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交 通 部 運 輸 研 究 所

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摘要：

本研究考慮深水重力波行進在一穩定三維剪流上，後者在垂直方向具有一強大剪應變但近乎線性分佈，而在水平方向則為緩慢變化。在此一情況下，我們仍可經由選擇一local co-ordinate system，使旋性擾動速度可輕易和非旋性者分離，以獲得波浪運動之WKBJ 解。而當旋性擾動速度未與非旋性者分離時，Voronovich (1976)曾應用一種攝動法導出一可適用於更複雜情況之波作用守恆方程式，但此一方程式在目前情況無法化減為與目前的調變方程式一致，故顯示在目前的情況下，波作用無法守恆，但在一較複雜的情況仍可守恆。由於目前的理論所涉及的 WKBJ 解僅包含一項，而適用於複雜情況下之波作用守恆方程式所涉及的 WKBJ 解則包含許多項，故目前的理論在應用上較方便。

當波作用守恆方程式不適用時，為深入瞭解調變方程式的物理意義，我們亦應用積分法將 Jonsson, Brink-Kjær & Thomas (1978)的理論加以延伸，其結果顯示即使旋性擾動速度和非旋性者具有相同的量級，前者亦不應在波能量方程式中加以考慮。而在一更普遍的情況，當積分法無法適用，且調變方程式和波作用守恆方程式無法一致時，上述波能量方程式僅需增加兩項即可和調變方程式一致，表有額外的波流交互作用存在。

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ABSTRACT:

Deep-water gravity waves propagating on a steady three-dimensional, strongly sheared current are studied for the case where the current varies slowly in the horizontal directions and deviates slightly from a linear profile in the vertical direction. In this case, a WKBJ description of wave propagation can still be developed by solution of the boundary-value problem in a local co-ordinate system in which the rotational perturbation velocity can easily be separated from the irrotational one. Without this separation, a perturbation scheme was applied by Voronovich (1976) to derive the action conservation equation in an even more complicated situation, but in the present case, this equation after reduction is inconsistent with the modulation equation of wave amplitude derived here, implying that the wave action is not conserved in the present case, but is conserved in a more complicated case.

The reason why the theory derived by Voronovich (1976) in a complicated case cannot be applied to a simpler case lies in the situation that the separation of the rotational perturbation velocity from the irrotational one is essential for the right use of the perturbation scheme (since the solution of the first-order equations in the hierarchy of equations in the perturbation scheme is featured of a potential motion in a certain sense) when the deviation of the current velocity from a linear profile in the vertical direction becomes small and the density-field inhomogeneities vanish, but not otherwise. The advantage of the present theory is that there exists a one-term WKBJ solution in the present case while the action conservation equation for a more complicated case is related to a WKBJ solution containing a large number of terms according to Shrira (1993).

When the wave action is not conserved, to obtain a physical insight into the modulation equation in the present case, the theory of Jonsson, Brink-Kjær & Thomas (1978) is extended by using the integral approach. The results indicate that the rotational perturbation velocity, even if it has the same order of magnitude as the irrotational one, should not be considered in the wave energy equation. In the more general case in which the integral approach is invalid and the modulation equation is inconsistent with the action conservation equation, this wave energy equation can be reconciled with the modulation equation by adding only two terms to the former, representing additional wave-mean flow interaction.

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1. Introduction

Previous studies of the wave-current interactions, well documented in the review articles by Peregrine (1976), Jonsson (1990) and Thomas & Klopman (1997), may be divided into two categories: the first is to study the interactions between the waves and the currents that are all horizontally uniform. In these studies, much attention has been given to the effects of the large amplitudes of waves and the strong shear of currents. Consequently, certain numerical calculations are often needed (see, for example, Simmen & Saffman 1985 and Teles da Silva & Peregrine 1988). However an analytical solution in terms of an infinite series in powers of a certain parameter ϵ, which characterizes the smallness of the deviation of the current velocity from a linear profile in the vertical direction was derived by Shrira (1993) for linear waves propagating obliquely on a steady, strongly sheared current. Since this series solution can be rapidly convergent in a practical situation, this solution, as pointed out by Shrira (1993), is useful to the study of the 'gradually varying problem', which is among the second category.

In the second category, the underlying current is allowed to vary slowly in the horizontal directions, which will certainly result in the corresponding slow modulations of the wave amplitudes and wave-numbers. Modern theories on this problem were begun by Longuet-Higgins & Stewart (1960, 1961), Whitham (1965), and Bretherton & Garrett (1968), in which the idea of radiation stress was introduced and the action conservation equation established for the case of an irrotational current. Although these theories can be applied to many practical situations (e.g. waves on the majority of tidal flows), there are situations (e.g. waves on a wind-drift current) in which a highly sheared current exists so that these theories may become invalid.

Extensions of modulation theories from irrotational currents to rotational ones have successfully been made by Jonsson *et al.* (1978) in a two-dimensional analysis using an integral approach and by Voronovich (1976) and White (1999) in three-dimensional analyses using perturbation schemes. However, in White's (1999) theory, the spatial scales of the current in the horizontal directions and in the vertical direction are assumed to be the same, meaning that the current varies slowly not only in the horizontal directions but also in the vertical direction. As a result, the dispersion relation and the action conservation equation derived in White (1999) are not different from those for an irrotational current, but a new equation for a spatially varying phase shift, which can displace the positions of the wave crests by a distance on the order of a wavelength, has been derived in this situation.

Conversely, in Voronovich's (1976) theory, slowness of the variation of the main motion is not assumed along the vertical co-ordinate and even the variation in the buoyancy of the fluid is allowed so that the action conservation equation derived by Voronovich (1976) in terms of the local solution of the wave motion involves the vorticity of the main motion as well as the Brunt-Väisälä frequency N. Therefore if the perturbation scheme applied by Voronovich (1976) remains valid in the present case in which $N = 0$ and $\epsilon \ll 1$, by substitution of the local solution in this case, which corresponding to the zeroth-order term of the series solution in Shrira (1993), into the action conservation equation, the slow modulation of the wave amplitude can be determined in the present case. However, according to the discussion in §4, a different perturbation scheme, which can exist only if ϵ is so small that the wave motion is nearly potential in a certain sense, can result in a modulation equation of wave amplitude that is very likely to be inconsistent with the action conservation equation derived by Voronovich (1976). This scheme, though it exists in theory, cannot be implemented in practice so that an alternative is pursued here.

In the present approach without a formal asymptotic expansion, it is unnecessary to explicitly introduce the ordering parameters to scale equations (for a demonstration of this strategy, see Shyu & Phillips 1990 and Shyu & Tung 1999). On the other hand, according to Shrira (1993), if the deviation of the current velocity from a linear profile in the vertical direction is small, the series solution derived by Shrira (1993) will converge very rapidly, which renders a one-term WKBJ solution possible. Therefore, in §2, we shall temporarily neglect the slow variations in the horizontal directions and the slight deviation from a linear profile in the vertical direction of the underlying current to obtain an exact solution of the linear waves in this situation. This solution coincides with the zeroth-order term of the series solution derived by Shrira (1993) and will hereafter be referred to as the basic solution. This basic solution, if allowing its parameters to slowly vary, represents the first-order WKBJ solution of the slowly varying wave train, although the variations of these parameters, especially that of the wave amplitude, remain to be determined, for which the discussion in §2 can also provide important information, including the features of the irrotational and rotational perturbation velocities.

Notice that if the variation of the current velocity with depth is near linear and rapid and if the component of this velocity in the direction toward which the waves propagate intrinsically, increases with depth, the critical layer where the current velocity is equal to the propagation speed of waves will always occur in deep water. When this critical layer occurs, the frequency of waves becomes complex so that waves will grow or decay (Morland, Saffman & Yuen 1991; Shrira 1993; Miles 2001). This instability problem is important for wave dynamics. However, since the existence of the imaginary part of the frequency will render the following analysis difficult if not impossible and since the main purpose of this study is to clarify the limits of the validity of the action conservation equation, the occurrences of the critical layers will be avoided in this study by assuming that the component of the current velocity in the direction toward which the waves propagate intrinsically, decreases with depth. This assumption is appropriate for the wave-current fields in which the waves and the strong shear of the current are all generated by the local wind.

Also we emphasize that in deep water, the slight deviation of the current velocity from a linear profile in the vertical direction can result in a great change of the current velocity from that of the linear profile in the region far away from the water surface. This change cannot be determined from a very limited number of quantities representing the properties of the current only at the mean water surface. Therefore, when the solution of the slowly varying wave train derived below involves only these quantities in connection with the current, it is suggested that the properties of the current in the region far away from the water surface have vanishingly small influence on the surface waves. Hence, although the following WKBJ description of wave modulations is derived under the assumption of a nearly linear shear current at all depths, one of the obvious applications of this work is to currents that can be approximated by a nearly linear shear current near to the surface, with zero or constant current below.

The WKBJ description of wave modulations will be deduced in §3 by solution of the boundary-value problem in the light of the basic solution. This three-dimensional analysis can be simplified significantly by using a local co-ordinate system to separate the irrotational and rotational perturbation velocities and to achieve other important purposes. The resulting modulation equation will in §4 be compared with the action conservation equation derived by Voronovich (1976), which indicates that these two equations cannot be consistent with each other unless the rotational perturbation velocity becomes negligible or its fast variation with depth $-z$ can be specified solely by the simple function e^{kz} where k represents the wave-number. The reason for this inconsistency is also illuminated in §4. From this comparison and discussion, it is concluded that the wave action is in general not conserved in the present case, though in the case that ϵ is not small or the influence of density-field inhomogeneities cannot be neglected, the theory of Voronovich (1976) is valid and the wave action is conserved.

When the wave action is not conserved, to obtain a physical insight into the modulation equation derived in §3, the theory of Jonsson *et al.* (1978) will in §5 and §6 be extended from two-dimensional flows to three-dimensional flows using an integral approach.

In §5, the wave energy equation will be derived by using this integral approach in a less general case and is indeed consistent with the modulation equation derived in §3 and the action conservation equation in Voronovich (1976). However in §6, it is found that even in a simple case, as long as the transverse rotational perturbation velocity v has the same order of magnitude as the longitudinal irrotational one $\partial\phi/\partial x$, the integral approach will become invalid. The reason for this invalidity is also given in §6.

When v has the same order of magnitude as $\partial\phi/\partial x$ but the fast variation with depth of v can be specified solely by e^{kz} so that the integral approach is invalid but the modulation equation derived in §3 remains consistent with the action conservation equation in Voronovich (1976), the latter equation can safely be applied to obtain the wave energy equation. In this case, no extra terms occur in the wave energy equation compared with that derived in §5 for the case when $v = 0$, implying that even when v has the same order of magnitude as $\partial\phi/\partial x$, the rotational perturbation velocity v should not be considered in the equation for the balance of wave energy. Therefore, in an even more general case in which the fast variation with depth of v can no longer be specified solely by e^{kz} and in which the modulation equation has been derived in §3, the wave energy equation can be reconciled with the modulation equation by adding only two terms to the former. This situation and its implications are discussed in §7.

2. The basic solution

In this section we shall describe the exact solution of the linear deep-water gravity waves propagating obliquely on a steady current $\mathbf{U}\{U(z), V(z), 0\}$ uniform in the horizontal directions but strongly and linearly sheared (constant vorticity) in the vertical direction. This solution, defined as the basic solution, is closely related to the WKBJ solution, because when the velocity and the vorticity of the underlying current become slowly varying in the horizontal directions and in the horizontal and vertical directions, respectively, the parameters in this basic solution will similarly vary slowly, resulting in the WKBJ solution which represents the first-order term of the asymptotic expansion of the solution for the 'gradually varying problem'.

Since in the present circumstances, due to the rigid rotation and the extension or contraction of the vortex-lines, one may expect that the oscillatory wave motion is no longer irrotational, we start with the Euler equation for perturbations of velocity $\mathbf{u}\{u, v, w\}$ and pressure p linearized upon the flow $\mathbf{U}$

$$\left.\begin{aligned} \frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + V\frac{\partial u}{\partial y} + w\frac{\partial U}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial x} &= 0 \\ \frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} + V\frac{\partial v}{\partial y} + w\frac{\partial V}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial y} &= 0 \\ \frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} + V\frac{\partial w}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial z} + g &= 0 \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0 \end{aligned}\right\} \tag{2.1}$$

where ρ is the density of the water and g the acceleration due to gravity. In (2.1), the choice of the directions of the $x-$ and $y-$ axes of the rectangular co-ordinates are at our disposal. On the other hand, since the underlying current $\mathbf{U}$ is uniform in the horizontal directions, the waves will not be refracted by the current. Therefore the $y-$ axis can be chosen to be parallel to the wave crests so that all variables are independent of y and the above system of equations then reduces to

$$\left.\begin{aligned} \frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + w\Omega_2 + \frac{1}{\rho}\frac{\partial p}{\partial x} &= 0 \\ \frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial z} + g &= 0 \\ \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0 \end{aligned}\right\} \tag{2.2a}$$

and

$$\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} - w\Omega_1 = 0 \qquad (2.2b)$$

where $\mathbf{\Omega}\{\Omega_1, \Omega_2, 0\}$ denotes the vorticity of the underlying current $\mathbf{U}$ with $\Omega_1 = -\mathrm{d}V/\mathrm{d}z$ and $\Omega_2 = \mathrm{d}U/\mathrm{d}z$ in the present situation, and suffices $(1, 2)$ indicate the vector components in the $x-$ and $y-$directions respectively.

Notice that the variables v and V as well as the constant Ω_1 are absent from (2.2a), meaning that if this situation also occurs to the free-surface boundary conditions, the solutions of u, w and p will not be affected by the convection in the $y-$direction V and its shear Ω_1. Nevertheless, if $\Omega_1 \neq 0$ and $w \neq 0$, according to (2.2b), the oscillatory velocity component v will occur, which is important for the development of the WKBJ description in the next section.

The boundary conditions at the free surface $z = \eta(x, y, t)$ transformed on the plane $z = 0$ which corresponding to the unperturbed free surface can be written as

$$\frac{\partial \eta}{\partial t} + U\frac{\partial \eta}{\partial x} = w, \qquad p = \rho g \eta, \qquad (2.3)$$

(see Shrira 1993) which are indeed free from v, V and Ω_1. Therefore, one can solve (2.2a) and (2.3) without consideration of v, after which v can be determined from (2.2b).

Differentiating the first and second equations in (2.2a) with respect to z and x respectively, combining the resulting equations into one to eliminate the pressure terms, and using the third of equations (2.2a), we obtain

$$\frac{\partial \omega_2}{\partial t} + U\frac{\partial \omega_2}{\partial x} = 0, \qquad (2.4)$$

where $\omega_2 \equiv \partial u/\partial z - \partial w/\partial x$ represents the vorticity component of the wave motion in the $y-$direction. Thus if initially $\omega_2 = 0$ everywhere, from (2.4) it will remain so in an inviscid fluid. Therefore a two-dimensional velocity potential $\phi(x, z, t)$ can be defined such that $u = \partial\phi/\partial x$ and $w = \partial\phi/\partial z$. The third of equations (2.2a) then requires

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0. \qquad (2.5)$$

Thus in deep water we have

$$\phi = A\mathrm{e}^{kz}\mathrm{e}^{\mathrm{i}(kx - n_0 t)}, \qquad (2.6)$$

where A is a constant, k the wave-number and n_0 the observed frequency of a chosen Fourier component.

If now the surface displacement

$$\eta = a\mathrm{e}^{\mathrm{i}(kx-n_0t)}, \tag{2.7}$$

where the amplitude a is a constant, from the boundary conditions (2.3), we obtain

$$A = -\mathrm{i}\frac{\sigma}{k}a \tag{2.8}$$

and

$$p|_{z=0} = \rho g a\mathrm{e}^{\mathrm{i}(kx-n_0t)},$$

where

$$\sigma \equiv n_0 - U_0 k \tag{2.9}$$

is the intrinsic frequency relative to the frame of reference in which the mean surface velocity equals zero, and $U_0 \equiv U|_{z=0}$. Substituting all these results into the first and second equations of (2.2a), we have respectively the dispersion relation

$$gk = \sigma^2 + \sigma\Omega_2 \tag{2.10}$$

and the pressure fluctuations

$$p = -\rho g z + \rho g a\mathrm{e}^{kz}\mathrm{e}^{\mathrm{i}(kx-n_0t)} - \rho\sigma\Omega_2 a z\mathrm{e}^{kz}\mathrm{e}^{\mathrm{i}(kx-n_0t)}. \tag{2.11}$$

The last term in (2.11), arising from the fact that $U = U_0 + \Omega_2 z$, cannot be found when the underlying current is irrotational, but is important for the analysis in §5. On the other hand, the dispersion relation (2.10) is identical with the zeroth-order term of the series solution derived by Shrira (1993).

