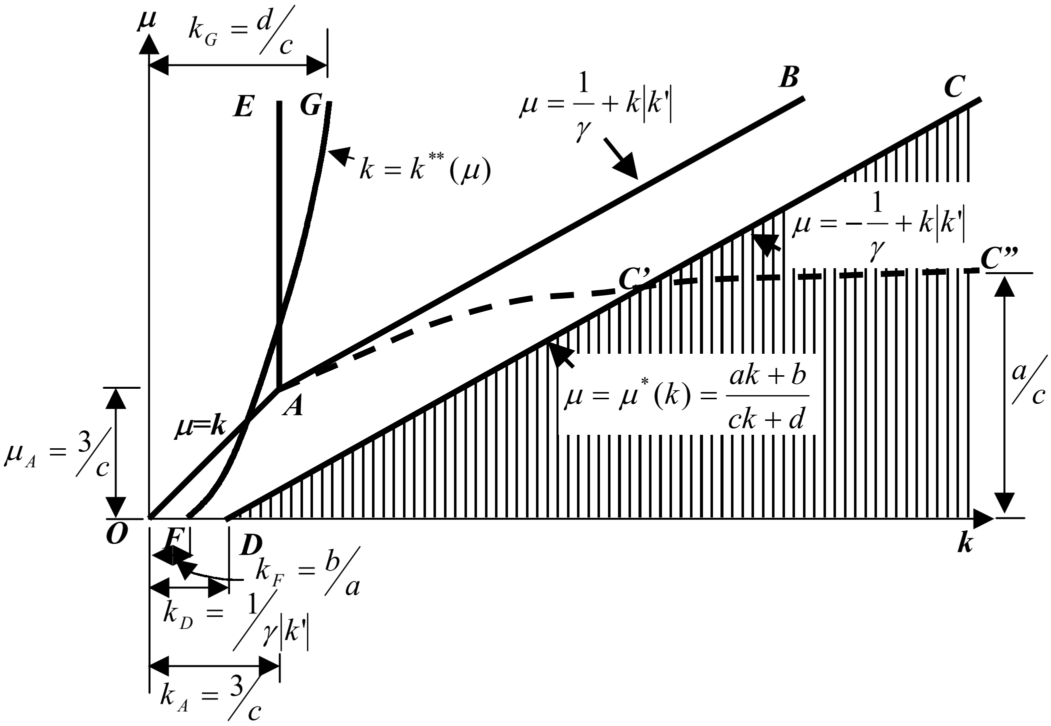

Fig.4.6b Slide-rock (about O') region, $|k'| \ge \frac{1+\gamma^2}{4+\gamma^2}$

IV.7. Discussion

The above constitutes an analysis of conditions governing initiation of various possible modes of response of a rigid body under the action of a horizontal force pointing to the left and applied below center of mass of the body.

Unlike the case in which the force F is applied above center of mass of the body, in the present case, some of the regions in the $k-\mu$ plane of the various modes of response overlap. This situation requires further analysis to determine how these overlapping regions should be dealt with.

In Fig.4.3, when values of k and μ correspond to a point in the shaded region, sliding is imminent. Reaction f_y acts within base OO' (Fig.4.1), away from edges O and O'. Other than for points in the $k-\mu$ plane whose coordinates k and μ are such that they fall on the boundary AB where f_y acts at O and at the boundary CD where f_y acts at O', other points inside the slide region all correspond to the case in which f_y lies between but not at O and O'. Thus, no rocking about O nor about O' can take place in this region other than for values of k and μ corresponding to points on the boundaries AB and DC. For this reason, region $BB'C'C$ for rock about O in Fig.4.4a and region $BAC'C$ in Fig.4.4b should be eliminated. Along same line of reasoning, points in sectors $CC'C''$ in Figs.4.4a and 4.4b do not give rise to rocking about O since, if rocking indeed takes place, it does so about O' because line DC ($C'C$) represents the case when reaction f_y acts at O' ($\xi = B$).