Finally, from (2.2b) we have

$$v = \frac{\sigma\Omega_1}{n_0 - Uk}a\mathrm{e}^{kz}\mathrm{e}^{\mathrm{i}(kx-n_0t)}, \tag{2.12}$$

meaning that when the wave profiles propagate obliquely on a horizontally uniform shear flow, a transverse rotational perturbation velocity will occur, which can be as large as $\partial\phi/\partial x$ and $\partial\phi/\partial z$ if Ω_1 has the same order of magnitude as σ. On the other hand, if Ω_2 has the same order of magnitude as σ, the two terms on the right-hand side of (2.10) also have the same order of magnitude. Therefore, in the following discussion we assume that Ω_1 and Ω_2 have the same order of magnitude as σ, representing a strongly sheared current.

Notice that since v varies with x and z, the vorticity components ω_1 and ω_3 of the oscillatory wave motion in the $x-$ and $z-$directions are non-zero. On the other hand, since U varies with depth, according to (2.12), the value of v becomes infinity at the critical layers where $n_0 - Uk = 0$. This result is in contradiction with the linear-wave assumption for application of the governing equations (2.1) and the boundary conditions (2.3). Therefore the solution derived here will become invalid if there exist critical layers.

When critical layers occur and the mean flow vorticity is constant everywhere, the solution of (2.1) and (2.3) that is analytic at any finite depth is not immediately clear, but if the mean flow vorticity becomes slowly varying in the vertical direction, the exact series solution of (2.1) and (2.3) in powers of a certain parameter which characterizes the smallness of the deviation of the current velocity from a linear profile in the vertical direction has been derived by Shrira (1993). In this solution, if critical layers exist, the frequency of the wave motion becomes complex so that waves will grow or decay. On the other hand, although the zeroth-order term of the series solution takes the same form as the solution derived here, since the frequency n_0 in Shrira's (1993) solution contains the imaginary part when critical layers occur, the denominator in (2.12) will not vanish at any finite depth in his solution.

Since the imaginary part of the frequency will render the following analysis difficult if not impossible and since the main purpose of this study is to clarify the limits of the validity of the action conservation equation, the occurrences of the critical layers will be avoided here by assuming that

$$U_0 - U(z) \geq 0 \tag{2.13}$$

at any depth $-z$. Since it follows from (2.9) that

$$n_0 - Uk = n_0 - U_0 k + (U_0 - U)k = \sigma + (U_0 - U)k$$

and since σ and k can be assumed to be positive without loss of generality (although in this situation n_0 may sometimes be negative), the assumption (2.13) can therefore ensure that the denominator in (2.12) never vanishes in the water so that the solution derived here remains valid at any depth.

From (2.13) it is clear that no critical layers will occur for wind waves propagating on a tidal current or an ocean current like the Gulf Stream in which the waves and the strong shear of the mean flow are all due to the wind. In fact, the condition (2.13) can be fulfilled by any wave-current fields in which the waves and the strong shear of the current are all generated by the local wind. Also, we emphasize that even if there exists a critical layer at

$z = z_c$, say, as long as $|z_c|$ is very large compared with the wavelength, the growth or decay rate of the waves will be very small, and in the meantime, v remains finite at $z = z_c$ because of the existence of the imaginary part of n_0, which will arise when the current profile is not perfectly linear. Therefore, even in this case, the variations of the wave motion can still be specified approximately by the modulation theory developed below without consideration of the instability. This situation is reminiscent of our earlier suggestion that the profile of the mean flow in the region far away from the water surface has little or no influence on the surface waves. Hence the present theory can have a wide application.

3. The modulation theory

In this section, the underlying current **U** similar to that described in §2 is allowed to vary slowly in both the $x-$ and $y-$directions, and its vorticity components Ω_1 and Ω_2 can gradually vary not only in the horizontal directions but also in the vertical direction (see figure 1). In this situation, since all these variations are slow in the sense that their length scales are large compared with the wavelength, the solution described in §2, when allowing its parameters to slowly vary, represents the first term of the asymptotic expansion of the exact solution for this 'gradually varying problem'. The modulation rates of these parameters will be derived in this section, which complete the so-called WKBJ solution in this case.

When a wave train propagates on a horizontally non-uniform current, the magnitude and the direction of the wave-number **k** will both change with distance. However, even in this case, the $x-$axis of the rectangular co-ordinates can still be chosen to be parallel to the local **k** at the position under consideration and the $y-$axis is therefore parallel to the local wave crest. On the other hand, when the underlying current is non-uniform in the horizontal directions, the mean water surface may not be horizontal, but according to Phillips (1981), Longuet-Higgins (1985, 1987) and Henyey *et al.* (1988), the effects of its slope and curvature on the wave motion are equivalent to those with a level mean surface and with the gravitational acceleration g being replaced by the effective gravitational acceleration g' defined by Phillips (1981). Therefore, by using g' instead of g and by using the co-ordinate system chosen above in which the xy plane is tangent to the mean water surface at the position under consideration, the solution described in §2 can directly be applied for derivation of the WKBJ solution.

The use of the local co-ordinate system in the analysis has many other advantages that will become clear in the following discussion. However, since a fixed co-ordinate system is usually employed to study and compute numerically global variations of a wave field, these two systems are linked in figure 2, in which the local system is distinguished from the fixed system by using the notations s, m instead of x, y, respectively.

Since the differentiation of the slowly varying parameters increases the order of magnitude by one each time, to derive the first-order WKBJ solution, the second-order derivatives of the slowly varying parameters and the products of any two first-order derivatives of these parameters can all be neglected in the following discussion. Similarly, since the underlying current velocity is slowly varying in the horizontal directions and its vorticity is slowly varying in all directions, this treatment can also be applied to the derivatives with respect to s or m of U, V, Ω_1 and Ω_2, and to the derivatives with respect to z of Ω_1 and Ω_2.

Also, we emphasize that when the first-order derivatives of the slowly varying parameters and quantities are taken into account, the second term of the asymptotic expansion of each unknown in §2 should also be considered implicitly, and in the meantime, some unknowns that vanish in §2 will now become non-zero. All of these extra terms, though one order of magnitude smaller than those obtained in §2 and eventually negligible within the present approximation, must be included in the analysis. This situation also occurs in a formal perturbation scheme in which the second term of the asymptotic expansion of each unknown is considered to derive the secular condition of the first term of the expansion, which leads to the modulation equation for the first term of the expansion (see, for example, Whitham 1974 and Mei 1983). After this, the second term of the asymptotic expansion can be neglected in the WKBJ solution. Thus the present approach is not separated from the perturbation scheme in which the perturbation parameter is the ratio of the wavelength to the length scale of the variations of U and V in the horizontal directions and the variations of Ω_1 and Ω_2 in all directions.

The quantities that are one order of magnitude smaller than their counterparts defined in §2 are distinguished from them by using the symbols with a hat. For example, although the component of the current velocity in the $z-$direction $\widehat{W}$ vanishes at the mean water surface at the position under consideration, from the continuity equation

$$\frac{\partial U}{\partial s} + \frac{\partial V}{\partial m} + \frac{\partial \widehat{W}}{\partial z} = 0, \tag{3.1}$$

and from the situation that $\partial U/\partial s \neq 0$ and $\partial V/\partial m \neq 0$, it follows that $\widehat{W}$ has a small but non-zero value at the depth within one wavelength where the current has a direct influence on the surface waves.

The non-uniformity of the current in the horizontal directions also implies that the vorticity component in the $z-$direction $\widehat{\Omega}_3 \equiv \partial V/\partial s - \partial U/\partial m$ is non-zero. On the other hand, the quantities $(\partial\widehat{W}/\partial s)_{z=0}$ and $(\partial\widehat{W}/\partial m)_{z=0}$, which represent the first-order derivatives of a smaller quantity, remain negligible locally. Finally, in the present case, the perturbation vorticity component $\widehat{\omega}_2 \equiv \partial\widehat{u}/\partial z - \partial\widehat{w}/\partial s$, though small, also becomes non-zero. Therefore, in addition to $\partial\phi/\partial s$ and $\partial\phi/\partial z$, the rotational velocity components of the wave motion $\widehat{u}$ and $\widehat{w}$ in the $s-$ and $z-$directions, respectively, also exist and are one order of magnitude smaller than $\partial\phi/\partial s$ and $\partial\phi/\partial z$ as well as v.

Notice that if $\widehat{w}$ does not vanish at the mean water surface, since its fast variation in the horizontal directions can locally be represented by the function $\exp[\mathrm{i}(ks - n_0 t)]$ and since its slow variation can be neglected within the present approximation, it is always possible to define an irrotational velocity field $\widehat{\phi}$ which takes the same form as (2.6) locally so that $\partial\widehat{\phi}/\partial z$ can have the same value as $\widehat{w}$ at each point on the mean water surface. Therefore, after subtracting $\partial\widehat{\phi}/\partial z$ from $\widehat{w}$, subtracting $\partial\widehat{\phi}/\partial s$ from $\widehat{u}$, and in the meantime, adding $\widehat{\phi}$ to ϕ, the new rotational perturbation velocity becomes horizontal at the mean water surface, and the new velocity potential still takes the same form as (2.6) except that the second term of the asymptotic expansion of A becomes different. Consequently, the boundary condition

$$\widehat{w} = 0 \qquad \text{at} \qquad z = 0 \tag{3.2}$$

can be applied to simplify the analysis importantly.

In order to describe both the fast and the slow variations, the new velocity potential can be written as

$$\phi = A(s, m) \exp\left[\int_0^z l(s, m, z)\, dz\right] e^{\mathrm{i}\chi(s,m,t)} \tag{3.3}$$

with

$$\mathbf{k} = \nabla_h \chi, \qquad n_0 = -\partial\chi/\partial t, \tag{3.4}$$

where $\nabla_h \equiv (\partial/\partial s, \partial/\partial m)$ represents the horizontal gradient operator, $\mathbf{k}\{k_1, k_2\}$ the wave-number vector, and $l(s, m, z)$ a slowly varying function of position. Since in the present co-ordinate system, $k_2 = 0$ at the position under consideration, and from the relation (3.9) given below, $l|_{z=0} \approx k$, the expression (3.3) together with (3.4) is indeed identical with (2.6) locally if the higher-order terms in the asymptotic expansions of A, $\mathbf{k}$, and l are neglected and their slow variations are ignored.

From the first of equations (3.4) it follows immediately that

$$\frac{\partial k_2}{\partial s} = \frac{\partial k_1}{\partial m}. \tag{3.5}$$

Also, from (3.4)

$$\partial \mathbf{k}/\partial t + \nabla_h n_0 = 0,$$

which is the kinematical conservation equation (Phillips 1977). Since in the present case the underlying current is steady, we have $\partial\mathbf{k}/\partial t = 0$ so that from the above equation n_0 is constant everywhere.

Substitution of (3.3) and (3.4) into the three-dimensional Laplace equation yields

$$-k_1^2 + \mathrm{i}\frac{\partial k_1}{\partial s} + 2\mathrm{i}k_1 \frac{1}{A}\frac{\partial A}{\partial s} + \mathrm{i}\frac{\partial k_2}{\partial m} + l^2 + \frac{\partial l}{\partial z} = 0 \qquad \text{at} \qquad z = 0 \tag{3.6}$$

in which the terms $(1/A)(\partial^2 A/\partial s^2)$ and $(1/A)(\partial^2 A/\partial m^2)$ have been neglected and the fact that $k_2 = 0$ locally has also been taken into account. Furthermore, since the variation of the wave motion in the $m-$direction at the position under consideration is slow, the second-order derivative $(\partial^2\phi/\partial m^2)_{z=0} = \mathrm{i}(\partial k_2/\partial m)A\mathrm{e}^{\mathrm{i}\chi} + (\partial^2 A/\partial m^2)\mathrm{e}^{\mathrm{i}\chi} \approx \mathrm{i}(\partial k_2/\partial m)A\mathrm{e}^{\mathrm{i}\chi}$ should also be negligible here, meaning that at the position under consideration

$$\partial k_2/\partial m = 0 \tag{3.7}$$

within the present approximation. This important suggestion will later be justified analytically in this section.

In (3.6), since both l and $\partial l/\partial z$ exist, one cannot express l in terms of other parameters and their derivatives without another equation. In Shyu & Tung (1999), the relation (see their (2.14))

$$\left.\frac{\partial l}{\partial z}\right|_{z=0} = -\mathrm{i}\frac{\partial k}{\partial s}$$

has been derived from the Laplace equation for the exactly two-dimensional case in which the wave crest is straight. Since the two terms in this relation both represent the derivatives of the slowly varying parameters and therefore are small, the small curvature of the wave crest occurred in the present case will impose a modification of this relation even smaller and therefore negligible within the present approximation. Thus, in the present case, we have

$$\left.\frac{\partial l}{\partial z}\right|_{z=0} = -\mathrm{i}\frac{\partial k}{\partial s} = -\mathrm{i}\left(\frac{k_1}{k}\frac{\partial k_1}{\partial s} + \frac{k_2}{k}\frac{\partial k_2}{\partial s}\right) = -\mathrm{i}\frac{\partial k_1}{\partial s} \tag{3.8}$$

at the position under consideration, because $k = (k_1^2 + k_2^2)^{1/2}$ and $k_2 = 0$ locally. Therefore, substituting (3.7) and (3.8) into (3.6), we obtain

$$l^2|_{z=0} = k_1^2 - 2\mathrm{i}k_1\frac{1}{A}\frac{\partial A}{\partial s}.$$

Squaring both sides of it and neglecting the term $(1/2k_1)(1/A)^2(\partial A/\partial s)^2$ and the even higher order terms, we finally have

$$l|_{z=0} = k - \mathrm{i}\frac{1}{A}\frac{\partial A}{\partial s}. \tag{3.9}$$

We next consider the kinematic free-surface condition, which in the present case can be written as

$$\frac{\partial\eta}{\partial t} + \left(\frac{\partial\phi}{\partial s} + \widehat{u} + U\right)\frac{\partial\eta}{\partial s} + \left(\frac{\partial\phi}{\partial m} + v + V\right)\frac{\partial\eta}{\partial m} = \frac{\partial\phi}{\partial z} + \widehat{w} + \widehat{W} \quad \text{at} \quad z = \eta.$$

After Taylor series expansions about $z = 0$, we have the linear-wave approximation

$$\frac{\partial \eta}{\partial t} + U\frac{\partial \eta}{\partial s} + V\frac{\partial \eta}{\partial m} = \frac{\partial \phi}{\partial z} - \eta\left(\frac{\partial U}{\partial s} + \frac{\partial V}{\partial m}\right) \qquad \text{at} \qquad z = 0 \tag{3.10}$$

in view of (3.1) and (3.2). Therefore if the surface displacement is

$$\eta = a(s, m)\mathrm{e}^{\mathrm{i}\chi(s,m,t)}, \tag{3.11}$$

substitution of (3.3), (3.4), (3.7), (3.9) and (3.11) into (3.10) yields

$$\frac{A}{a} = -\mathrm{i}\frac{\sigma}{k} + \frac{1}{ak}\left(U\frac{\partial a}{\partial s} + V\frac{\partial a}{\partial m}\right) + \frac{1}{k}\left(\frac{\partial U}{\partial s} + \frac{\partial V}{\partial m}\right) + \mathrm{i}\frac{1}{ak}\frac{\partial A}{\partial s} \quad \text{at} \quad z = 0 \tag{3.12}$$

in virtue of (2.9). From (3.12), neglecting the smaller terms containing the derivatives of the slowly varying functions, we have

$$A/a \approx -\mathrm{i}\sigma/k. \tag{3.13}$$

Its differentiation with respect to s yields

$$\frac{\partial A}{\partial s} = -\mathrm{i}\frac{\sigma}{k}\frac{\partial a}{\partial s} - \mathrm{i}\frac{a}{k}\frac{\partial \sigma}{\partial s} + \mathrm{i}a\frac{\sigma}{k^2}\frac{\partial k_1}{\partial s} \tag{3.14}$$

because $\partial k/\partial s = \partial k_1/\partial s$ (see (3.8)) and $\partial k/\partial m = \partial k_1/\partial m$. Therefore, by substituting (3.14) into (3.12) for $\partial A/\partial s$, one can express A in terms of other quantities and their derivatives within the present approximation. Note that without consideration of the second term of the asymptotic expansion of each quantity, (3.13) is again identical with (2.8), though the parameters A, a, σ and k are now slowly varying.

Finally, the dynamical free-surface condition is imposed by the requirement that the pressure in the water at the free surface is equal to the atmospheric pressure which is assumed to be constant here. Therefore, if at a certain point at the free surface, the component of the equation of motion in the $\xi-$direction in figure 3 is under consideration, which is tangent to the instantaneous free surface at this point and perpendicular to the local wave crest, the pressure gradient in this equation will vanish. The rest of the terms in this equation, though originally involving the components of the quantities in the $\xi-$, $m-$ and $\nu-$directions in figure 3, can all be transformed into the terms containing the components in the $s-$, $m-$ and $z-$directions. This can be done without difficulty, because in figure 3, $\cos\alpha \approx 1$ and $\sin\alpha \approx \partial\eta/\partial s$ for linear waves. The resulting equation can then be expressed as Taylor series expansions about the mean water surface $z = 0$, so that after neglecting the higher-order terms of the asymptotic expansions $\widehat{\Omega}_3 \partial\phi/\partial m$, $\widehat{u}\partial U/\partial s$, $(\partial V/\partial s)\partial\phi/\partial m$ and $V\partial\widehat{u}/\partial m$, and neglecting the nonlinear terms of the oscillatory wave motion, we have

$$\left\{g\frac{\partial\overline{\zeta}}{\partial s}+U\frac{\partial U}{\partial s}+V\frac{\partial U}{\partial m}\right\}+\left\{g'\frac{\partial\eta}{\partial s}+\frac{\partial^2\phi}{\partial s\partial t}+U\frac{\partial^2\phi}{\partial s^2}+\frac{\partial U}{\partial s}\frac{\partial\phi}{\partial s}+U\frac{\partial\widehat{u}}{\partial s}+V\frac{\partial^2\phi}{\partial s\partial m}+U\frac{\partial U}{\partial z}\frac{\partial\eta}{\partial s}+\frac{\partial U}{\partial s}\frac{\partial U}{\partial z}\eta\right.$$
$$\left.+U\frac{\partial^2 U}{\partial s\partial z}\eta+\frac{\partial\widehat{u}}{\partial t}+\frac{\partial U}{\partial z}\frac{\partial\eta}{\partial t}+\frac{\partial U}{\partial m}\frac{\partial V}{\partial z}\eta+V\frac{\partial^2 U}{\partial m\partial z}\eta+V\frac{\partial U}{\partial z}\frac{\partial\eta}{\partial m}+\frac{\partial U}{\partial m}v\right\}=0 \quad \text{at} \quad z=0, \tag{3.15}$$

where $\overline{\zeta}$ is the height of the mean water surface; see figure 3.

Notice that in deriving (3.15), the vector identity $\mathbf{v}\times(\nabla\times\mathbf{v})=(1/2)\nabla(\mathbf{v}\cdot\mathbf{v})-\mathbf{v}\cdot\nabla\mathbf{v}$ and the fact that the velocity component in the $\nu-$direction at the instantaneous free surface equals $\partial\eta/\partial t$ for linear waves have been utilized. Furthermore, the situation that η represents the surface displacement of the waves in the $z-$direction in figure 3 has resulted in the replacement of g by g' in the second braces in (3.15), which in the present case is defined as $g\cos\theta$.

The above equation can also be deduced directly from the component of the equation of motion in the $s-$direction evaluated directly at the mean water surface (so that no Taylor series expansion about $z=0$ is required) if the validity of the second of equations (2.3) is assumed. However the present derivation involves no such assumption so that this derivation is preferred and can serve as a proof of the validity of the second of equations (2.3) in the present circumstances.

In (3.15), the terms in the first braces are time-independent while the expression in the second braces represents a linear combination of the time-harmonic terms. Therefore the latter itself should vanish, resulting in

$$g'\frac{\partial\eta}{\partial s}+\frac{\partial^2\phi}{\partial s\partial t}+U\frac{\partial^2\phi}{\partial s^2}+\frac{\partial U}{\partial s}\frac{\partial\phi}{\partial s}+V\frac{\partial^2\phi}{\partial s\partial m}+U\frac{\partial U}{\partial z}\frac{\partial\eta}{\partial s}+\frac{\partial U}{\partial z}\frac{\partial\eta}{\partial t}+V\frac{\partial U}{\partial z}\frac{\partial\eta}{\partial m}$$
$$+\left(\frac{\partial U}{\partial s}\frac{\partial U}{\partial z}+U\frac{\partial^2 U}{\partial s\partial z}+\frac{\partial U}{\partial m}\frac{\partial V}{\partial z}+V\frac{\partial^2 U}{\partial m\partial z}\right)\eta=R \quad \text{at} \quad z=0, \tag{3.16}$$

where

$$R\equiv-\frac{\partial\widehat{u}}{\partial t}-U\frac{\partial\widehat{u}}{\partial s}-v\frac{\partial U}{\partial m} \tag{3.17}$$

also evaluated at $z=0$.

In (3.16), the terms on the right-hand side represented by R all contain the rotational perturbation velocity component $\widehat{u}$ or v. The sum of these terms can be related to those involving only the steady flow and the irrotational part of the wave motion. To achieve this purpose, we consider the component of the vorticity equation in the m – direction evaluated at the mean water surface for the entire flow. After neglecting the higher-order terms $\widehat{u}\partial\Omega_2/\partial s$, $(\partial\Omega_2/\partial m)\partial\phi/\partial m$, $\widehat{\omega}_2\partial V/\partial m$, $\Omega_2\partial^2\phi/\partial m^2$, $(\partial V/\partial z)\partial\widehat{u}/\partial m$ and $\widehat{\Omega}_3\partial^2\phi/\partial z\partial m$, and neglecting the nonlinear terms of the oscillatory wave motion, this equation becomes

$$\left\{U\frac{\partial^2 U}{\partial s\partial z}+V\frac{\partial^2 U}{\partial m\partial z}+\frac{\partial U}{\partial m}\frac{\partial V}{\partial z}-\frac{\partial V}{\partial m}\frac{\partial U}{\partial z}\right\}+\left\{\frac{\partial}{\partial z}\left(\frac{\partial\widehat{u}}{\partial t}+U\frac{\partial\widehat{u}}{\partial s}+v\frac{\partial U}{\partial m}\right)\right.$$

$$\left.+\frac{\partial U}{\partial z}\frac{\partial\widehat{w}}{\partial z}+\frac{\partial^2 U}{\partial s\partial z}\frac{\partial\phi}{\partial s}+\frac{\partial^2 U}{\partial z^2}\frac{\partial\phi}{\partial z}+\frac{\partial V}{\partial z}\frac{\partial^2\phi}{\partial s\partial m}\right\}=0 \qquad \text{at} \qquad z=0 \tag{3.18}$$

in view of (3.2) and the fact that

$$\frac{\partial\widehat{u}}{\partial s}+\frac{\partial v}{\partial m}+\frac{\partial\widehat{w}}{\partial z}=0. \tag{3.19}$$

Notice that although $\widehat{w}|_{z=0}=0$, $(\partial\widehat{w}/\partial z)_{z=0}$ is in general non-zero.

In (3.18), the time-independent terms and the time-harmonic terms have again been separated so that we have for the steady flow

$$U\frac{\partial^2 U}{\partial s\partial z}+V\frac{\partial^2 U}{\partial m\partial z}+\frac{\partial U}{\partial m}\frac{\partial V}{\partial z}-\frac{\partial V}{\partial m}\frac{\partial U}{\partial z}=0 \qquad \text{at} \qquad z=0, \tag{3.20}$$

and for the oscillatory wave motion

$$\frac{\partial}{\partial z}\left(-\frac{\partial\widehat{u}}{\partial t}-U\frac{\partial\widehat{u}}{\partial s}-v\frac{\partial U}{\partial m}\right)=\frac{\partial U}{\partial z}\frac{\partial\widehat{w}}{\partial z}+\frac{\partial^2 U}{\partial s\partial z}\frac{\partial\phi}{\partial s}+\frac{\partial^2 U}{\partial z^2}\frac{\partial\phi}{\partial z}+\frac{\partial V}{\partial z}\frac{\partial^2\phi}{\partial s\partial m} \qquad \text{at} \qquad z=0. \tag{3.21}$$

The expression in the parentheses in (3.21) is identical to that represented by R in (3.17). From (3.3), (3.5), (3.9) and the situation that $k_2=0$ locally, it is not difficult to prove that the solution

$$R=\frac{\partial U}{\partial z}\widehat{w}+\frac{\partial^2 U}{\partial s\partial z}(\mathrm{i}\phi)+\frac{\partial^2 U}{\partial z^2}\phi+\frac{\partial V}{\partial z}\left(\mathrm{i}\frac{\partial\phi}{\partial m}\right) \qquad \text{at} \qquad z=0 \tag{3.22}$$

can satisfy (3.21) within the present approximation. In addition, the uniqueness of this solution can be substantiated as follows.

First, the differentiation of $(\partial U/\partial z)\partial\phi/\partial z$ (instead of $(\partial^2 U/\partial z^2)\phi$ which appears in (3.22)) with respect to z will result in not only the term $(\partial^2 U/\partial z^2)\partial\phi/\partial z$ occurring in (3.21) but also the term $(\partial U/\partial z)\partial^2\phi/\partial z^2$ which is absent from (3.21). The latter term is even one order of magnitude larger than the former term. Next, the derivatives with respect to z of the terms $(\partial U/\partial s)\partial\phi/\partial s$ and $V\partial^2\phi/\partial s\partial m$, instead of $(\partial^2 U/\partial s\partial z)(\mathrm{i}\phi)$ and $(\partial V/\partial z)(\mathrm{i}\partial\phi/\partial m)$ chosen in (3.22), also contain each an extra term that is not negligible even if the steady flow $\mathbf{U}$ becomes irrotational so that $\widehat{u}$, v and $\widehat{w}$ in (3.21) vanish. This is certainly impossible. Finally, the choice of $(\partial U/\partial z)\widehat{w}$ rather than $U\partial\widehat{w}/\partial z$ in (3.21) is because the differentiation of the latter with respect to z also produces two terms which have the same order of magnitude but only one of which really occurs in (3.21), while the differentiation of the term $(\partial U/\partial z)\widehat{w}$ with respect to z results in $(\partial U/\partial z)\partial\widehat{w}/\partial z$ and $(\partial^2 U/\partial z^2)\widehat{w}$; the latter is indeed negligible or even vanishes at $z = 0$ according to (3.2). More important, the latter term can also be found in the original equation that leads to (3.18). This and the fact that the fast variations in the $z-$direction of the last three terms in both (3.21) and (3.22) are simply specified by the function e^{kz} can put even more confidence in the solution (3.22).

Since $\widehat{w}|_{z=0} = 0$, the expression (3.22) reduces to

$$R = \frac{\partial^2 U}{\partial s\partial z}(\mathrm{i}\phi) + \frac{\partial^2 U}{\partial z^2}\phi + \frac{\partial V}{\partial z}\left(\mathrm{i}\frac{\partial\phi}{\partial m}\right) \qquad \text{at} \qquad z = 0 \tag{3.23}$$

of which the right-hand side is indeed devoid of the rotational perturbation velocity.

Notice that the derivative of the right-hand side of (3.17) with respect to m is definitely negligible so that the derivative of the right-hand side of (3.23) with respect to m, which involves the term $(\partial V/\partial z)\mathrm{i}(\partial^2\phi/\partial m^2)_{z=0}$, should also be negligible. This can justify (3.7) analytically because $(\partial^2\phi/\partial m^2)_{z=0} \approx \mathrm{i}(\partial k_2/\partial m)A\mathrm{e}^{\mathrm{i}\chi}$.

The equation (3.16) together with (3.23) involves only η and ϕ as the unknowns. Therefore, substituting (3.3) and (3.11) into (3.16), using (3.12) and (3.14) to eliminate A in favour of a, neglecting the terms containing the second-order derivatives of the slowly varying functions or the products of any two first-order derivatives of these functions, and then crossing out the common factor, we obtain

$$\begin{aligned}&\left\{\left(g' + 2\sigma U + U\frac{\partial U}{\partial z}\right)\frac{1}{a}\frac{\partial a}{\partial s} + \left(2\sigma V + V\frac{\partial U}{\partial z}\right)\frac{1}{a}\frac{\partial a}{\partial m} + U\frac{\partial\sigma}{\partial s} + V\frac{\partial\sigma}{\partial m} + 2\sigma\frac{\partial U}{\partial s} + \sigma\frac{\partial V}{\partial m}\right.\\ &\left.+\frac{\partial U}{\partial s}\frac{\partial U}{\partial z} + \frac{\partial U}{\partial m}\frac{\partial V}{\partial z} + U\frac{\partial^2 U}{\partial s\partial z} + V\frac{\partial^2 U}{\partial m\partial z} - \frac{\sigma}{k}\frac{\partial^2 U}{\partial s\partial z} - \left(\frac{\sigma}{k}\frac{1}{a}\frac{\partial a}{\partial m} + \frac{1}{k}\frac{\partial\sigma}{\partial m} - \frac{\sigma}{k^2}\frac{\partial k_1}{\partial m}\right)\frac{\partial V}{\partial z}\right\}\\ &+\mathrm{i}\left\{g'k - \sigma^2 - \sigma\frac{\partial U}{\partial z} + \frac{\sigma}{k}\frac{\partial^2 U}{\partial z^2}\right\} = 0 \qquad \text{at} \qquad z = 0.\end{aligned} \tag{3.24}$$

Since without loss of generality, the amplitude $a(s,m)$ in (3.11) can be defined as a real function, from the imaginary and real parts of (3.24) we finally have the dispersion relation

$$g'k = \sigma^2 + \sigma\Omega_{20} - \frac{\sigma}{k}\frac{\partial^2 U}{\partial z^2}\bigg|_{z=0} \tag{3.25}$$

and the modulation equation of the wave amplitude

$$\begin{aligned}(g' + 2\sigma U_0 + \Omega_{20}U_0)\frac{1}{a}\frac{\partial a}{\partial s} + (2\sigma + \Omega_{20})V_0\frac{1}{a}\frac{\partial a}{\partial m} + U_0\frac{\partial \sigma}{\partial s} + V_0\frac{\partial \sigma}{\partial m} + 2\sigma\frac{\partial U_0}{\partial s} + \sigma\frac{\partial V_0}{\partial m} \\ + \left(\frac{\partial U_0}{\partial s} + \frac{\partial V_0}{\partial m}\right)\Omega_{20} - \frac{\sigma}{k}\frac{\partial^2 U}{\partial s \partial z}\bigg|_{z=0} + \left(\frac{\sigma}{k}\frac{1}{a}\frac{\partial a}{\partial m} + \frac{1}{k}\frac{\partial \sigma}{\partial m} - \frac{\sigma}{k^2}\frac{\partial k_1}{\partial m}\right)\Omega_{10} = 0\end{aligned} \tag{3.26}$$

in virtue of (3.20), where U_0, V_0, Ω_{10} and Ω_{20} denote the values of U, V, Ω_1 and Ω_2 at the mean water surface respectively, so that

$$\Omega_{10} \equiv \frac{\partial \widehat{W}}{\partial m}\bigg|_{z=0} - \frac{\partial V}{\partial z}\bigg|_{z=0} = -\frac{\partial V}{\partial z}\bigg|_{z=0}, \qquad \Omega_{20} \equiv \frac{\partial U}{\partial z}\bigg|_{z=0} - \frac{\partial \widehat{W}}{\partial s}\bigg|_{z=0} = \frac{\partial U}{\partial z}\bigg|_{z=0}$$

within the present approximation.

Notice that since Ω_{20} represents the component of vorticity perpendicular to the local $\mathbf{k}$ and the latter may vary in the $s-$direction, we have

$$\begin{aligned}\frac{\partial \Omega_{20}}{\partial s} = \frac{\partial}{\partial s}\left[\frac{\partial}{\partial z}\left(\frac{k_1}{k}U + \frac{k_2}{k}V\right)\right]_{z=0} &= \frac{\partial}{\partial s}\left[\frac{k_1}{k}\,\frac{\partial U}{\partial z}\bigg|_{z=0} + \frac{k_2}{k}\,\frac{\partial V}{\partial z}\bigg|_{z=0}\right] \\ &= \frac{\partial^2 U}{\partial s \partial z}\bigg|_{z=0} + \frac{1}{k}\frac{\partial k_2}{\partial s}\,\frac{\partial V}{\partial z}\bigg|_{z=0} \\ &= \frac{\partial^2 U}{\partial s \partial z}\bigg|_{z=0} - \frac{1}{k}\frac{\partial k_1}{\partial m}\Omega_{10}\end{aligned} \tag{3.27}$$

at the position under consideration, meaning that the quantities $\partial\Omega_{20}/\partial s$ and $(\partial^2 U/\partial s \partial z)_{z=0}$ are generally not equal to each other.