For the same reason, in the case of slide-rock about O in Fig.4.5a, regions $BB'C'C$ and $CC'C''$ should be deleted. For the case represented by Fig.4.5b, slide-rock about O simply can not happen and the shaded region $BAC'C''$ should not be counted. For the case of slide-rock about O', the entire shaded regions shown in Figs.4.6a and 4.6b are valid representations of mode of slide-rock about O'.

The final result is shown in Figs.4.7a and 4.7b for the cases of $|k'| \le \frac{1+\gamma^2}{4+\gamma^2}$ and $|k'| \ge \frac{1+\gamma^2}{4+\gamma^2}$. Here for brevity, the modes are represented by symbols *RE* (rest), *SL* (slide), *RO* (rock about point O), SRO_- (slide-rock about O with $\dot{x} \le 0$), and SRO'_- (slide-rock about O' with $\dot{x} \le 0$).

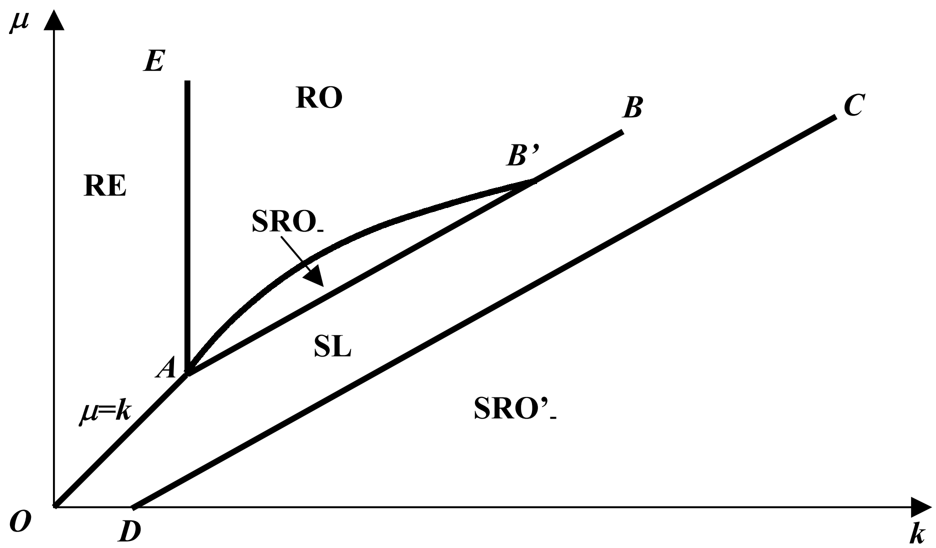

Fig.4.7a All modes $|k'| \le \frac{1+\gamma^2}{4+\gamma^2}$

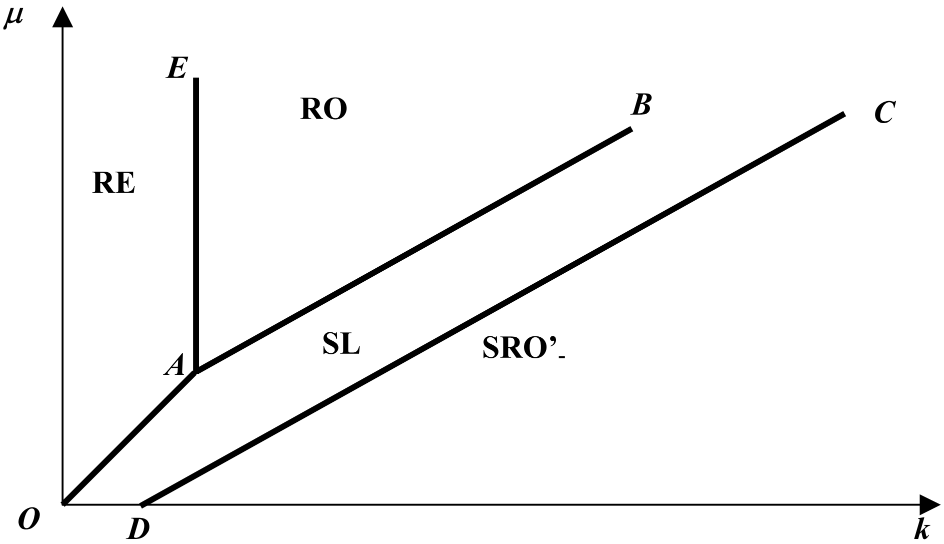

Fig.4.7b All modes $|k'| \ge \frac{1+\gamma^2}{4+\gamma^2}$

V. Concluding Remarks

This study is motivated by the desire to know how a caisson would respond to the action of a breaking wave.

In this study, as an initial step, we have achieved to identify and derive conditions for initiation of the various modes of response of a caisson, modeled as a rigid body placed on a frictional base subjected to a horizontal force, mimicking the action of a breaking wave.

The equations used are the three equations of motion of a plane rigid body. Simple as the model is, the analysis reveals the complex behavior of the body. Although the derivation of these conditions is quite involved, the results have been presented in graphical form in the $k-\mu$ plane (k being related to the magnitude of the force F and μ, coefficient of friction between the body and the base).giving, in a clear manner, the conditions under which the modes of response the body would be initiated into.

VI. Acknowledgements

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VII. Reference

Shenton, HW. Criteria for initiation of slide, rock and slide-rock rigid body modes. Journal of Engineering Mechanics(ASCE) 1996;122(7):690-693