Also we emphasize that if the higher-order term $-(\sigma/k)(\partial^2 U/\partial z^2)_{z=0}$ and those inherent in g' are neglected, the dispersion relation (3.25) is again identical with (2.10), though the quantities k, σ, and Ω_{20} are now slowly varying. These higher-order terms, when substituting in (3.26), will produce even higher-order terms and therefore are negligible within the present approximation. On the other hand, the slow variations of $\mathbf{k}$ and σ corresponding to those of the underlying current can be determined approximately from

(2.9), (2.10), (3.5) and the fact that n_0 = const.. Thus the last term in (3.25) can be discarded and g' can be replaced by g in the WKBJ solution. Therefore, we have

$$gk = \sigma^2 + \sigma\Omega_{20}. \tag{3.28}$$

Using (2.9), (3.5), (3.28), and the fact that $n_0 = \text{const.}$, the values of σ, k_1 and k_2 at each point can be solved in either a local co-ordinate system or a fixed co-ordinate system. Therefore, in (3.26), all quantities except a become known so that the value of a at each point can be determined numerically from (3.26), after which the values of A, p and v at the same point can also be determined by using (3.13), (2.11) and (2.12) respectively. All of these represent the first-order WKBJ solution of the waves propagating obliquely on a steady three-dimensional, strongly sheared current that varies slowly in the horizontal directions and deviates slightly from a linear profile in the vertical direction.

Finally we note that if the component of the equation of motion in the $m-$direction evaluated at the instantaneous free surface is considered, following the same approach that leads to (3.24) and using the component of the vorticity equation in the $z-$direction evaluated at $z = 0$, one can obtain an equation similar to (3.24). This equation can completely be cancelled out by using the equation

$$U\frac{\partial^2 V}{\partial s \partial z} + V\frac{\partial^2 V}{\partial m \partial z} + \frac{\partial V}{\partial s}\frac{\partial U}{\partial z} - \frac{\partial U}{\partial s}\frac{\partial V}{\partial z} = 0 \qquad \text{at} \qquad z = 0 \tag{3.29}$$

derived from the time-independent terms of the component of the vorticity equation in the $s-$direction for the entire flow. Therefore, the dynamical free-surface condition can indeed be satisfied by the present solution.

4. Comparison with the action conservation equation

The modulation equation (3.26) is complicated and will become even more complicated in a fixed co-ordinate system when a transformation of axes is performed. Therefore from this equation one cannot immediately know whether the wave action is conserved in the present case. This difficulty can however be overcome by expansion of the action conservation equation derived by Voronovich (1976) in a fixed co-ordinate system, by expressing the result in the local co-ordinate system, and then by comparison of the resulting equation with (3.26) to see whether or not these two equations are consistent with each other in the present case.

With one exception that will become clear in the following discussion, the situation under which Voronovich (1976) has derived the action conservation equation is far more general than the present case. For example, in Voronovich (1976), the slowness of the variations of Ω_1 and Ω_2 along the vertical co-ordinate is not assumed and the variation in the buoyancy of the fluid is allowed so that the propagation of internal gravity waves has also been considered. The results can, in tensor notation, be written as

$$\frac{\partial I}{\partial t} + \frac{\partial}{\partial x_\alpha}\left(C_{gr}^{(\alpha)} I\right) = 0, \tag{4.1}$$

where I represents the wave action density and $C_{gr}^{(\alpha)} I$ the wave action flux due to the mean surface velocity $\mathbf{U_0} \equiv \mathbf{U}|_{z=0}$ and the group velocity $\mathbf{C_g} \equiv \partial\sigma/\partial\mathbf{k}$ (so that $\mathbf{C_{gr}} \equiv \mathbf{U_0} + \mathbf{C_g}$).

The expressions of I and $C_{gr}^{(\alpha)} I$ have also been deduced by Voronovich (1976) in terms of the mean square value of the first term of the asymptotic expansion of the wave vertical velocity. Since in the present case, this mean square value at each depth $-z$ is $(\sigma^2 a^2/2)\mathrm{e}^{2kz}$ according to (3.3), (3.9) and (3.13), substitution into the expressions (17) and (18) in Voronovich (1976) for I and $C_{gr}^{(\alpha)} I$ and neglect of the buoyancy terms and the terms relating to the variations of Ω_1 and Ω_2 along the vertical co-ordinate, which are negligible in the present case (in which $\partial\Omega_1/\partial z$ and $\partial\Omega_2/\partial z$ are small but non-zero), lead to

$$I = \frac{\rho}{2}\left(\frac{g}{\sigma} - \frac{k_\alpha}{2k^2}\frac{\partial U_\alpha}{\partial z}\right) a^2 = \frac{\rho}{2k}\left(\sigma + \frac{1}{2}\frac{k_\alpha}{k}\frac{\partial U_\alpha}{\partial z}\right) a^2 \tag{4.2}$$

and

$$C_{gr}^{(\alpha)} I = \left[U_0^{(\alpha)} + \left(\frac{\sigma^2}{k}\frac{k_\alpha}{k} + 2\frac{\sigma}{k}\frac{k_\alpha}{k}\frac{k_\beta}{k}\frac{\partial U_\beta}{\partial z} - \frac{\sigma}{k}\frac{\partial U_\alpha}{\partial z}\right) \Big/ 2\left(\sigma + \frac{1}{2}\frac{k_\beta}{k}\frac{\partial U_\beta}{\partial z}\right)\right] \cdot \frac{\rho}{2k}\left(\sigma + \frac{1}{2}\frac{k_\beta}{k}\frac{\partial U_\beta}{\partial z}\right) a^2 \tag{4.3}$$

in virtue of (3.28). (Here and hereafter, all the quantities involving the differentiation with respect to z represent those evaluated at $z = 0$.)

Notice that to obtain (4.2) and (4.3), the dispersion relation (3.28) must be rewritten as

$$gk = \sigma^2 + \sigma \frac{k_\alpha}{k} \frac{\partial U_\alpha}{\partial z} \tag{4.4}$$

before substitution. In (4.4), the last term represents a zeroth-order tensor and is therefore independent of the choice of co-ordinate axes. By differentiation of (4.4) with respect to k_α, we also have the group velocity

$$C_g^{(\alpha)} = \left(\frac{\sigma^2}{k} \frac{k_\alpha}{k} + 2 \frac{\sigma}{k} \frac{k_\alpha}{k} \frac{k_\beta}{k} \frac{\partial U_\beta}{\partial z} - \frac{\sigma}{k} \frac{\partial U_\alpha}{\partial z} \right) \bigg/ 2 \left(\sigma + \frac{1}{2} \frac{k_\beta}{k} \frac{\partial U_\beta}{\partial z} \right), \tag{4.5}$$

which indeed coincides with that implied in (4.3).

The action conservation equation (4.1) together with (4.2) and (4.3) can now be expanded by carrying out the differentiation with respect to x_α while the derivative with respect to t can be discarded due to the steadiness of the wave train. After differentiation, the local co-ordinate system (s, m, z) is chosen so that the terms containing k_2 or $\partial k_2/\partial m$ can be eliminated (see (3.7)). The resulting equation can be reduced further by substitution of (3.20) and the equations arising from differentiation of the kinematical conservation equation $n_0 = \sigma + U_\alpha k_\alpha =$ const. in the present case and from differentiation of the dispersion relation (4.4), resulting in

$$\begin{aligned}
(g' + 2\sigma U_0 + \Omega_{20} U_0) \frac{1}{a} \frac{\partial a}{\partial s} + (2\sigma + \Omega_{20}) V_0 \frac{1}{a} \frac{\partial a}{\partial m} + U_0 \frac{\partial \sigma}{\partial s} + V_0 \frac{\partial \sigma}{\partial m} + 2\sigma \frac{\partial U_0}{\partial s} + \sigma \frac{\partial V_0}{\partial m} \\
+ \left(\frac{\partial U_0}{\partial s} + \frac{\partial V_0}{\partial m} \right) \Omega_{20} - \frac{\sigma}{k} \frac{\partial^2 U}{\partial s \partial z} \bigg|_{z=0} + \left(\frac{\sigma}{k} \frac{1}{a} \frac{\partial a}{\partial m} + \frac{1}{k} \frac{\partial \sigma}{\partial m} - \frac{\sigma}{k^2} \frac{\partial k_1}{\partial m} \right) \Omega_{10} \\
= -\frac{1}{2} \frac{\sigma}{k} \frac{\partial^2 U}{\partial s \partial z} + \frac{1}{2} \frac{\sigma}{k} \frac{\partial^2 V}{\partial m \partial z} + \frac{\partial V}{\partial z} \frac{\partial U_0}{\partial m}.
\end{aligned} \tag{4.6}$$

The terms on the left-hand side of (4.6) are exactly the same as those on the left-hand side of (3.26) so that the three terms on the right-hand side of (4.6) represent the differences between (3.26) and (4.6). The reason for these differences may lie in the situation that in Voronovich (1976), the rotational perturbation velocity has not been separated from the irrotational one. Therefore the rotational perturbation velocity components $\widehat{u}$ and $\widehat{w}$ defined in §3, which are of the second order, can in Voronovich (1976) be inferred from the solutions of the second-order equations in the hierarchy of equations arising from substitution of the asymptotic expansions of the unknowns into the original boundary-value problem. However $\widehat{u}$ and $\widehat{w}$ can also be determined directly from the component of the vorticity equation in the $m-$direction and the continuity equation (see (3.18) and (3.19)) in terms of $\partial\phi/\partial s$, $\partial\phi/\partial z$ and v. Thus the rotational and irrotational parts of the perturbation velocity can be separated from each other before asymptotic expansion, and each term of the asymptotic expansions of $\widehat{u}$ and $\widehat{w}$ can in theory be expressed in terms of the lower-order terms in the expansions of $\partial\phi/\partial s$, $\partial\phi/\partial z$ and v and should therefore be considered as known in the equations in the hierarchy, meaning that extra terms will occur on the right-hand side of the second-order inhomogeneous equations in the hierarchy compared with those in Voronovich (1976). Consequently, in this new perturbation scheme, the solvability condition of the second-order equations and therefore the modulation equation of the first-order terms of expansions will be different from those in Voronovich (1976), although these first-order terms of expansions in these two schemes represent the same quantities.

Since $\widehat{u}$ and $\widehat{w}$ can in general be expressed explicitly in terms of $\partial\phi/\partial s$, $\partial\phi/\partial z$ and v only at $z = 0$ (see (3.17) and (3.23)), the new perturbation scheme cannot be implemented in practice (so that an alternative has been applied in §3). However, when $\widehat{u}$ and $\widehat{w}$ become even smaller or the first terms of their expansions represent the homogeneous solutions of the second-order inhomogeneous equations in the hierarchy, no extra terms will occur on the right-hand side of these equations when the rotational perturbation velocity is separated from the irrotational one. Thus in these two situations, the modulation equation of the first-order terms of expansions in the new perturbation scheme can also be obtained by using the old perturbation scheme, resulting in (4.6). Therefore in these two situations, one may expect that (4.6) can coincide with (3.26), which will be scrutinized below.

The two situations may separately occur in four cases:

(1) Waves propagate on a steady two-dimensional current with strong but constant shear and are in a direction parallel with this current. In this case (which has also been considered in Jonsson *et al.* (1978)), the perturbation velocity remains irrotational so that $\widehat{u} = \widehat{w} = 0$ indeed. On the other hand, since $V = 0$, it follows from (3.20) that $\partial^2 U/\partial s\partial z = 0$. Thus the three terms on the right-hand side of (4.6) all vanish. Therefore (4.6) indeed coincides with (3.26) in this case.

(2) Waves propagate on a steady three-dimensional, strongly sheared current in which $\partial V/\partial z = 0$ in the local co-ordinate system so that $v = 0$ according to (2.12). When this occurs at a certain position and if initially $\widehat{\omega}_2 \equiv \partial\widehat{u}/\partial z - \partial\widehat{w}/\partial s \approx 0$ here, from the vorticity

equation, $\widehat{\omega}_2$ and therefore $\widehat{u}$, $\widehat{w}$ will remain negligible at this position if $\partial U/\partial z$, though large, remains unchanged along the vertical co-ordinate. On the other hand, since $\partial V/\partial z = 0$, differentiation of (3.20) with respect to z leads to

$$\frac{\partial U}{\partial z}\frac{\partial^2 U}{\partial s \partial z} = 0 \tag{4.7}$$

within the present approximation so that the three terms on the right-hand side of (4.6) can be neglected. Therefore in this case, (4.6) indeed coincides with (3.26).

(3) Waves propagate perpendicularly on a steady two-dimensional current with strong but constant shear so that $U = 0$, $\partial U/\partial z = 0$ but $\partial V/\partial z \neq 0$. In this case, according to (2.12), v has the same order of magnitude as $\partial\phi/\partial s$, but the fast variation of v with depth can now be specified solely by the simple function e^{kz} because $\partial U/\partial z = 0$. Next, from the component of the vorticity equation in the $m-$direction, which in this case reduces to $\partial\widehat{\omega}_2/\partial t = -(\partial^2\phi/\partial s\partial m)(\partial V/\partial z)$ (see (3.21)) within the present approximation, and from the continuity equation (3.19), it is clear that the fast variations with depth of $\widehat{u}$ and $\widehat{w}$ are also specified solely by e^{kz}. Thus $\widehat{u}$ and $\widehat{w}$ represent the homogeneous solution of the second-order inhomogeneous equations in the hierarchy derived by Voronovich (1976). Therefore in this case, (4.6) remains unchanged when the rotational perturbation velocity is separated from the irrotational one. On the other hand, when $U = 0$, it follows from (3.29) that $\partial^2 V/\partial m\partial z = 0$. Thus the three terms on the right-hand side of (4.6) all vanish. Therefore in this case, though the rotational perturbation velocity is no longer negligible, still (4.6) coincides with (3.26) indeed.

(4) Waves propagate on a steady three-dimensional, strongly shear current in which $\partial U/\partial z = 0$ but $\partial V/\partial z \neq 0$ in the local co-ordinate system. In this case, for the same reason as that in case (3), the fast variation of v with depth can be specified solely by e^{kz}. Furthermore, it follows from (3.20) that $\partial U/\partial m = 0$ (since $\partial V/\partial z \neq 0$). Therefore in this case, the component of the vorticity equation in the $m-$direction reduces to $\partial\widehat{\omega}_2/\partial t + U(\partial\widehat{\omega}_2/\partial x) = -(\partial^2\phi/\partial s\partial m)(\partial V/\partial z)$ within the present approximation. This equation and the continuity equation (3.19) indicate that the fast variations of $\widehat{u}$ and $\widehat{w}$ with depth are also specified solely by e^{kz} because at the position under consideration, U is independent of z. Therefore in this case, (4.6) also remains unchanged when the rotational perturbation velocity is separated from the irrotational one. On the other hand, since $\partial U/\partial z = 0$, differentiation of (3.29) with respect to z results in $(\partial V/\partial z)(\partial^2 V/\partial m\partial z) = 0$ within the present approximation. Therefore in this case, the three terms on the right-hand side of (4.6) can all vanish (recall that $\partial U/\partial m = 0$) so that (4.6) indeed coincides with (3.26).

The above discussion and the situation that the derivation of (3.26) is straightforward can put great confidence in the validity of (3.26) in the general case. Therefore from the comparison between (3.26) and (4.6) it is very likely that the wave action is not conserved in the general case. However the wave action remains conserved in a situation more general than that considered in Jonsson *et al.* (1978). Therefore it is interesting to see whether this conclusion can also be reached by an extension of the theory of Jonsson *et al.* (1978) using an integral approach, which given in the next section, can directly result in the wave energy equation and can therefore apply to other purposes.

We wind up the present section by emphasizing that although there exists a term containing $\partial^2 U/\partial z^2$ on the left-hand side of each field equation in the hierarchy derived by Voronovich (1976), this term is one order of magnitude smaller than Voronovich (1976) originally assumed it to be when the deviation of the current velocity from a linear profile in the vertical direction is small. Thus the term containing $\partial^2 U/\partial z^2$ in the first-order equations in the hierarchy in Voronovich (1976) should now occur on the right-hand side of the second-order equations instead, meaning that in this case, another extra term will occur on the right-hand side of the second-order equations compared with those in Voronovich (1976), but the solution of the first-order equations can still be expressed in terms of a two-dimensional velocity potential ϕ and a transverse rotational perturbation velocity v. Therefore in this case, the new perturbation scheme with the rotational perturbation velocity being separated from the irrotational one is still required and the wave action defined in (4.2) is not conserved. This situation will alter when the deviation of the current velocity from a linear profile in the vertical direction is not small (or when the density-field inhomogeneities exist). Therefore in this more complicated case, it is unnecessary to separate the rotational perturbation velocity from the irrotational one, meaning that the action conservation equation derived by Voronovich (1976) is correct. However, according to Shrira (1993), when $\partial^2 U/\partial z^2$ is not small, the solution of the first-order equations in the hierarchy, which will be substituted into the expressions (17) and (18) in Voronovich (1976) for I and $C_{gr}^{(\alpha)} I$, is itself a series solution with a large number of terms, which can render the applications of this action conservation equation difficult.

5. Extension of Jonsson, Brink-Kjær & Thomas' theory

The approach by Jonsson *et al.* (1978) is based on the integral properties of the combined wave and current motion across a fixed vertical section. This approach was first applied by Longuet-Higgins & Stewart (1960) in their derivation of the radiation stress tensor and was applied by Phillips (1977) to derive the expressions for the conservation of mass, momentum and energy when a wave train propagates obliquely on a variable irrotational current. The special arrangement made by Jonsson *et al.* (1978) is particularly suitable for a rotational current.

Since the vertical integrals involving the rotational perturbation velocity v cannot in general be evaluated in terms of simple functions owing to the fact that the denominator in (2.12) varies rapidly, the situation that $\Omega_1 = 0$ (but $\Omega_2 \neq 0$ and $V \neq 0$), which ensuring that $v = 0$ according to (2.12), is considered first, which corresponds to case (2) in §4. On the other hand, according to (3.25), (3.26) and the discussion above (3.28), the slow variations in the vertical direction of the quantities about the current cannot affect the WKBJ solution; therefore the vertical integrals in this analysis will be evaluated without consideration of these variations (so that $\widehat{W} = 0$ at each depth). Also, in this integral approach, unlike the perturbation scheme, it is unnecessary to consider the second-order terms of asymptotic expansions, meaning that the components of the rotational perturbation velocity $\hat{u}$ and $\hat{w}$ can also be disregarded in the vertical integrals even in a general case. Therefore in the present case, it suffices to substitute all the solutions in §2 except v into the vertical integrals for the local properties of the wave motion.

Following the precedent of Jonsson *et al.* (1978) (and Phillips (1977) for a three-dimensional analysis), we first define the radiation stress tensor

$$S_{\alpha\beta} = \delta_{\alpha\beta}\overline{\int_{-h}^{\eta} p\,dz} - \delta_{\alpha\beta}\int_{-h}^{0}(-\rho g z)\,dz + \rho\overline{\int_{-h}^{\eta}\tilde{u}_\alpha\tilde{u}_\beta\,dz} - \rho\int_{-h}^{0}\widetilde{U}_\alpha\widetilde{U}_\beta\,dz, \qquad (\alpha,\beta = 1,2), \tag{5.1}$$

where a overbar denotes averaging over the (constant) observed period, $\delta_{\alpha\beta}$ is the unit tensor ($\delta_{\alpha\beta} = 1$ if $\alpha = \beta$ and vanishes otherwise), h the local mean water depth,

$$\tilde{u}_1 \equiv \partial\phi/\partial s + U, \qquad \tilde{u}_2 \equiv \partial\phi/\partial m + V = V \tag{5.2}$$

the total horizontal velocity components, and $\widetilde{U}_1$, $\widetilde{U}_2$ the s, m components of a 'formal current velocity', the profiles of which are defined as

$$\widetilde{U}_1(z) = \widetilde{U}_{10} + \Omega_2 z, \qquad \widetilde{U}_2(z) = \widetilde{U}_{20}, \tag{5.3}$$

where $\widetilde{U}_{10}$, $\widetilde{U}_{20}$, and Ω_2 are independent of z. The relations between $\widetilde{U}_1$ and U and between $\widetilde{U}_2$ and V can be established from the requirement that

$$\int_{-h}^{0} \widetilde{U}_1(z)\,dz = \overline{\int_{-h}^{\eta} \widetilde{u}_1(z)\,dz}, \qquad \int_{-h}^{0} \widetilde{U}_2(z)\,dz = \overline{\int_{-h}^{\eta} \widetilde{u}_2(z)\,dz}. \tag{5.4}$$

Substituting (2.6), (2.8), (5.2) and (5.3) into (5.4) and recalling that $U = U_0 + \Omega_2 z$ and $V = V_0$, we obtain

$$\widetilde{U}_{10} = U_0 + \frac{2\sigma + \Omega_2}{4h}a^2, \qquad \widetilde{U}_{20} = V_0 \tag{5.5}$$

correct to the second order in (ak). To achieve these results, the mean water depth h is assumed to be large compared with the wavelength so that the solutions (2.6) and (2.8) for deep-water waves can be applied here.

In order to evaluate the first integral in (5.1) to the second order in (ak), the mean pressure distribution

$$\overline{p} = -\rho g z - \frac{\rho}{2}\sigma^2 a^2 \mathrm{e}^{2kz}, \tag{5.6}$$

correct to the second order in (ak) and valid even for a vortical flow (see (3.2.17) in Phillips (1977)) is also required in addition to (2.11). By substituting all these results and the solutions given in §2 into (5.1), we obtain

$$\left.\begin{aligned} S_{11} &= \frac{\rho}{4}ga^2 + \frac{\rho}{4}\Omega_2(2\sigma + \Omega_2)a^2 h \\ S_{22} &= \frac{\rho}{4k}\sigma\Omega_2 a^2 \\ S_{12} &= S_{21} = 0 \end{aligned}\right\} \tag{5.7}$$

correct to the second order in (ak).

Next the total mean energy flux per unit area

$$F_\alpha = \overline{\int_{-h}^{\eta} \left[p + \rho g(z+b) + \frac{\rho}{2}(\widetilde{u}_1^2 + \widetilde{u}_2^2 + \widetilde{u}_3^2)\right] \widetilde{u}_\alpha\,dz}, \qquad (\alpha = 1, 2) \tag{5.8}$$

where $\widetilde{u}_3 \equiv \partial\phi/\partial z + \widehat{W} = \partial\phi/\partial z$ and $b = b(s, m)$ represents the height of the mean water surface above a reference level (see figure 4) specified for determination of the potential energy. By substitution and after some lengthy manipulations, we obtain

$$\begin{aligned} F_1 =& \frac{\rho}{4k}\sigma^2 U_0 a^2 + \frac{\rho}{4k}g\sigma a^2 + \frac{\rho}{2}gU_0 a^2 - \frac{3}{8}\rho a^2\left(-2\sigma\Omega_2 U_0 h + \frac{2}{3}\sigma\Omega_2^2 h^2 - \Omega_2^2 U_0 h + \frac{1}{3}\Omega_2^3 h^2\right) \\ &+ \frac{\rho}{2}\int_{-h}^{0} \widetilde{U}_1^3\,dz + \frac{\rho}{2}\int_{-h}^{0} \widetilde{U}_2^2 \widetilde{U}_1\,dz + \rho g b h \overline{U}_1 \end{aligned} \tag{5.9}$$

$$F_2 = \frac{\rho}{2}gV_0 a^2 + \frac{\rho}{4}a^2\left(\sigma\Omega_2 V_0 h + \frac{1}{2}\Omega_2^2 V_0 h\right) + \frac{\rho}{2}\int_{-h}^{0} \widetilde{U}_1^2 \widetilde{U}_2\,dz + \frac{\rho}{2}\int_{-h}^{0} \widetilde{U}_2^3\,dz + \rho g b h \overline{U}_2 \tag{5.10}$$

where $\overline{U}_1 \equiv (1/h)\overline{\int_{-h}^{\eta} \widetilde{u}_1(z)\,dz} = \widetilde{U}_{10} - \Omega_2 h/2$ and $\overline{U}_2 \equiv (1/h)\overline{\int_{-h}^{\eta} \widetilde{u}_2(z)\,dz} = \widetilde{U}_{20}$ (see (5.3) and (5.4)), the average-over-depth velocity.

On the other hand, the mean total momentum flux $M_{\alpha\beta}$ per unit area equals the sum of the first and third terms on the right-hand side of (5.1). Thus

$$M_{\alpha\beta} = S_{\alpha\beta} + \frac{\rho}{2} g h^2 \delta_{\alpha\beta} + \rho \int_{-h}^{0} \widetilde{U}_\alpha \widetilde{U}_\beta \, dz, \qquad (\alpha, \beta = 1, 2). \tag{5.11}$$

The horizontal components of the mean total pressure force acting on the fluid at the bed per unit length in the $s-$ and $m-$ directions are $P_1 = \rho g h(\partial D/\partial s)$ and $P_2 = \rho g h(\partial D/\partial m)$, respectively; see figure 4. The equations $-\partial M_{\alpha 1}/\partial s - \partial M_{\alpha 2}/\partial m + P_\alpha = 0, (\alpha = 1, 2)$ of total momentum conservation therefore take the form

$$\frac{\partial S_{11}}{\partial s} + \frac{\partial S_{12}}{\partial m} + \frac{\partial}{\partial s}\left(\rho \int_{-h}^{0} \widetilde{U}_1 \widetilde{U}_1 \, dz\right) + \frac{\partial}{\partial m}\left(\rho \int_{-h}^{0} \widetilde{U}_1 \widetilde{U}_2 \, dz\right) + \rho g h \frac{\partial b}{\partial s} = 0, \tag{5.12}$$

$$\frac{\partial S_{21}}{\partial s} + \frac{\partial S_{22}}{\partial m} + \frac{\partial}{\partial s}\left(\rho \int_{-h}^{0} \widetilde{U}_1 \widetilde{U}_2 \, dz\right) + \frac{\partial}{\partial m}\left(\rho \int_{-h}^{0} \widetilde{U}_2 \widetilde{U}_2 \, dz\right) + \rho g h \frac{\partial b}{\partial m} = 0. \tag{5.13}$$

Also the equation expressing total energy conservation is simply

$$\frac{\partial F_1}{\partial s} + \frac{\partial F_2}{\partial m} = 0, \tag{5.14}$$

where F_1 and F_2 are given by (5.9) and (5.10) respectively.

The equations (5.12)–(5.14) can be combined into one equation to eliminate the terms devoid of the wave amplitude a. To achieve this purpose, we multiply (5.12) by $\overline{U}_1$ and (5.13) by $\overline{U}_2$, and then subtract the resulting equations from (5.14), so that the terms originated from the last terms in (5.9), (5.10), (5.12) and (5.13) can immediately be cancelled out in this operation, considering the mass conservation equation

$$\frac{\partial}{\partial s}(\overline{U}_1 h) + \frac{\partial}{\partial m}(\overline{U}_2 h) = 0. \tag{5.15}$$

The integrals in (5.9), (5.10), (5.12) and (5.13) can also yield the terms free from a in this operation. However, by using (5.15) repeatedly and in consideration of (3.20), (5.3), (5.4) and (5.5), it can be proved that

$$\begin{aligned}
&\frac{\partial}{\partial s}\left(\frac{1}{2}\int_{-h}^{0} \widetilde{U}_1^3 \, dz + \frac{1}{2}\int_{-h}^{0} \widetilde{U}_1 \widetilde{U}_2^2 \, dz\right) + \frac{\partial}{\partial m}\left(\frac{1}{2}\int_{-h}^{0} \widetilde{U}_1^2 \widetilde{U}_2 \, dz + \frac{1}{2}\int_{-h}^{0} \widetilde{U}_2^3 \, dz\right) - \overline{U}_1 \frac{\partial}{\partial s}\left(\int_{-h}^{0} \widetilde{U}_1^2 \, dz\right) \\
&\quad - \overline{U}_1 \frac{\partial}{\partial m}\left(\int_{-h}^{0} \widetilde{U}_1 \widetilde{U}_2 \, dz\right) - \overline{U}_2 \frac{\partial}{\partial s}\left(\int_{-h}^{0} \widetilde{U}_1 \widetilde{U}_2 \, dz\right) - \overline{U}_2 \frac{\partial}{\partial m}\left(\int_{-h}^{0} \widetilde{U}_2^2 \, dz\right) \\
&\qquad = \frac{1}{12} h^3 \Omega_2 \frac{\partial \Omega_2}{\partial s}\left(\frac{2\sigma + \Omega_2}{4h} a^2 - \frac{1}{2}\Omega_2 h\right).
\end{aligned} \tag{5.16}$$

Therefore, if $\partial\Omega_2/\partial s \neq 0$ (and $\Omega_2 \neq 0$), the terms originated from the integrals in (5.9), (5.10), (5.12) and (5.13) cannot be cancelled out in this operation and will yield a term which is free from a and cannot therefore be balanced by other terms remaining in the equation (imaging the case when $a = 0$). This situation implies that when $\Omega_1 = 0$ and dissipation is neglected (so that the energy conservation equation (5.14) together with (5.8) is valid), we have $\partial\Omega_2/\partial s \approx 0$. This result is consistent with (4.7) in view of (3.27).

The rest of the terms in the equation resulting from this operation all contain the wave amplitude a and can be divided into three groups. The first group is devoid of h while each term in the second and third groups contains respectively h and h^2 as the common factor. However, if $\partial\Omega_2/\partial s = 0$, the terms in the third group are completely cancelled out and in the meantime, the two expressions from the first and second groups without consideration of their common factors coincide with each other exactly. Therefore, for an arbitrary value of h, the original equation reduces to

$$(g + 2\sigma U_0 + \Omega_2 U_0)\frac{1}{a}\frac{\partial a}{\partial s} + (2\sigma + \Omega_2)V_0\frac{1}{a}\frac{\partial a}{\partial m} + U_0\frac{\partial \sigma}{\partial s} + V_0\frac{\partial \sigma}{\partial m} + 2\sigma\frac{\partial U_0}{\partial s} + \sigma\frac{\partial V_0}{\partial m} + \left(\frac{\partial U_0}{\partial s} + \frac{\partial V_0}{\partial m}\right)\Omega_2 = 0, \tag{5.17}$$

which is indeed consistent with (3.26) and (4.6) when both Ω_1 and $\partial\Omega_2/\partial s$ in these two equations vanish in this case.

In order to obtain a physical insight into (5.17), the mean wave energy density

$$E \equiv \overline{\int_{-h}^{\eta} \rho g z\, dz} - \int_{-h}^{0} \rho g z\, dz + \frac{\rho}{2}\overline{\int_{-h}^{\eta} (\widetilde{u}_1^2 + \widetilde{u}_2^2 + \widetilde{u}_3^2)\, dz} - \frac{\rho}{2}\int_{-h}^{0} (\widetilde{U}_1^2 + \widetilde{U}_2^2)\, dz \tag{5.18}$$

defined by Jonsson *et al.* (1978) is also evaluated here. Substitution of (5.2), (5.3) and (5.5) into (5.18) yields

$$E = \frac{\rho}{2k}a^2\sigma\left(\sigma + \frac{\Omega_2}{2}\right) + \frac{\rho}{4}a^2\Omega_2 h\left(\sigma + \frac{\Omega_2}{2}\right). \tag{5.19}$$

On the other hand, it follows from (4.5) that

$$C_g^{(1)} = \frac{\sigma}{2k}(\sigma + \Omega_2)\Big/\left(\sigma + \frac{\Omega_2}{2}\right), \qquad C_g^{(2)} = 0 \tag{5.20}$$

in the local co-ordinate system in the present case. Thus (5.9) and (5.10) can also be written as

$$
\begin{aligned}
F_1 =&(U_0 + C_g^{(1)})E + \overline{U}_1 S_{11} + \frac{\rho}{2}\int_{-h}^{0} \widetilde{U}_1^3\, dz + \frac{\rho}{2}\int_{-h}^{0} \widetilde{U}_2^2 \widetilde{U}_1\, dz + \rho g b h \overline{U}_1,\\
F_2 =&(V_0 + C_g^{(2)})E + \overline{U}_2 S_{22} + \frac{\rho}{2}\int_{-h}^{0} \widetilde{U}_1^2 \widetilde{U}_2\, dz + \frac{\rho}{2}\int_{-h}^{0} \widetilde{U}_2^3\, dz + \rho g b h \overline{U}_2.
\end{aligned}
$$

Consequently, the combination of equations (5.12)-(5.14), which has previously led to (5.17), produces

$$
\frac{\partial}{\partial s}\left[(U_0 + C_g^{(1)})E\right] + \frac{\partial}{\partial m}\left[(V_0 + C_g^{(2)})E\right] + S_{11}\frac{\partial \overline{U}_1}{\partial s} + S_{22}\frac{\partial \overline{U}_2}{\partial m} = 0, \tag{5.21}
$$

which is certainly equivalent to (5.17).