LIST OF SYMBOLS

a	quantity defined in (3.18) and (4.42)
B	half width of body
b	quantity defined in (3.19) and (4.43)
C	center of mass of body
c	quantity defined in (3.20) and (4.44)
d	quantity defined in (3.21) and (4.45)
F	horizontal applied force
f_x	horizontal reaction
f_y	vertical reaction
g	gravitational acceleration
H	half height of body
h	distance between line of action of force F and center of mass C of body
I	mass moment of inertia of body about center of mass C
k	non-dimensional non-negative quantity used to express magnitude of force F in terms of weight of body
k'	non-dimensional non-negative quantity used to express distance of force F from center of mass C in terms of H
k_A	value of k corresponding to the point A in $k-\mu$ plane
$k_{B'}$	abscissa of point B', intersection of $\mu^*(k)$ and line AB in Figs. 4.4(a), 4.5(a), 4.6(a) and 4.7(a)
m	total mass of body
RE	symbol used to represent region of rest mode
RO	symbol used to represent region of rock (about O) mode
SL	symbol used to represent region of slide mode
SRO_-	symbol used to represent region of slide-rock (about O) mode with $\ddot{x}_O \leq 0$
SRO_+	symbol used to represent region of slide-rock (about O) mode with $\ddot{x}_O \geq 0$
x	horizontal displacement of center of mass C of body, positive to the right

x_O	horizontal displacement of O at base of body, positive to the right
$x_{O'}$	horizontal displacement of O' at base of body, positive to the right
y	vertical displacement of center of mass C, positive upward
Δ	slope of $\mu^*(k)$ at A minus slope $\|k'\|$ of line AB referred to in Figs.4.4(a), 4.4(b), 4.6(a) and 4.6 (b)
γ	$= H/B$, aspect or slenderness ratio of body
θ	rotation of body , positive counterclockwise
μ	coefficient of friction between body and base
$\mu^*(k)$	curve defined in (3.24) and (4.40)
$\mu^{**}(k)$	curve defined in (4.96) and (4.97)
$\mu_{B'}$	ordinate of B', intersection of $\mu^*(k)$ and line AB in Figs. 4.4(a), 4.5(a), 4.6(a) and 4.7(a)
ξ	distance between line of action of f_y and center of mass C of body
$\| \|$	absolute value sign

List of Figures