The above equation indicates that in case (2) in §4, the slow variation of the wave amplitude can be specified by not only the action conservation equation but also the energy balance equation involving the rate of working by the radiation stress $S_{\alpha\beta}$ against the mean rate of strain. Therefore the situation here is quite similar to that in an irrotational current, but in the present case, according to (4.2) and (5.19), the relation between E and I becomes $I = E/\sigma_m$, where $\sigma_m \equiv n_0 - k\overline{U}_1 = \sigma + \Omega_2 kh/2 + O(a^2)$ is the frequency of waves relative to the frame of reference moving with the average-over-depth velocity component $\overline{U}_1$. This relation is different from that established by Bretherton & Garrett (1968) in which σ_m is replaced by the intrinsic frequency σ, which is also the frequency in a frame of reference moving with the surface velocity component U_0. However, according to the footnote in page 412 in Jonsson *et al.* (1978), an anonymous referee of their paper has pointed out that redefining the wave energy density as E' by calculating the kinetic energy solely from the perturbation particle velocities, resulting in

$$
E' = \frac{1}{4}\rho a^2 \left(2g - \Omega_2 \frac{\sigma}{k}\right) = \frac{\rho}{2k} a^2 \sigma \left(\sigma + \frac{\Omega_2}{2}\right) \tag{5.22}
$$

in view of (2.10), the wave action density I can again be written in terms of the intrinsic frequency σ as E'/σ. This situation remains to be true in the present case. In addition, equation (5.21) is equivalent to

$$
\frac{\partial}{\partial s}\left[(U_0 + C_g^{(1)})E'\right] + \frac{\partial}{\partial m}\left[(V_0 + C_g^{(2)})E'\right] + S'_{11}\frac{\partial U_0}{\partial s} + S'_{22}\frac{\partial V_0}{\partial m} = 0, \tag{5.23}
$$

where

$$\left.\begin{aligned} S'_{11} &= \frac{\rho}{4} g a^2 \\ S'_{22} &= \frac{\rho}{4k} \sigma \Omega_2 a^2 \\ S'_{12} &= S'_{21} = 0 \end{aligned}\right\} \tag{5.24}$$

is the new radiation stress arising from (5.7) by neglect of the term involving h. This result is not really surprising in view of the statements above (5.17) and considering that the difference of E' from E also lies in the term containing h.

Equation (5.23) together with (5.22) and (5.24) is simpler than (5.21) together with (5.19) and (5.7), but as pointed out by Jonsson *et al.* (1978), the physical interpretation of E' (and also $S'_{\alpha\beta}$) is not obvious because the wave energy equation (5.23) does not follow directly from the overall energy equation (5.14) in the integral approach. However, the integral approach itself will become invalid even in the simple cases (3) and (4) in §4 in which $\partial U/\partial z = 0$ but $\partial V/\partial z \neq 0$. This situation and the significance of (5.23) together with (5.22) and (5.24) will be discussed in the next section.

6. The limitations of the integral approach

We start the discussion by considering the case (3) in §4. In this simple case, the waves propagate perpendicularly on a steady two-dimensional current with strong and constant shear so that the properties of the wave train and the water depth will not vary in the $s-$direction. Furthermore, since $U = 0$ and $v \neq 0$, the total horizontal velocity components become $\widetilde{u}_1 = \partial\phi/\partial s, \widetilde{u}_2 = v + V$ where $V = V_0 - \Omega_1 z$, and the s, m components of the 'formal current velocity' can now be defined as

$$\widetilde{U}_1(z) = \widetilde{U}_{10}, \qquad \widetilde{U}_2(z) = \widetilde{U}_{20} - \Omega_1 z, \tag{6.1}$$

where

$$\widetilde{U}_{10} = \frac{\sigma}{2h}a^2, \qquad \widetilde{U}_{20} = V_0 + \frac{\Omega_1}{4h}a^2 \tag{6.2}$$

by using (5.4). Now substituting all these results and the solutions given in §2 into (5.1) and (5.8), we obtain

$$\left.\begin{aligned} S_{11} &= \frac{\rho}{4}ga^2 \\ S_{22} &= \frac{\rho}{4k}\Omega_1^2 a^2 - \frac{\rho}{4}\Omega_1^2 a^2 h \\ S_{12} = S_{21} &= \frac{\rho}{4k}\sigma\Omega_1 a^2 - \frac{\rho}{4}\sigma\Omega_1 a^2 h \end{aligned}\right\} \tag{6.3}$$

and

$$\begin{aligned} F_2 =& \frac{\rho}{2}gV_0 a^2 + \frac{\rho}{4k}g\Omega_1 a^2 + \frac{3\rho}{8k}V_0\Omega_1^2 a^2 + \frac{3\rho}{16k^2}\Omega_1^3 a^2 - \frac{3\rho}{8}V_0\Omega_1^2 a^2 h - \frac{\rho}{8}\Omega_1^3 a^2 h^2 \\ &+ \frac{\rho}{2}\int_{-h}^{0}\widetilde{U}_2^3\,dz + \rho g b h \overline{U}_2, \end{aligned} \tag{6.4}$$

Where

$$\overline{U}_1 \equiv (1/h)\overline{\int_{-h}^{\eta}\widetilde{u}_1(z)\,dz} = \widetilde{U}_{10} = (\sigma/2h)a^2 \tag{6.5}$$

$$\overline{U}_2 \equiv (1/h)\overline{\int_{-h}^{\eta}\widetilde{u}_2(z)\,dz} = V_0 + (\Omega_1/4h)a^2 + \Omega_1 h/2 \tag{6.6}$$

by virtue of (5.4), (6.1) and (6.2).

The component F_1 of the total mean energy flux in the $s-$direction is not given here because in the mean total energy conservation equation (5.14), F_1 is differentiated with respect to s, which in the present case will vanish, so that (5.14) reduces to

$$\partial F_2/\partial m = 0. \tag{6.7}$$

For the same reason, (5.12) and (5.13) reduce to

$$\frac{\partial S_{12}}{\partial m} + \frac{\partial}{\partial m}\left(\rho \int_{-h}^{0} \widetilde{U}_1 \widetilde{U}_2 \, dz\right) = 0, \tag{6.8}$$

$$\frac{\partial S_{22}}{\partial m} + \frac{\partial}{\partial m}\left(\rho \int_{-h}^{0} \widetilde{U}_2^2 \, dz\right) + \rho g h \frac{\partial b}{\partial m} = 0 \tag{6.9}$$

in which

$$\rho \int_{-h}^{0} \widetilde{U}_1 \widetilde{U}_2 \, dz = \rho\sigma V_0 a^2/2 + \rho\sigma\Omega_1 a^2 h/4 \tag{6.10}$$

from (6.1) and (6.2).

Notice that after substituting (6.3) and (6.10) into (6.8), the terms devoid of a are absent from the resulting equation. Therefore, to eliminated these terms from the rest of the conservation equations, only (6.7) and (6.9) will be combined into one equation. This can be achieved by multiplying (6.9) by $\overline{U}_2$ and then by subtracting the resulting equation from (6.7), which by means of (5.15) can immediately cancel out the last term of (6.9) and the terms originated from the last term of (6.4). Moreover, since it follows from (6.1), (6.2) and (6.6) that

$$\int_{-h}^{0} \widetilde{U}_2^3 \, dz = \overline{U}_2^3 h + \overline{U}_2 \Omega_1^2 h^3/4,$$

$$\int_{-h}^{0} \widetilde{U}_2^2 \, dz = \overline{U}_2^2 h + \Omega_1^2 h^3/12$$

and since $\partial(\overline{U}_2 h)/\partial m = 0$ from (5.15), the integral terms in the combination of (6.7) and (6.9)

$$\frac{\partial}{\partial m}\left(\frac{\rho}{2} \int_{-h}^{0} \widetilde{U}_2^3 \, dz\right) - \overline{U}_2 \frac{\partial}{\partial m}\left(\rho \int_{-h}^{0} \widetilde{U}_2^2 \, dz\right) = \frac{\rho}{12} \overline{U}_2 \Omega_1 h^3 \frac{\partial \Omega_1}{\partial m} \tag{6.11}$$

Since the rest of the terms in the combination of (6.7) and (6.9) all contain a, the result (6.11) implies that in the absence of wave motion and dissipation, it is necessary that $\partial\Omega_1/\partial m = 0$ when $U = 0$, which is consistent with (3.29). If this situation remains so when $a \neq 0$, after the above cancellation, the combination of (6.7) and (6.9) finally results in

$$\frac{1}{k}\left[\rho a \frac{\partial a}{\partial m}\left(\frac{1}{2} g\Omega_1 + gkV_0 + \frac{1}{4} V_0 \Omega_1^2 + \frac{3}{8k}\Omega_1^3\right) + \rho a^2 \frac{\partial V_0}{\partial m}\left(\frac{1}{2} gk + \frac{3}{8}\Omega_1^2\right)\right]$$

$$- h\left[\rho a \frac{\partial a}{\partial m}\left(\frac{1}{4} V_0 \Omega_1^2 + \frac{1}{4k}\Omega_1^3\right) + \rho a^2 \frac{\partial V_0}{\partial m}\left(\frac{1}{4}\Omega_1^2\right)\right] = 0. \tag{6.12}$$

The two expressions in the two square brackets in (6.12) cannot possibly coincide with each other so that (6.12) cannot reduce to an equation devoid of h, which is contrary to the situation resulting in (5.17) and can only show the invalidity of the integral approach in this case.

The reason for this invalidity is that in the present case in which $v \neq 0$, the nonlinear term $v\partial^2 v/\partial m\partial z$ in the component of the vorticity equation in the $s-$direction will result in an extra term in the time-independent equation (3.29). This term and the other similar terms have been ignored in the analysis in §3 in which the terms of $O(a^2)$ are all neglected, but in the integral approach, these terms (which will vanish in case (2) in §4) have the same order of magnitude as the terms contributing to (5.17) or (5.21) and cannot therefore be ignored. Thus in the present case, when $a \neq 0$, the terms on the right-hand side of (6.11) will not vanish so that equation (6.12) cannot be achieved. More important, the remaining terms on the right-hand side of (6.11), which should be included in (6.12), cannot be determined owing to the fact that the variation of $v\partial^2 v/\partial m\partial z$ with depth is specified by e^{2kz} while the term $V\partial^2 V/\partial m\partial z$ in (3.29) is constrained to vary linearly with depth in the present analysis. This situation will also occur in case (4) in §4, but in case (3), since there is no need to eliminate the terms devoid of a in (6.8) by using (3.29), from (6.8) it is possible to obtain

$$\frac{\partial}{\partial m}\left[\left(V_0 + \frac{\Omega_1}{2k}\right)\frac{\rho}{2}\sigma a^2\right] = 0. \tag{6.13}$$

Since $k = \text{const.}$ in the present case, this equation can be rewritten as

$$\frac{\partial}{\partial m}\left[\left(V_0 + \frac{\Omega_1}{2k}\right)\frac{\rho}{2k}\sigma a^2\right] = 0, \tag{6.14}$$

which corresponding to the action conservation equation (4.1) in view of (4.3) and considering that the action flux will not vary in the $s-$direction in the present case. Furthermore, since $\sigma = n_0 = \text{const.}$ in the present case, equation (6.13) can also be rewritten as

$$\frac{\partial}{\partial m}\left[\left(V_0 + \frac{\Omega_1}{2k}\right)\frac{\rho}{2k}\sigma^2 a^2\right] = 0, \tag{6.15}$$

which can be recognized as the wave energy equation if the wave energy density E' and the radiation stress $S'_{\alpha\beta}$ are defined by (5.22) and (5.24) respectively. This result implies that the transverse rotational perturbation velocity v, even if it has the same order of magnitude as $\partial\phi/\partial s$, should be excluded from the definitions of the wave energy density and the radiation stress.

In case (4) in §4, since $U \neq 0$ (though $\partial U/\partial z = 0$), the equation corresponding to (6.8) in case (3) will involve the terms devoid of a so that it is impossible to obtain an equation like (6.13) in this case by using the integral approach. However, since in this case, the action conservation equation derived by Voronovich (1976) is still consistent with (3.26), from (4.1), (4.3) and (4.5) it follows that

$$\frac{\partial}{\partial s}\left[(U_0 + C_g^{(1)})\frac{\rho}{2k}\sigma^2 a^2\right] + \frac{\partial}{\partial m}\left[(V_0 + C_g^{(2)})\frac{\rho}{2k}\sigma^2 a^2\right] - (U_0 + C_g^{(1)})\frac{\rho}{2k}\sigma a^2 \frac{\partial \sigma}{\partial s}$$
$$- (V_0 + C_g^{(2)})\frac{\rho}{2k}\sigma a^2 \frac{\partial \sigma}{\partial m} = 0 \tag{6.16}$$

in the present case. Furthermore, since $C_g^{(1)} = \sigma/2k$ and $C_g^{(2)} = \Omega_1/2k$ in the present case according to (4.5), from the equations derived by differentiation of the kinematical conservation equation and the dispersion relation, the last two terms of (6.16)

$$-(U_0 + C_g^{(1)})\frac{\rho}{2k}\sigma a^2 \frac{\partial \sigma}{\partial s} - (V_0 + C_g^{(2)})\frac{\rho}{2k}\sigma a^2 \frac{\partial \sigma}{\partial m} = \frac{\rho}{4k}\sigma^2 a^2 \frac{\partial U_0}{\partial s} = \frac{\rho}{4} g a^2 \frac{\partial U_0}{\partial s}$$

in virtue of (4.4) with $\partial U/\partial z = 0$. (Recall that in case (4) in §4, $\partial U_0/\partial m = 0$.) Therefore (6.16) reduces to

$$\frac{\partial}{\partial s}\left[(U_0 + C_g^{(1)})\frac{\rho}{2k}\sigma^2 a^2\right] + \frac{\partial}{\partial m}\left[(V_0 + C_g^{(2)})\frac{\rho}{2k}\sigma^2 a^2\right] + \frac{\rho}{4} g a^2 \frac{\partial U_0}{\partial s} = 0, \tag{6.17}$$

which again can be recognized as the wave energy equation if the wave energy density E' and the radiation stress $S'_{\alpha\beta}$ are defined by (5.22) and (5.24) respectively.

When the underlying mean flow becomes more complicated by allowing that Ω_2 has the same order of magnitude as σ and Ω_1, the action conservation equation derived by Voronovich (1976), the wave energy equation (5.23) together with (5.22) and (5.24), and the modulation equation (3.26) are in general inconsistent with one another. However, since Ω_2 has already been included in the definitions of E' and $S'_{\alpha\beta}$ in (5.22) and (5.24), one may expect that with a slight modification, the wave energy equation (5.23) can still specify the slow modulation of the wave amplitude in a general case, which will be discussed in the next section.

7. Modification of the wave energy equation

In this section, the wave energy equation (5.23) will be modified to reconcile it with the modulation equation (3.26) in a general case, but before this, equation (5.23) together with (5.22) and (5.24) will be expressed in a fixed co-ordinate system and in tensor notation. This equation will then be expanded, followed by the choice of the local co-ordinate system in which (3.26) has been derived so that the resulting equation can be compared with (3.26). After this comparison, the modification of the wave energy equation will be conducted and explained in the fixed co-ordinate system, which is useful in practical applications.

In the fixed co-ordinate system and in tensor notation, the wave energy equation (5.23) together with (5.22) and (5.24) can be written as

$$\frac{\partial}{\partial x_\alpha}\left[(U_0^{(\alpha)} + C_g^{(\alpha)})E\right] + S_{\alpha\beta}\frac{\partial U_0^{(\beta)}}{\partial x_\alpha} = 0, \tag{7.1}$$

where the wave energy density

$$E = \frac{\rho}{2k}a^2\sigma\left(\sigma + \frac{1}{2}\frac{k_\alpha}{k}\frac{\partial U_\alpha}{\partial z}\right) \tag{7.2}$$

and the radiation stress

$$S_{\alpha\beta} = a_{\alpha\gamma}a_{\beta\delta}S'_{\gamma\delta}, \tag{7.3}$$

in which the matrix

$$A \equiv \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} k_1/k & -k_2/k \\ k_2/k & k_1/k \end{bmatrix} \tag{7.4}$$

and the radiation stress components $S'_{\alpha\beta}$, $(\alpha, \beta = 1, 2)$ are given by (5.24).

Expression (7.3) together with (7.4) represents the transformation of the radiation stress components $S'_{\alpha\beta}$, $(\alpha, \beta = 1, 2)$ in the local co-ordinate system (s, m, z) into the components $S_{\alpha\beta}$, $(\alpha, \beta = 1, 2)$ in the co-ordinate system (x_1, x_2, z) in which k_1 and k_2 are the components of $\mathbf{k}$. Substituting (4.5) for $C_g^{(\alpha)}$ in (7.1) and following the procedure which leads to (4.6), we have

$$\begin{aligned}(g' + 2\sigma U_0 + \Omega_{20}U_0)\frac{1}{a}\frac{\partial a}{\partial s} + (2\sigma + \Omega_{20})V_0\frac{1}{a}\frac{\partial a}{\partial m} + U_0\frac{\partial \sigma}{\partial s} + V_0\frac{\partial \sigma}{\partial m} + 2\sigma\frac{\partial U_0}{\partial s} + \sigma\frac{\partial V_0}{\partial m} \\ + \left(\frac{\partial U_0}{\partial s} + \frac{\partial V_0}{\partial m}\right)\Omega_{20} - \frac{\sigma}{k}\frac{\partial^2 U}{\partial s \partial z}\bigg|_{z=0} + \left(\frac{\sigma}{k}\frac{1}{a}\frac{\partial a}{\partial m} + \frac{1}{k}\frac{\partial \sigma}{\partial m} - \frac{\sigma}{k^2}\frac{\partial k_1}{\partial m}\right)\Omega_{10} \\ = -\frac{1}{2}\frac{\sigma}{k}\frac{\partial^2 U}{\partial s \partial z} + \frac{1}{2}\frac{\sigma}{k}\frac{\partial^2 V}{\partial m \partial z}\end{aligned} \tag{7.5}$$

The difference between (7.5) and (4.6) is clear and both equations are inconsistent with (3.26) in a general case. However, since the two terms on the right-hand side of (7.5), which represent the difference between (7.5) and (3.26), can in tensor notation be written as

$$-\frac{\sigma}{k}\frac{k_\alpha}{k}\frac{k_\beta}{k}\frac{\partial^2 U_\beta}{\partial x_\alpha \partial z}+\frac{1}{2}\frac{\sigma}{k}\frac{\partial^2 U_\alpha}{\partial x_\alpha \partial z}, \tag{7.6}$$

which multiplied by the common factor $\rho\sigma a^2/2k$ arise directly from the expansion of the term

$$\frac{\partial}{\partial x_\alpha}[C_g^{(\alpha)}E]=\frac{\partial}{\partial x_\alpha}\left[\frac{\rho}{2k}a^2\sigma\left(\frac{1}{2}\frac{\sigma^2}{k}\frac{k_\alpha}{k}+\frac{\sigma}{k}\frac{k_\alpha}{k}\frac{k_\beta}{k}\frac{\partial U_\beta}{\partial z}-\frac{1}{2}\frac{\sigma}{k}\frac{\partial U_\alpha}{\partial z}\right)\right], \tag{7.7}$$

the new wave energy equation that is consistent with (3.26) in a general case is

$$\frac{\partial}{\partial x_\alpha}\left[(U_0^{(\alpha)}+C_g^{(\alpha)})E\right]-\left(\frac{\rho}{2k^2}a^2\sigma^2\frac{k_\alpha}{k}\frac{k_\beta}{k}\frac{\partial^2 U_\beta}{\partial x_\alpha \partial z}-\frac{\rho}{4k^2}a^2\sigma^2\frac{\partial^2 U_\alpha}{\partial x_\alpha \partial z}\right)+S_{\alpha\beta}\frac{\partial U_0^{(\beta)}}{\partial x_\alpha}=0. \tag{7.8}$$

The extra terms in (7.8) certainly represent the wave-mean flow interaction, the consequence of which is simply that although the wave energy flux due to the group velocity is explicitly dependent on $\partial U_\alpha/\partial z$ (see (7.7)), the net effect of this flux and the wave-mean flow interaction is independent of $\partial U_\alpha/\partial z$.

8. Conclusions

We have derived the modulation equation (3.26) for the amplitudes of deep-water gravity waves propagating obliquely on a steady three-dimensional, strongly sheared current that varies slowly in the horizontal directions and deviates slightly from a linear profile in the vertical direction. This equation is consistent with the action conservation equation derive by Voronovich (1976) as long as the rotational perturbation velocity becomes negligible or its fast variation with depth can be specified solely by the simple function e^{kz}. The reason for the action conservation equation being inconsistent with (3.26) in a general case can be explained satisfactorily so that the wave action is not likely to be conserved in this general case. However, in an even more general case in which the deviation of the current velocity from a linear profile in the vertical direction is not small or the density-field inhomogeneities exist, the reason for the invalidity of the action conservation equation is absent so that the wave action is conserved in this case.

When the action conservation equation is inconsistent with (3.26), to provide a physical insight into the latter, the theory of Jonsson *et al.* (1978) has been extended by using the integral approach, although for an apparent reason this approach also becomes invalid when the rotational perturbation velocity is no longer negligible. The results indicate that the wave energy equation (5.23) together with (5.22) and (5.24) derived by using this approach in a case in which the rotational perturbation velocity is negligible can remain consistent with (3.26) and the action conservation equation even when the rotational perturbation velocity is no longer negligible but varies with depth as e^{kz}, implying that even if the transverse rotational perturbation velocity v has the same order of magnitude as the longitudinal irrotational one, v should not be included in the definitions of the wave energy density and the radiation stress.

In the general case in which the rotational perturbation velocity is not negligible and its fast variation with depth cannot be specified solely by e^{kz}, the wave energy equation (5.23) can still be reconciled with (3.26) by adding two extra terms to the former, indicating that the definitions of the wave energy density and the radiation stress without consideration of v remain valid in this case but additional wave-mean flow interaction occurs now. Its consequence is simply that although the wave energy flux due to the group velocity is explicitly dependent on the vertical shear of the current velocity, the net effect of this flux and the wave-mean flow interaction is independent of this vertical shear.

Finally, we emphasize that to obtain (3.26), equations (3.20) and (3.29) have been used constantly, meaning that the modulation equation (3.26) and therefore the modified wave energy equation (7.8) can hold only if the background motion satisfies the original, fully non-linear hydrodynamic equations of an inviscid fluid within the present approximation. For example, even if the term $(\mu/\rho)(\partial^2/\partial z^2)(\partial U/\partial z)$ in the vorticity equation due to the viscosity is non-zero, as long as this term is at least one order of magnitude smaller than the dominant terms in (3.20), equations (3.26) and (7.8) can still hold. This situation remains true for the action conservation equation derived by Voronovich (1976), although in this case, the background motion is allowed to vary slowly with time. Therefore, depending on how small the deviation of the current velocity from a linear profile in the vertical direction is, one equation among the action conservation equation (together with the series solution of Shrira (1993)) and the modified wave energy equation can be chosen to combine with the theories of wind-wave and wave-wave interactions and other theories to estimate the wave field on a three-dimensional, strongly sheared current. Specifically, the modified wave energy equation can be applied to estimate the wave fields in the regions outside Hualien and Su-Auo harbours on the eastern coast of Taiwan during typhoons, because in these events, the waves and the strong shear of the mean flow can be generated on the Kuroshio, which itself will vary in the regions not far from the coastline. On the other hand, since the high-frequency wave components, of which the wavelengths are small compared with the horizontal length scale of the Kuroshio, can excite the low-frequency resonances in a harbour according to Chen, Mei & Chang (2006), the present theory may benefit the study to reduce the resonances of Hualien and Su-Auo harbours.

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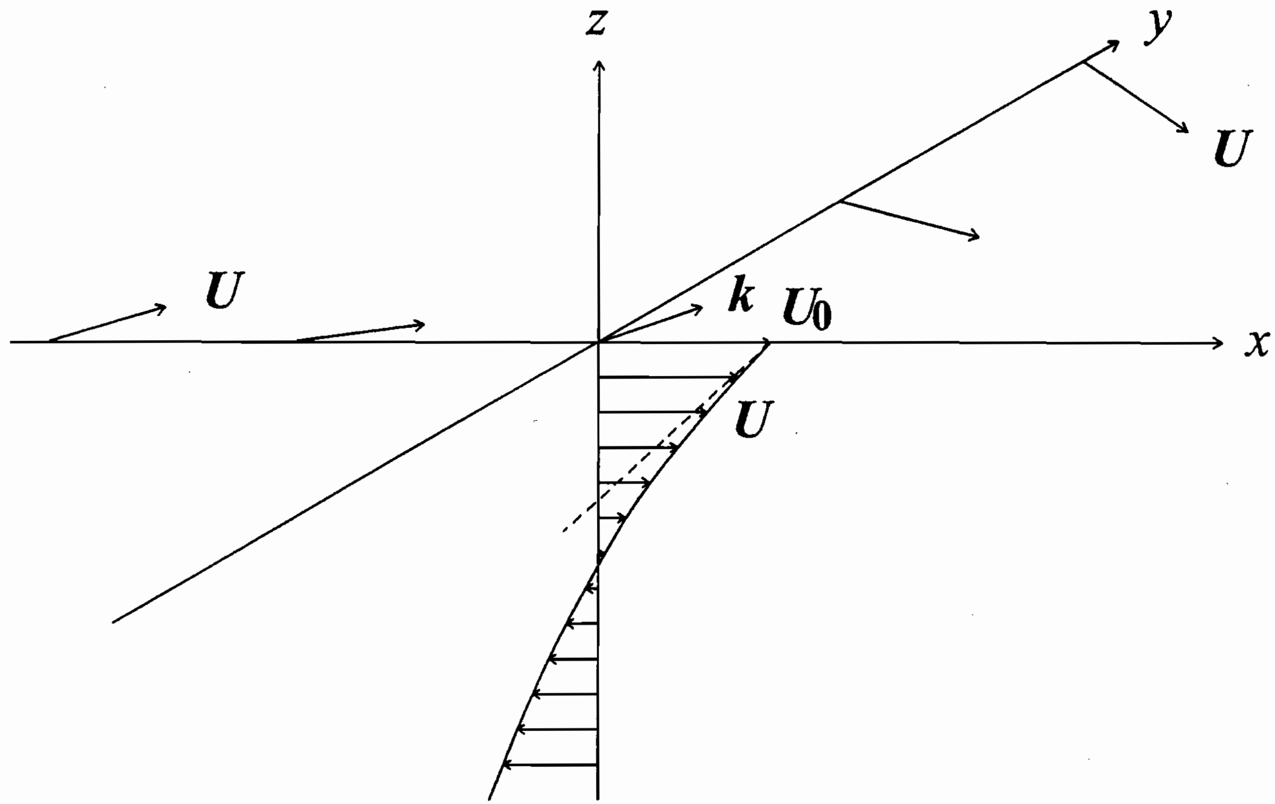


Fig. 1. The variations of the current field.

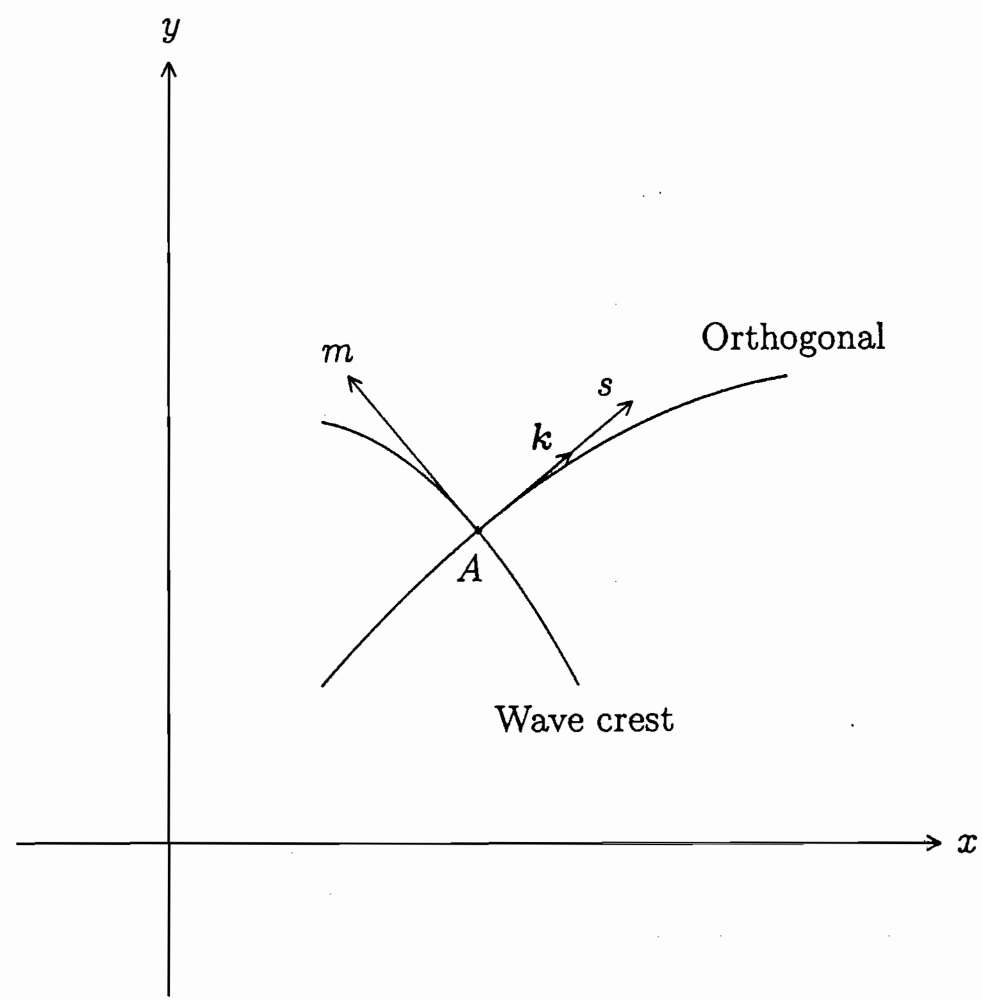


Fig. 2. A diagram to link the fixed co-ordinate system x, y and the local co-ordinate system s, m at point A. The vertical co-ordinate z in these two systems remains unchanged.

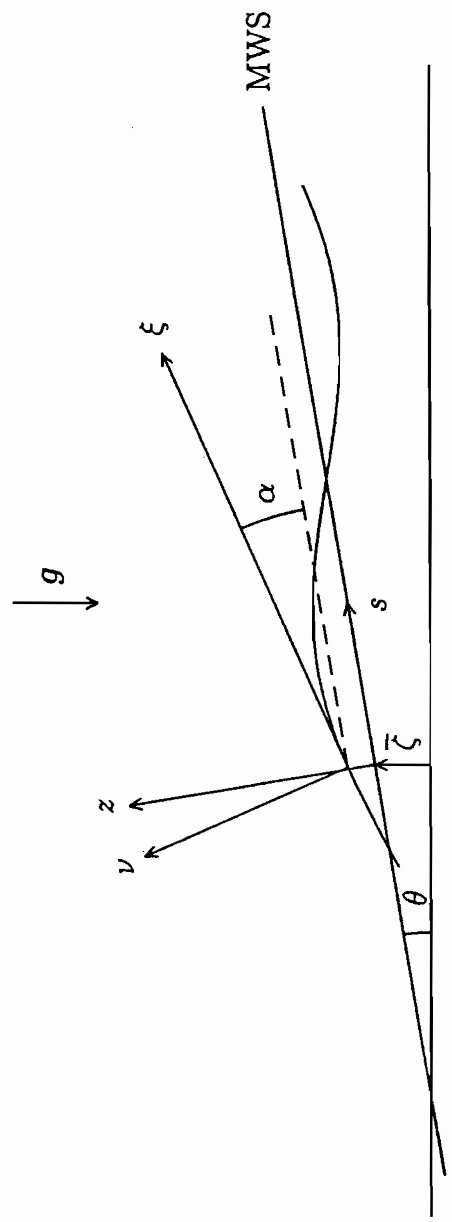


Fig. 3. Definition sketch.

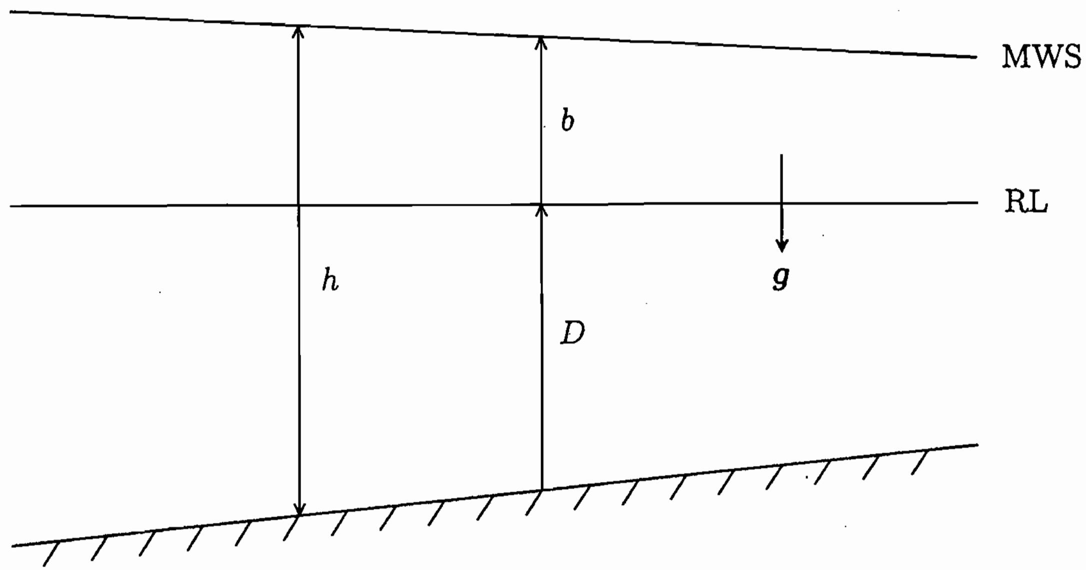


Fig. 4. Definition sketch.

交通部運輸研究所

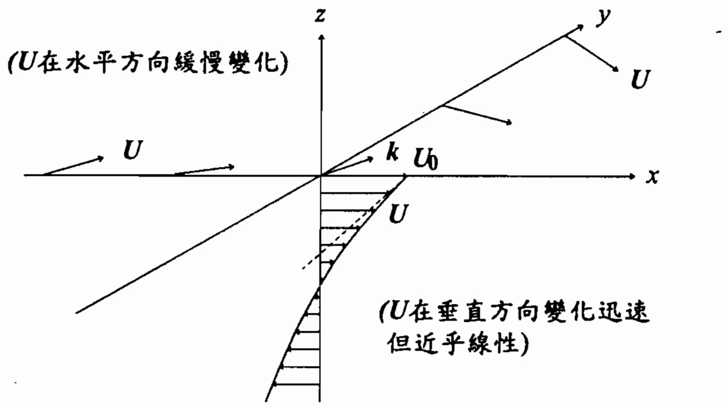
「臺灣東岸港口共振現象改善方案研究(2/4)」

九十五年度科技計畫審查會審查意見答覆及辦理情形

審查委員及審查意見	答覆及辦理情形
林銘崇教授： 波流互制作用中，波與流之相對強度在推導處理過程是否有所考量或範圍條件設定?	在目前的推導處理過程中，未對波與流之相對強度做任何範圍條件設定，僅在第三節開始對流的速度在水平方向上的變化率以及流的旋度在水平和垂直方向上的變化率設定條件，故目前的理論可適用於任何流速大小。
李忠潘教授： 1. Euler equation 應用於無黏性的流場，理論上當然可以加上任何給定的流場，如文中的 $U(U(z),V(z),0)$，但是一個垂直方向 strongly sheared 的流場似乎無法假設為無黏性，故需要加以說明其實用性，譬如，能否用於邊界層。 2. 文內如能像簡報一樣多加上一些圖來說明，將較容易理解。 3. 一般 shear flow 指的(或有此意思)是 shear stress 造成的，故容易被視為有黏性的流體內的流場，與 Euler eq. 不符，需加說明。	1. 一個垂直方向 strongly sheared 的流場雖然實際大多由黏滯力所造成，但在無黏性流中仍可維持存在(表滿足 Euler equation)，目前的理論即針對此種可「存在」於無黏性流中的剪流，導出波受其影響發生變化之方程式。此種剪流包含甚廣，且可做為許多實際流場的近似，故目前的結果可適用於包括邊界層在內的許多流場，只要這些流場能滿足第三節開始所設定的條件。 2. 目前已採納李教授的意見，將簡報中的一張圖加在文章內做為 figure 1，但另外一張圖，由於需簡報者以手勢輔助說明，方能達到其目的，故未加入文章內。 3. 此一問題以及第一個問題已在 Conclusions 最後新加一段文字說明。

李兆芳教授： 1. 報告內容為波浪和水流互相作用之理論解析。包含基本解、JBKT 理論延伸、及理論限制之討論，整個內容敘述完整。 2. 報告內容和計畫名稱“東岸港口共振現象改善”之相關和應用可以在文中加強。 3. 計畫為四年之第二年，內容中可以加入整個計畫之內容，各年之預期內容和相關性。	1. 感謝肯定。 2. 為加強報告內容和計畫名稱之相關和應用，已在 Conclusions 最後新加一段文字說明。 3. 由於本篇報告的內容雖然對計畫未來的發展有幫助，但仍有其獨立性，且內容十分複雜，故為避免分散讀者的注意力，在報告內容中未將整個計畫之內容以及各年之預期內容加入。
郭一羽教授： 1. 計畫名稱與內容不易銜接，報告內容應對此多加解釋。 2. 此理論的發展在應用上請多加說明。	1. 為將報告內容與計畫名稱銜接，已在 Conclusions 最後新加一段文字說明。 2. 上述新增加的說明亦和此理論如何應用於實際問題有關。另外在報告的第 3 頁及第 8 頁對此亦有詳細的說明。
簡仲璟科長： 1. 本研究利用 local co-ordinate system 將旋性與非旋性擾動分離，而獲得波浪解，是一創新的方法。惟本研究結果對於港口共振改善的探討有何助益？建議可補充說明。 2. 報告內容完整，說明詳盡，但可應用的波浪情況可再加描述。例如旋性流的流速大小與波速之相較。	1. 為補充說明本研究結果對於港口共振改善的探討之助益，目前已在 Conclusions 最後新加一段文字說明。 2. 由於在目前的理論推導中，對流的速度本身未做任何的假設，表流的速度(和波速相比)沒有任何限制，而波浪亦可為任何的深水重力波，僅流的速度在水平方向的變化率及流的旋度在水平和垂直方向的變化率，如第三節開始所說，有所限制。

Propagation of deep-water gravity waves on three-dimensional, strongly sheared currents



以往的理論：(非旋流)

1. 2D wave energy equation (Longuet-Higgins & Stewart 1960, 1961)

$$\frac{\partial E}{\partial t}+\frac{\partial}{\partial x_\alpha}\left\{E[U_\alpha+C_g^{(\alpha)}]\right\}+S_{\alpha\beta}\frac{\partial U_\beta}{\partial x_\alpha}=0$$

(E : wave energy density, $S_{\alpha\beta}$: radiation stress)

2. 2D action conservation equation (Bretherton & Garrett 1969)

$$\frac{\partial I}{\partial t}+\frac{\partial}{\partial x_\alpha}\left\{I[U_\alpha+C_g^{(\alpha)}]\right\}=0$$

($I\equiv E/\sigma$: wave action density, σ : intrinsic frequency)

以往的理論：(旋性流)

1. 2D action conservation equation (Voronovich 1976)

$$\frac{\partial I}{\partial t}+\frac{\partial}{\partial x_\alpha}\left\{I[U_0^{(\alpha)}+C_g^{(\alpha)}]\right\}=0$$

(I 包含平均流之 vorticity Ω , 且 $\Omega=O(\sigma)$)

2. 1D action conservation equation (Jonsson *et al.* 1978)

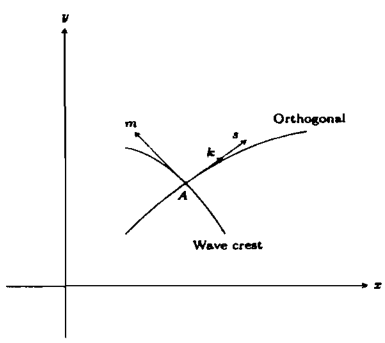
$$\frac{d}{dx}\{I[U_0+C_g]\}=0 \quad (I \text{ 包含}\Omega\text{，且}\Omega=O(\sigma))$$

3. 2D action conservation equation (White 1999)

$$\frac{\partial I}{\partial t}+\frac{\partial}{\partial x_\alpha}\left\{I[U_\alpha+C_g^{(\alpha)}]\right\}=0 \quad (\Omega\ll\sigma\text{, 故和非旋流者相同})$$

目前的理論：

採用一種local co-ordinate system (s, m, z)

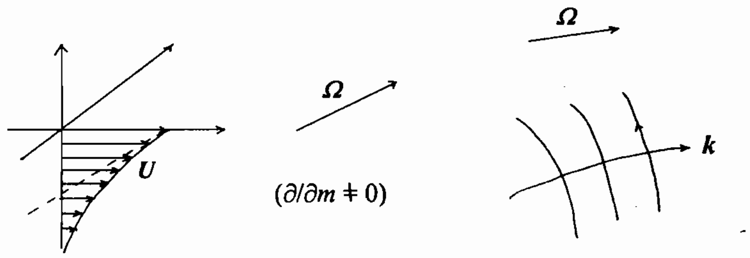


使 rotational and irrotational perturbation velocities 可以分離

⟹ Perturbation method 中的 secular conditions 因而改變

但 rotational perturbation velocity 無法獲得封閉形式的解

⟹ 傳統的 perturbation method 無法適用！



第一階近似： $\dfrac{\partial \phi^{(1)}}{\partial s}$, $v^{(1)}$, $\dfrac{\partial \phi^{(1)}}{\partial z}$

第二階修正： $\dfrac{\partial \phi^{(2)}}{\partial s} + u^{(2)}$, $v^{(2)}$, $\dfrac{\partial \phi^{(2)}}{\partial z} + w^{(2)}$

傳統的perturbation method： (Voronovich 1976)

經由asymptotic expansion 獲得第二階 eq.

field eq.：

$$w^{(2)\prime\prime} + \left(\frac{k^2 \mu}{\omega_d^2} - \frac{\omega_d''}{\omega_d} - k^2 \right) w^{(2)} = F$$

B.C.：

$$\begin{cases} \omega_d \dfrac{\partial w^{(2)}}{\partial z} - \left(\dfrac{g k^2}{\omega_d} + \omega_d' \right) w^{(2)} = G_1 & \text{at} \quad z = 0 \\ w^{(2)} = G_2 & \text{at} \quad z = -h \end{cases}$$

由 F, G_1, G_2 決定第一階近似解之secular condition

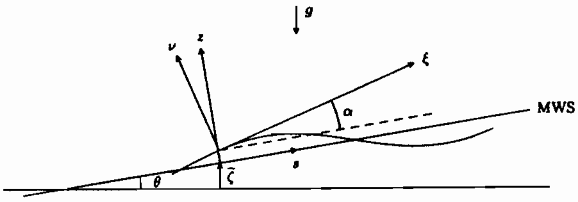
若 $w^{(2)}$ 用 $\dfrac{\partial \phi^{(2)}}{\partial z} + w^{(2)}$ 取代，則 secular condition 將改變！

但 $w^{(2)}$ 實際無法獲得封閉形式的解

⟹ 傳統的 perturbation method 無法適用！

故採用 Shyu & Phillips (1990) 和 Shyu & Tung (1999) 的方法

⟹ 可避免正式的asymptotic expansion ⟹ 免用secular condition



僅需決定平均水面處的 $u^{(2)}$, $v^{(2)}$, $w^{(2)}$，故可獲得

$$(g' + 2\sigma U_0 + \Omega_{20} U_0)\frac{1}{a}\frac{\partial a}{\partial s} + (2\sigma + \Omega_{20})V_0\frac{1}{a}\frac{\partial a}{\partial m} + U_0\frac{\partial \sigma}{\partial s} + V_0\frac{\partial \sigma}{\partial m} + 2\sigma\frac{\partial U_0}{\partial s} + \sigma\frac{\partial V_0}{\partial m}$$
$$+\left(\frac{\partial U_0}{\partial s} + \frac{\partial V_0}{\partial m}\right)\Omega_{20} - \frac{\sigma}{k}\frac{\partial^2 U}{\partial s \partial z}\bigg|_{z=0} + \left(\frac{\sigma}{k}\frac{1}{a}\frac{\partial a}{\partial m} + \frac{1}{k}\frac{\partial \sigma}{\partial m} - \frac{\sigma}{k^2}\frac{\partial k_1}{\partial m}\right)\Omega_{10} = 0$$

和Voronovich (1976) 的結果比較：

將 action conservation equation 展開，並進行座標轉換後，獲得

$$(g' + 2\sigma U_0 + \Omega_{20} U_0)\frac{1}{a}\frac{\partial a}{\partial s} + (2\sigma + \Omega_{20})V_0\frac{1}{a}\frac{\partial a}{\partial m} + U_0\frac{\partial \sigma}{\partial s} + V_0\frac{\partial \sigma}{\partial m} + 2\sigma\frac{\partial U_0}{\partial s} + \sigma\frac{\partial V_0}{\partial m}$$
$$+\left(\frac{\partial U_0}{\partial s} + \frac{\partial V_0}{\partial m}\right)\Omega_{20} - \frac{\sigma}{k}\frac{\partial^2 U}{\partial s \partial z}\bigg|_{z=0} + \left(\frac{\sigma}{k}\frac{1}{a}\frac{\partial a}{\partial m} + \frac{1}{k}\frac{\partial \sigma}{\partial m} - \frac{\sigma}{k^2}\frac{\partial k_1}{\partial m}\right)\Omega_{10} \neq 0$$
$$= -\frac{1}{2}\frac{\sigma}{k}\frac{\partial^2 U}{\partial s \partial z} + \frac{1}{2}\frac{\sigma}{k}\frac{\partial^2 V}{\partial m \partial z} + \frac{\partial V}{\partial z}\frac{\partial U_0}{\partial m}$$

證明：當 $\Omega = O(\sigma)$ 時，wave action 一般無法守恆！

修改wave energy equation：

$$(g' + 2\sigma U_0 + \Omega_{20} U_0)\frac{1}{a}\frac{\partial a}{\partial s} + (2\sigma + \Omega_{20})V_0 \frac{1}{a}\frac{\partial a}{\partial m} + U_0\frac{\partial \sigma}{\partial s} + V_0\frac{\partial \sigma}{\partial m} + 2\sigma\frac{\partial U_0}{\partial s} + \sigma\frac{\partial V_0}{\partial m}$$
$$+ \left(\frac{\partial U_0}{\partial s} + \frac{\partial V_0}{\partial m}\right)\Omega_{20} - \frac{\sigma}{k}\frac{\partial^2 U}{\partial s \partial z}\bigg|_{z=0} + \left(\frac{\sigma}{k}\frac{1}{a}\frac{\partial a}{\partial m} + \frac{1}{k}\frac{\partial \sigma}{\partial m} - \frac{\sigma}{k^2}\frac{\partial k_1}{\partial m}\right)\Omega_{10} = 0$$

$$\frac{\partial}{\partial x_\alpha}\left[(U_0^{(\alpha)} + C_g^{(\alpha)})E\right] - \left(\frac{\rho}{2k^2}a^2\sigma^2\frac{k_\alpha}{k}\frac{k_\beta}{k}\frac{\partial^2 U_\beta}{\partial x_\alpha \partial z} - \frac{\rho}{4k^2}a^2\sigma^2\frac{\partial^2 U_\alpha}{\partial x_\alpha \partial z}\right) + S_{\alpha\beta}\frac{\partial U_0^{(\beta)}}{\partial x_\alpha} = 0$$

I　　　　　　　　II　　　　　　　　III

註：1. 上式中的第二項 (II) 代表新增的項。

2. 由於 $\frac{\partial}{\partial x_\alpha}[C_g^{(\alpha)}E] = \frac{\partial}{\partial x_\alpha}\left[\frac{\rho}{2k}a^2\sigma\left(\frac{1}{2}\frac{\sigma^2}{k}\frac{k_\alpha}{k} + \frac{\sigma}{k}\frac{k_\alpha}{k}\frac{k_\beta}{k}\frac{\partial U_\beta}{\partial z} - \frac{1}{2}\frac{\sigma}{k}\frac{\partial U_\alpha}{\partial z}\right)\right]$

表由羣速度所產生的wave energy flux隨流的剪應變變化，但此一現象會被第二項 (II) 抵消。

結論：

1. 當流中的垂直剪應變很大時，wave action一般無法守恆。
2. 在上述情況下wave energy equation亦須修改，使包含額外的項。
3. 新增加的項可使由羣速度所產生的wave energy flux之變化率和流中剪應變變化率無關。
4. 傳統的perturbation method及積分法在目前的情況下無法適用，其原因可加以解釋。